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CORE INVERSE OF OPERATORS IN HILBERT SPACES

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Abstract. This article includes the core inverses of operators on Hilbert spaces. In addition, the group inverses and Moore-Penrose inverses of these operators are characterised.

1. Introduction

Let $\mathscr{H},\mathscr{K},\mathscr{L}$ be the Hilbert spaces and $\mathscr{B}(\mathscr{H},\mathscr{K})$ denotes the algebra of bounded linear operators from $\mathscr{H}$ to $\mathscr{K}$. For an operator $A\in\mathscr{B}(\mathscr{H},\mathscr{K})$, if there exists an operator $B\in\mathscr{B}(\mathscr{K},\mathscr{H})$ such that $ABA=A$, then B is called a generalized inverse of A. Many researchers made efforts to explore different types of generalized inverses of different operators and established relationships between them. In 2005, Honk-Ke Du and Chun-Yuan Deng [5] explored the Drazin inverses of operators on a Hilbert space. In this paper, they showed the existence of Drazin inverse of bounded linear operator under Hilbert space decomposition and proved its uniqueness. In 2014, Dragana S. Rakic, Nebojsa C. Dincic, Dragana S. Djordevic [7] defined core and group inverses for bounded linear operators on $\mathscr{H}$ and found the sufficient conditions for existence of these inverses. V. Pavlovic, D. S. Cvetkovic-IIic [8] studied representations for regular operators $A\in\mathscr{B}(\mathscr{H},\mathscr{K})$ and $B\in\mathscr{B}(\mathscr{K},\mathscr{L})$. They also established generalized inverses for theses operators. Recently, J. M. Mwanzia, M. Kavila and J. M. Khalagai [6] studied Moore-Penrose inverses of closed range linear bounded operators on $\mathscr{H}$. A good survey of resources for generalized inverses are also found in [2, 9, 10].

These ideas of generalized inverses propelled us to think about the following questions.

a) Does there exist bounded linear operators on Hilbert space whose core inverses could be found?

b) If yes, then how are they interlinked with other generalized inverses?

The answers to these questions are validated. Section-1 includes the brief introduction and the literature review associated with this work. In section-2, we introduce several notations and preliminaries related to this work. In section-3, we introduce the matrix representations of idempotent operators and closed range normal operators under some Hilbert space decompositions of $\mathscr{H}$ and also deduce the core inverse, group

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inverse and Moore-Penrose inverse of these operators. Some relationships between operators and their generalized inverses are established. The existence of core inverse is explained through the right shift operator on $l^2(\mathbb{Z})$. Section-4 includes bounded linear operators A which has the operator matrix representation under the Hilbert space decomposition $\mathscr{H} = M \oplus M^{\perp}$, where M is the closed invariant subspace of $\mathscr{H}$ under A. Further, the above generalized inverses for posinormal operators are studied. In the last section, we establish matrix representations and generalized inverses for regular operators $A, B \in \mathscr{B}(\mathscr{H}, \mathscr{K})$.

2. Preliminaries and notations

If $\mathscr{H} = \mathscr{K}$, then $\mathscr{B}(\mathscr{H}, \mathscr{K})$ is denoted as $\mathscr{B}(\mathscr{H})$. If $A \in \mathscr{B}(\mathscr{H})$, then $Ran(A)$, $Nul(A)$ represent the range and null space of A respectively. The set of all idempotent operators, set of all closed range normal operators are denoted by $\mathscr{S}_{idm}$ and $\mathscr{S}_{cn}$ respectively. $\mathscr{S}_{ci}(A)$ represents the set of all closed invariant subspaces of $\mathscr{H}$ under A, where $A \in \mathscr{B}(\mathscr{H})$. Here $\mathscr{S}_{M_i}$ and $\mathscr{S}_{M_i^{\perp}}$ denote the set of all invertible operators on M and $M^{\perp}$ respectively.

DEFINITION 2.1. An operator $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ is said to have core inverse $X \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ if the following conditions will hold.

1. $AXA = A$ 2. $Ran(A) = Ran(X)$ 3. $Nul(A^*) = Nul(X)$.

Throughout this paper, we denote core inverse of A as $A^{\copyright}$.

DEFINITION 2.2. An operator $A^{\sharp} \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ will be the group inverse of $A \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ if the following properties are satisfied.

1. $AA^{\sharp}A = A$ 2. $A^{\sharp}AA^{\sharp} = A^{\sharp}$ 3. $AA^{\sharp} = A^{\sharp}A$.

DEFINITION 2.3. For $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$, the Moore-Penrose inverse $A^{\dagger} \in \mathscr{B}(\mathscr{K}, \mathscr{H})$ of A is an unique generalized inverse which satisfies the following Moore-Penrose equations.

1. $AA^{\dagger}A = A$ 2. $A^{\dagger}AA^{\dagger} = A^{\dagger}$ 3. $(AA^{\dagger})^* = AA^{\dagger}$ 4. $(A^{\dagger}A)^* = A^{\dagger}A$. An operator $A \in \mathscr{B}(\mathscr{H}, \mathscr{K})$ having Moore-Penrose inverse is called regular operator.

In this paper, the core, group and Moore-Penrose inverses of operators on the Hilbert spaces have been studied using the Hilbert space decompositions and different properties of these operators are established. In fact, some illustrations have been demonstrated for the existence of core inverses of some operators.

An operator $A \in \mathscr{B}(\mathscr{H})$ is called idempotent if $A^2 = A$, normal if $AA^* = A^*A$ and posinormal if there exists a positive operator $S \in \mathscr{B}(\mathscr{H})$ such that $AA^* = A^*SA$. A closed range operator $A \in \mathscr{B}(\mathscr{H})$ is termed as equal-projection (EP-operator) if one of the following conditions is satisfied:

$Ran(A) = Ran(A^*)$ or $Nul(A) = Nul(A^*)$ or $AA^{\dagger} = A^{\dagger}A$ or $A^{\dagger} = A^{\sharp}$. $A \in \mathscr{B}(\mathscr{H})$ is posinormal if $Ran(A) \subseteq Ran(A^*)$. If $A \in \mathscr{B}(\mathscr{H})$, then $\sigma(A)$, $\sigma_{ess}(A)$, $\sigma_p(A)$ and $\rho(A)$ represent spectrum, essential spectrum, point spectrum and resolvent of A respectively. Let $D(\sigma(A))$ denotes the set of limit points of the spectrum $\sigma(A)$.

Throughout this paper, for any $A \in \mathscr{B}(\mathscr{H})$ and $\mathscr{H} = H_1 \oplus H_2$, where H_1, H_2 are Hilbert subspaces of $\mathscr{H}$, we consider the matrix representation of A as

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \to \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

where

$$A_1 : H_1 \to H_1, \quad A_2 : H_2 \to H_1,$$
$$A_3 : H_1 \to H_2, \quad A_4 : H_2 \to H_2.$$

3. Main results

In this section, we investigate the core inverse, group inverse and Moore-Penrose inverse of the idempotent operator $A \in \mathscr{B}(\mathscr{H})$. Also, we study the equivalent conditions for EP-ness of the idempotent operators.

THEOREM 3.1. *If $A \in \mathscr{B}(\mathscr{H})$ be the idempotent operator, then the core inverse of A is the projection operator.*

Proof. Let $A : \mathscr{H} \to \mathscr{H}$ be the idempotent operator i.e $A^2 = A$. Now, $\forall y \in Ran(A)$,

$$\begin{aligned} &Ay = A(A(x)) = A^2(x) = A(x) = y \ (\because A(x) = y) \\ &\Rightarrow Ay = y \ \forall y \in Ran(A) \\ &\Rightarrow A = I \quad \text{on} \quad Ran(A). \end{aligned}$$

Now, $ker(I - A) = Ran(A)$ implies $Ran(A)$ is closed, then we have $H = Ran(A) \oplus (Ran(A))^{\perp} = Ran(A) \oplus Nul(A^*)$. Let $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$, where

$$A_1 : Ran(A) \to Ran(A), \quad A_2 : Nul(A^*) \to Ran(A),$$
$$A_3 : Ran(A) \to Nul(A^*), \quad A_4 : Nul(A^*) \to Nul(A^*).$$

For $x \in Ran(A)$, $A_3(x) = A|_{Ran(A)}(x) \in Ran(A)$ and $A_3(x) \in Nul(A^*)$.

$\Rightarrow A_3(x) = 0 \ \forall x \in Ran(A)$ implies $A_3 = 0$. Similarly $A_4 = 0$. So $A = \begin{bmatrix} I & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$. For the core invertible, $AXA = A$, $Ran(X) = Ran(A)$ and $Nul(X) = Nul(A^*)$. Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$. Now,

$$\begin{aligned} &\begin{bmatrix} I & A_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} I & A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & A_2 \\ 0 & 0 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} X_1 + A_2X_3 & (X_1 + A_2X_3)A_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & A_2 \\ 0 & 0 \end{bmatrix} \\ &\Rightarrow X_1 + A_2X_3 = I \quad \text{and} \quad (X_1 + A_2X_3)A_2 = A_2. \end{aligned}$$

Again,

$$Ran(X) = Ran(A)$$
$$\Rightarrow \left\{ X \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} : h_1 \in Ran(A),\ h_2 \in Nul(A^*) \right\} = \left\{ A \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} : k_1 \in Ran(A), k_2 \in Nul(A^*) \right\}$$

leads to $X_3h_1 + X_4h_2 = 0\ \forall\ h_1 \in Ran(A),\ h_2 \in Nul(A^*)$ which implies $X_3 = 0$ and $X_4 = 0$.

So, $X = \begin{bmatrix} X_1 & X_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$.

Now,

$$Nul(A^*) = \left\{ \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \in \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} : A^* \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ k_2 \end{bmatrix} : k_2 \in Nul(A^*) \right\}.$$

From $Nul(X) = Nul(A^*)$, we get $X_2k_2 = 0\ \forall\ k_2 \in Nul(A^*) \Rightarrow X_2 = 0$. Since $X_1 + A_2X_3 = I$, then $X_1 = I$. Thus the core inverse of the idempotent operator A is $A^{©} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, which is the projection operator on $Ran(A)$. □

THEOREM 3.2. *The group inverse of idempotent operator $A \in \mathscr{B}(\mathscr{H})$ is itself.*

Proof. If A is the idempotent operator, then from theorem (3.1), A has the representation

$$A = \begin{bmatrix} I & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}.$$

Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$.

Now,

$$AXA = A \quad \text{implies} \quad X_1 + A_2X_3 = I \quad \text{and} \quad (X_1 + A_2X_3)A_2 = A_2. \tag{3.1}$$

Again from

$$XAX = X, \quad \text{we have} \quad X_3(X_2 + A_2X_4) = X_4. \tag{3.2}$$

From the equation $AX = XA$, one can deduce the following equations

$$X_3 = 0, \quad X_1 + A_2X_3 = X_1, \quad X_2 + A_2X_4 = X_1A_2. \tag{3.3}$$

Since $X_3 = 0$, then from equations (3.1) and (3.2), $X_1 = I$, $X_4 = 0$. Also from equation (3.3), $X_2 = A_2$. Hence the group inverse of the idempotent operator A is $A^\sharp = \begin{bmatrix} I & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$, which is A. □

THEOREM 3.3. *The Moore-Penrose inverse of idempotent operator* $A = \begin{bmatrix} I & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$ *is* $A^\dagger = \begin{bmatrix} I - A_2X_3 & 0 \\ X_3 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$, *where* X_3A_2, A_2X_3 *are self adjoint and* $X_3^* = (I - A_2X_3)A_2$.

Proof. The matrix representation of the idempotent operator is

$$A = \begin{bmatrix} I & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}.$$

Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$.

Now,

$$AXA = A \Rightarrow X_1 + A_2X_3 = I \quad \text{and} \quad (X_1 + A_2X_3)A_2 = A_2. \tag{3.4}$$

Again

$$XAX = X \Rightarrow X_3(X_2 + A_2X_4) = X_4. \tag{3.5}$$

From, $(AX)^* = AX \Rightarrow X_2 + A_2X_4 = 0$. Using this result in equation (3.5), we get $X_4 = 0$ and also $X_2 = 0$. Now from the given conditions, one can get $(XA)^* = XA$. Using the equation (3.4), we get the required result. □

COROLLARY 3.4. *If* $A \in \mathscr{B}(\mathscr{H})$ *is an idempotent operator, then the following conditions are equivalent.*

a) A is a Projection.

b) A is EP.

c) A is normal.

Proof. $(a) \Rightarrow (b)$ Let A is a projection, then

$$A = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}.$$

Here $Ran(A) = Ran(A^*) \Rightarrow A$ is EP.

$(b) \Rightarrow (c)$ Let A is EP, then $A^\dagger = A^\sharp \Rightarrow A_2 = 0 \Rightarrow A = P_{Ran(A)}$. So A is normal.

$(c) \Rightarrow (a)$ If A is normal $\Rightarrow A_2 = 0 \Rightarrow A = P_{Ran(A)}$. □

Now, core inverses of the closed range normal operators are established under a Hilbert space decomposition. In addition, we study group inverses and Moore-Penrose inverses of these operators. Also the existence of core inverses of normal Fredholm operators on the Hilbert space is given.

THEOREM 3.5. *If* $A \in \mathscr{B}(\mathscr{H})$ *is a closed range normal operator and* $0 \notin \sigma_p(A)$, *then the core inverse of* A *is*

$$A^{\copyright} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix}.$$

Proof. Let $A \in \mathscr{B}(\mathscr{H})$. Since A is normal, then $Nul(A) = Nul(A^*)$ and $Ran(A) = Ran(A^*)$. As $Ran(A)$ is closed in $\mathscr{H}$, then $\mathscr{H} = Ran(A) \oplus Ran(A)^{\perp} = Ran(A) \oplus Nul(A^*) = Ran(A) \oplus Nul(A)$.

Let

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix},$$

where

$$A_1 : Ran(A) \to Ran(A), \quad A_2 : Nul(A) \to Ran(A),$$

$$A_3 : Ran(A) \to Nul(A), \quad A_4 : Nul(A) \to Nul(A).$$

For $x \in Ran(A)$, $(x) = A|_{Ran(A)}(x) \in Ran(A)$ and $A_3(x) \in Nul(A)$ leads to $A_3(x) = 0\ \forall x \in Ran(A)$. So $A_3 = 0$. Similarly $A_4 = 0$ and $A_2 = 0$.

Thus, $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix}$. Since $0 \notin \sigma_p(A)$, then $ker(A) = \{0\} \Rightarrow ker(A_1) = 0$ and also $Ran(A) = Ran(A_1)$, implies A_1 is invertible.

If $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix}$. Now, from $AXA = A$, we can get $X_1 = A_1^{-1}$.

From $Ran(X) = Ran(A)$, one can compute $X_3 = 0$ and $X_4 = 0$.

If $Nul(X) = Nul(A^*)$, then $X_2 = 0$.

Thus $A^{©} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix}$, which is the core inverse of A. □

EXAMPLE 3.6. Let $I : \mathscr{H} \to \mathscr{H}$ be the identity operator which is closed range normal operator and $0 \notin \sigma_p(I)$,, then the core inverse of I is the projection operator under the Hilbert space decomposition $\mathscr{H} \oplus \{0\}$.

COROLLARY 3.7. *The group inverse and the Moore-Penrose inverse of the closed range normal operator $A \in \mathscr{B}(\mathscr{H})$ with $0 \notin \sigma_p(A)$, equals its core inverse.*

Proof. If A is closed range normal operator and $0 \notin \sigma_p(A)$, then from theorem (3.5), A has the representation

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix}.$$

Then, the result follows from [7]. □

COROLLARY 3.8. *If $A \in \mathscr{B}(\mathscr{H})$ is normal and $0 \notin \sigma_{ess}(A)$, then core inverse of A exists.*

Proof. Since $0 \notin \sigma_{ess}(A)$, then A is Fredholm operator. So $Ran(A)$ is closed. As A is normal, then $\mathscr{H} = Ran(A) \oplus Nul(A)$. Then from the theorem, A has the representation $A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A) \end{bmatrix}$.

Since $Nul(A_1) = Ran(A) \cap Nul(A) = \{0\}$ and $Ran(A_1) = Ran(A)$. Hence result follows from the theorem. □

EXAMPLE 3.9. Let $\mathscr{H} = l^2(\mathbb{Z}) = \{u = (\cdots u_{-2}, u_{-1}, u_0, u_1, u_2, \cdots) : \sum_{i=-\infty}^{\infty} u_i^2 < \infty\}$. The right shift operator on $\mathscr{H}$ defined by $\mathbf{R}(\mathbf{f_i}) = \mathbf{f_{i+1}}$, $i \in \mathbb{Z}$, where $\{f_i\}$ is an orthonormal basis for $l^2(\mathbb{Z})$. As $\mathbf{R}$ is normal and fredholm, then the core inverse of R is $R^{©} = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(\mathbf{R}) \\ Nul(\mathbf{R}) \end{bmatrix} \to \begin{bmatrix} Ran(\mathbf{R}) \\ Nul(\mathbf{R}) \end{bmatrix}$, under the decomposition $\mathscr{H} = \mathscr{H} \oplus \{0\}$ and $A_1 : \mathscr{H} \to \mathscr{H}$ is any invertible operator.

REMARK 3.10. If $A_1 : \mathscr{H} \to \mathscr{H}$ be the identity operator, then the core inverse of right shift operator is the projection operator.

4. Core inverses of operators associated with invariant subspaces

In this section, we obtain the core inverses, group inverses and Moore-Penrose inverses of some operators on $\mathscr{H} = M \oplus M^{\perp}$, where M is invariant under A.

THEOREM 4.1. *If $M \in \mathscr{S}_{ci}(A)$, $0 \in \rho(A_1) \cap \rho(A_4)$, then the core inverse of A is*

$$A^{©} = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} : \begin{bmatrix} M \\ M^{\perp} \end{bmatrix} \to \begin{bmatrix} M \\ M^{\perp} \end{bmatrix},$$

where $Ran(A_2) = M$.

Proof. Since M is a closed invariant subspace of H under A, then the operator A has the representation of the form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_4 \end{bmatrix} : \begin{bmatrix} M \\ M^{\perp} \end{bmatrix} \to \begin{bmatrix} M \\ M^{\perp} \end{bmatrix}.$$

Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} M \\ M^{\perp} \end{bmatrix} \to \begin{bmatrix} M \\ M^{\perp} \end{bmatrix}$. From $AXA = A$, one can deduce the equations:

$$(A_1X_1 + A_2X_3)A_1 = A_1, \quad (A_1X_1 + A_2X_3)A_2 + (A_1X_2 + A_2X_4)A_4 = A_2,$$

$$A_4X_3A_1 = 0 \quad \text{and} \quad A_4X_3A_2 + A_4X_4A_4 = A_4.$$

As $0 \in \rho(A_1) \cap \rho(A_4) \Rightarrow A_1$ and A_4 are invertible, then $X_3 = 0$, $X_1 = A_1^{-1}$, $X_4 = A_4^{-1}$ and $X_2 = -A_1^{-1}A_2A_4^{-1}$.

Now, $X = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} : \begin{bmatrix} M \\ M^{\perp} \end{bmatrix} \to \begin{bmatrix} M \\ M^{\perp} \end{bmatrix}$.

Here $Ran(X) = Ran(A)$, as $Ran(A_2) = M$. Since $0 \in \rho(A_1) \cap \rho(A_4) \Rightarrow A_1$ and A_4 are invertible, then $Nul(A^*) = Nul(X) = \{0\}$. So X is the core inverse of A. Hence the result follows. □

COROLLARY 4.2. *If $A \in \mathscr{B}(\mathscr{H})$ be the closed range posinormal operator, $0 \notin D(\sigma(A_1))$, $\|I - A_4\|_{Nul(A)} < 1$, then the core inverse of A under the decomposition $\mathscr{H} = N(A)^\perp \oplus Nul(A)$ is*

$$A^{©} = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} : \begin{bmatrix} Nul(A)^\perp \\ Nul(A) \end{bmatrix} \to \begin{bmatrix} Nul(A)^\perp \\ Nul(A) \end{bmatrix}.$$

Proof. Since 0 is not a limit point of spectrum of A_1, then from [1], A_1 is invertible. As $\|I - A_4\|_{Nul(A)} < 1$, then $A_4|_{Nul(A)^\perp}$ is invertible. Hence result follows. □

COROLLARY 4.3. *If M reduces A, $A_1 \in \mathscr{S}_{M_i}$ and $A_4 \in \mathscr{S}_{M_i^\perp}$, then the core inverse of A is*

$$A^{©} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & A_4^{-1} \end{bmatrix} : \begin{bmatrix} M \\ M^\perp \end{bmatrix} \to \begin{bmatrix} M \\ M^\perp \end{bmatrix}.$$

Proof. Since M reduces A, then $A_2 = 0$. Hence the result follows from the theorem. □

COROLLARY 4.4. *If $M \in \mathscr{S}_{ci}(A)$, $A_1 \in \mathscr{S}_{M_i}$, $A_4 \in \mathscr{S}_{M_i^\perp}$, then the group inverse of A is same as core inverse.*

Proof. From the theorem, if $AXA = A$, then X has the form

$$X = \begin{bmatrix} A_1^{-1} & -A_1^{-1}A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix} : \begin{bmatrix} M \\ M^\perp \end{bmatrix} \to \begin{bmatrix} M \\ M^\perp \end{bmatrix}.$$

Since $XA = I$, then $XAX = X$ and $XA = AX$. Hence the result follows. □

COROLLARY 4.5. *If $A, B \in \mathscr{B}(H)$ with $AB = BA$ and $Nul(B)$ reduces B, then*
a) The representation of A is $A = \begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix} : \begin{bmatrix} \overline{Ran(B^*)} \\ Nul(B) \end{bmatrix} \to \begin{bmatrix} \overline{Ran(B^*)} \\ Nul(B) \end{bmatrix}$.
b) If A_1 and A_4 are invertible, then the core inverse of A is

$$A^{©} = \begin{bmatrix} A_1^{-1} & 0 \\ -A_4^{-1}A_3A_1^{-1} & A_4^{-1} \end{bmatrix} : \begin{bmatrix} \overline{Ran(B^*)} \\ Nul(B) \end{bmatrix} \to \begin{bmatrix} \overline{Ran(B^*)} \\ Nul(B) \end{bmatrix},$$

where $Ran(A_3) = Nul(A)$.

Proof. a) Since A is an operator in the commutant of B and $N(B)$ reduces B, then from [1], the result follows.

b) As A_1 and A_4 are invertible with $Ran(A_3) = Nul(A)$, the result emanates from the theorem. □

COROLLARY 4.6. *If $A,B \in \mathscr{B}(H)$ with $AB = BA$ and B is posinormal, then the core inverse of A is*

$$A^{\copyright} = \begin{bmatrix} A_1^{-1} & 0 \\ -A_4^{-1}A_3A_1^{-1} & A_4^{-1} \end{bmatrix} : \begin{bmatrix} \overline{Ran(B^*)} \\ Nul(B) \end{bmatrix} \to \begin{bmatrix} \overline{Ran(B^*)} \\ Nul(B) \end{bmatrix},$$

where $Ran(A_3) = Nul(A)$.

Proof. Since B is posinormal, then $Ran(B)$ reduces B. Hence the result follows from the theorem. $\square$

5. Core inverses of regular operators

In this section, we establish the core inverses, group inverses and Moore-Penrose inverses of regular operators on the Hilbert space.

THEOREM 5.1. *If $A,B \in \mathscr{B}(H,K)$ be the two regular operators with $Ran(B) = Ran(A)$, then*

a) The representation of A is $A = \begin{bmatrix} A_1 & A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix} \to \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix}$.

b) If A_2 is invertible, then the core inverse of A is $A^{\copyright} = \begin{bmatrix} C_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix}$, where $Ran(C_1 - A_1) = Ran(A)$, $A_1C_1 = I$.

Proof. a) Since A,B are regular, then AX and BX are projectors and $Ran(AX) = Ran(A)$, $Ran(BX) = (B) \Rightarrow Ran(A)$ and $Ran(B)$ are closed. Then $H = Ran(B) \oplus Nul(B^*)$, $K = Ran(A) \oplus Nul(A^*)$. Hence the result follows.

b) Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix}$. From $AXA = A$, we get

$$(A_1X_1 + A_2X_3)A_1 = A_1 \quad \text{and} \quad (A_1X_1 + A_2X_3)A_2 = A_2. \tag{5.1}$$

As $Ran(B) = Ran(A)$ and $Nul(B^*) = Nul(A^*)$, then from $Ran(X) = Ran(A)$, one can deduce $X_3 = 0$ and $X_4 = 0$. Similarly from invertibility of A_2 and $Nul(X) = Nul(A^*)$, $X_2 = 0$. Then, $X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix}$.

Now, since $Ran(X_1 - A_1) = Ran(A)$ and $A_1X_1 = I$ for any $X_1 : Ran(A) \to Ran(B)$, then the result follows. $\square$

COROLLARY 5.2. *If $A_1X_1 = X_1A_1 = I$, then the core inverse of A is*

$$A^{\copyright} = \begin{bmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix},$$

where A_2 is surjective.

THEOREM 5.3. *If A, B are two regular operators with $Ran(B) = Ran(A)$, then*

$$A^{\dagger} = \begin{bmatrix} A_1^{-1}(I - A_2X_3) & 0 \\ X_3 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix},$$

where $0 \notin \sigma(A_1)$, $A_1^{-1}(I - A_2X_3)A_1$, X_3A_2 are self adjoint and $(X_3A_1)^* = X_3A_2$.

Proof. Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix}$, be the Moore-Penrose inverse of A. Then from $AXA = A$ we will have

$$(A_1X_1 + A_2X_3)A_1 = A_1 \quad \text{and} \quad A_1X_2 + A_2X_4 = A_2. \tag{5.2}$$

As $XAX = X$, then will get the equations

$$\begin{aligned} X_1(A_1X_1 + A_2X_3) = X_1, \quad X_1(A_1X_2 + A_2X_4) = X_2 \\ X_3(A_1X_1 + A_2X_3) = X_3, \quad X_3(A_1X_2 + A_2X_4) = X_4. \end{aligned} \tag{5.3}$$

Now $(AX)^* = AX$ implies

$$(A_1X_1 + A_2X_3)^* = A_1X_1 + A_2X_3, \quad (A_1X_2 + A_2X_4) = 0. \tag{5.4}$$

Substituting equation (5.4) in equation (5.3), we get $X_4 = 0$ and also from equation (5.4), $X_2 = 0$. Again as $0 \notin \sigma(A)$, then from (5.2), $X_1 = A_1^{-1}(I - A_2X_3)$. Finally, the conditions X_1A_1, X_3A_2 are self adjoint and $(X_3A_1)^* = X_3A_2$, lead to $(XA)^* = XA$. So $A^{\dagger} = \begin{bmatrix} A_1^{-1}(I - A_2X_3) & 0 \\ X_3 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix}$. □

THEOREM 5.4. *If $A, B \in \mathscr{B}(H, K)$ be the regular operators with $Ran(B) = Ran(A)$ and A_1 is invertible, then the group inverse of A is*

$$A^{\sharp} = \begin{bmatrix} A_1^{-1} & (A_1^{-1})^2A_2 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix}.$$

Proof. If $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} Ran(A) \\ Nul(A^*) \end{bmatrix} \to \begin{bmatrix} Ran(B) \\ Nul(B^*) \end{bmatrix}$. From $AXA = A$, we get

$$(A_1X_1 + A_2X_3)A_1 = A_1 \quad \text{and} \quad (A_1X_1 + A_2X_3)A_2 = A_2. \tag{5.5}$$

Again, from $XAX = X$,

$$\begin{aligned} X_1(A_1X_1 + A_2X_3) = X_1, \quad X_1(A_1X_2 + A_2X_4) = X_2 \\ X_3(A_1X_1 + A_2X_3) = X_3, \quad X_3(A_1X_2 + A_2X_4) = X_4. \end{aligned} \tag{5.6}$$

From, $AX = XA$,

$$\begin{aligned} (A_1X_1 + A_2X_3) = X_1A_1, \quad (A_1X_2 + A_2X_4) = X_1A_2 \\ X_3A_1 = 0, \quad X_3A_2 = 0. \end{aligned} \tag{5.7}$$

As A_1 is invertible, then from (5.7), $X_3 = 0$ and form (5.5), $X_1 = A_1^{-1}$. Now from (5.6), $X_4 = 0$. Also from (5.7), $X_2 = (A_1^{-1})^2A_2$. Hence the result follows. □

THEOREM 5.5. *If* $A \in \mathscr{B}(H_1 \oplus H_1^{\perp})$ *with* $A = \begin{bmatrix} 0 & 0 \\ A_3 & A_4 \end{bmatrix}$, *where* H_1 *is a Hilbert space, then*

a) $A^{©} = \begin{bmatrix} 0 & 0 \\ 0 & A_4^{-1} \end{bmatrix} : \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix} \to \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix}$, *where* $\|I - A_4\|_{H_1^{\perp}} < 1$ *and* $Ran[A_3, A_4] = H_1^{\perp}$.

b) If $C \in \mathscr{B}(H_1^{\perp})$ *and* $0 \notin \sigma_p(C)$, *then*

$$A^{©} = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} : \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix} \to \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix},$$

where $\|I - A_3\| < 1$, $A_4C = I$ *and* $Ran[A_3, A_4] = Ran(C)$.

Proof. a) Let $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix} \to \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix}$, be the core inverse of A. Then from $AXA = A$, we have

$$(A_3X_2 + A_4X_4)A_3 = A_3 \quad \text{and} \quad (A_3X_2 + A_4X_4)A_4 = A_4. \tag{5.8}$$

If $Ran(X) = Ran(A)$, then one can deduce

$$X_1h_1 + X_2h_2 = 0 \quad \text{and} \quad X_3h_1 + X_4h_2 = A_3h_1 + A_4h_2 \ \forall h_1 \in H_1, \ \ h_2 \in H_1^{\perp} \tag{5.9}$$

implies $X_1 = X_2 = 0$. Now $X = \begin{bmatrix} 0 & 0 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix} \to \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix}$.

As $\|I - A_4\|_{H_1^{\perp}} < 1$, then A_4 is invertible. So $Nul(A^*) = \left\{ \begin{bmatrix} k_1 \\ 0 \end{bmatrix} : k_1 \in H_1 \right\}$. If $Nul(X) = Nul(A^*)$, then we get $X_3 = 0$. As A_4 is invertible, then from equation (5.8), we have $X_4 = A_4^{-1}$. Since $Ran[A_3, A_4] = H_1^{\perp}$, then equation (5.9) holds, implies $Ran(X) = Ran(A)$. Hence $A^{©} = \begin{bmatrix} 0 & 0 \\ 0 & A_4^{-1} \end{bmatrix} : \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix} \to \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix}$.

b) If $X = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} : \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix} \to \begin{bmatrix} H_1 \\ H_1^{\perp} \end{bmatrix}$. As $A_4C = I$, then $AXA = A$. Again from $\|I - A_3\| < 1$, A_3 is invertible implies $Nul(A^*) = \left\{ \begin{bmatrix} k_1 \\ 0 \end{bmatrix} : k_1 \in H_1 \right\}$. Since $0 \notin \sigma_p(C)$, then $Nul(X) = Nul(A^*)$. Finally, given condition $Ran[A_3, A_4] = Ran(C)$ implies $Ran(X) = Ran(A)$. Thus X is the core inverse of A. $\square$

6. Conclusion

In this work, we have focused on the generalized inverses of operators which have the different operator matrix representations with respect to the several Hilbert space decompositions. The core inverses of operators are used to solve the linear system of equations, signal processing such as image restoration, noise reduction and deconvolution. In addition, core inverse techniques are applied for regularization and constrained optimization tasks in optimization problems.

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