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NONLINEAR SKEW LIE TYPE HIGHER DERIVATIONS ON SOME OPERATOR ALGEBRAS

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Abstract. Let $\mathscr{A}$ be a unital $*$-algebra. Let $p_n(A_1,A_2,\cdots,A_n)$ be the polynomial defined by n indeterminates $A_1,A_2,\cdots,A_n \in \mathscr{A}$ and their multiple skew Lie product, and $\mathbb{N}$ be the set of non-negative integers. In this paper, under some mild conditions on $\mathscr{A}$, it is shown that if $\mathscr{D}=\{d_m\}_{m\in\mathbb{N}}$ is the family of maps $d_m:\mathscr{A}\to\mathscr{A}$ such that $d_0=id_{\mathscr{A}}$, the identity map on $\mathscr{A}$ satisfying

$$d_m(p_n(A_1,A_2,\cdots,A_n))=\sum_{i_1+i_2+\cdots+i_n=m} p_n(d_{i_1}(A_1),d_{i_2}(A_2),\cdots,d_{i_n}(A_n))$$

for all $A_1,A_2,\cdots,A_n\in\mathscr{A}$ and for each $m\in\mathbb{N}$, then $\mathscr{D}=\{d_m\}_{m\in\mathbb{N}}$ is an additive $*$-higher derivation. Moreover, we apply the above result to prime $*$-algebras, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras and standard operator algebras.

1. Introduction

Let $\mathscr{A}$ be a $*$-algebra over the complex field $\mathbb{C}$. For $A,B\in\mathscr{A}$, denote by $[A,B]_*=AB-BA^*$ the skew Lie product of A and B. The skew Lie product is found playing a more and more important role in some research topics, and its study has recently attracted many authors' attention (for example, see [2, 6, 12, 14–17, 21]). The product is extensively studied because it naturally arises in the problem of representing quadratic functionals with sesquilinear functionals (see [14–16]) and in the problem of characterizing ideals (see [2, 12]).

Recall that an additive map $\Phi:\mathscr{A}\to\mathscr{A}$ is said to be an additive derivation if $\Phi(AB)=\Phi(A)B+A\Phi(B)$ for all $A,B\in\mathscr{A}$. We say that Φ is an additive $*$-derivation if it is an additive derivation and satisfies $\Phi(A^*)=\Phi(A)^*$ for all $A\in\mathscr{A}$. A map $\Phi:\mathscr{A}\to\mathscr{A}$ is said to be a nonlinear skew Lie derivation if

$$\Phi([A,B]_*)=[\Phi(A),B]_*+[A,\Phi(B)]_*$$

for all $A,B\in\mathscr{A}$. More generally, we say that a map $\Phi:\mathscr{A}\to\mathscr{A}$ is a nonlinear skew Lie triple derivation if

$$\Phi([[A,B]_*,C]_*)=[[\Phi(A),B]_*,C]_*+[[A,\Phi(B)]_*,C]_*+[[A,B]_*,\Phi(C)]_*$$

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Paper OaM-18-34

for all $A,B,C \in \mathscr{A}$.

Given the consideration of skew Lie derivations and skew Lie triple derivations, we can further develop them in one natural way. Suppose that $n \geqslant 2$ is a fixed positive integer. Let us see a sequence of polynomials with $*$

$$p_1(A_1) = A_1,$$

$$p_1(A_1,A_2) = [A_1,A_2]_*,$$
$$p_1(A_1,A_2,A_3) = [[A_1,A_2]_*,A_3]_*,$$
$$\cdots,$$
$$p_n(A_1,A_2,\cdots,A_n) = [p_{n-1}(A_1,A_2,\cdots,A_{n-1}),A_n]_*.$$

According, a nonlinear skew Lie n-derivation is a map $\Phi : \mathscr{A} \to \mathscr{A}$ satisfying the condition

$$\Phi(p_n(A_1,A_2,\cdots,A_n) = \sum_{k=1}^{n} p_n(A_1,\cdots,A_{k-1},\Phi(A_k),A_{k+1},\cdots,A_n)$$

for all $A_1,A_2,A_3,\cdots,A_n \in \mathscr{A}$. Clearly, every nonlinear skew Lie derivation is a nonlinear skew Lie 2-derivation and every nonlinear skew Lie triple derivation is a nonlinear skew Lie 3-derivation. These type of nonlinear skew Lie derivations, nonlinear skew Lie triple derivations and nonlinear skew Lie n-derivations collectively known as nonlinear skew Lie type derivations.

Nonlinear skew Lie-type derivations in different backgrounds are extensively studied by several authors (see [3, 5, 7–11, 19–21, 24]). Yu and Zhang in [21] proved that every nonlinear skew Lie derivation between factor von Neumann algebras is an additive $*$-derivation. This result was extended to the case of nonlinear skew Lie triple derivations by Li et al. [8]. Let $\mathscr{A}$ be a standard operator algebra which is closed under the adjoint operation. Lin [10] proved that every nonlinear skew Lie type derivation Φ on $\mathscr{A}$ is automatically linear. Moreover, Φ is an inner $*$-derivation. Lin [9] also proved that every nonlinear skew Lie type derivation between von Neumann algebras without central abelian projections is an additive $*$-derivation. Under some mild conditions on a unital $*$-algebra $\mathscr{A}$, Madni et al. [11] proved that every nonlinear skew Lie type derivation Φ on $\mathscr{A}$ is additive. Moreover, if $\Phi(i\frac{I}{2})$ is selfadjoint, then Φ is an additive $*$-derivation.

Let $\mathbb{N}$ be the set of non-negative integers and $\mathscr{D} = \{d_m\}_{m\in\mathbb{N}}$ be a family of maps $d_m : \mathscr{A} \to \mathscr{A}$ (not necessarily linear) such that $d_0 = id_{\mathscr{A}}$, the identity map on $\mathscr{A}$. $\mathscr{D}$ is called an additive higher derivation if for each $m \in \mathbb{N}$, d_m is additive and satisfies the condition

$$d_m(AB) = \sum_{i+j=m} d_i(A)d_j(B)$$

for all $A,B \in \mathscr{A}$. A family $\mathscr{D} = \{d_m\}_{m\in\mathbb{N}}$ of maps $d_m : \mathscr{A} \to \mathscr{A}$ is said to be:

(i) a nonlinear skew Lie higher derivation on $\mathscr{A}$ if for each $m \in \mathbb{N}$,

$$d_m([A,B]_*) = \sum_{i+j=m} [d_i(A),d_j(B)]_*$$

$A, B \in \mathscr{A}$.

(ii) a nonlinear skew Lie triple higher derivation on $\mathscr{A}$ if for each $m \in \mathbb{N}$,

$$d_m([[A,B]_*,C]_*) = \sum_{i+j+k=m} [[d_i(A),d_j(B)]_*,d_k(C)]_*$$

$A, B, C \in \mathscr{A}$.

(iii) a nonlinear skew Lie n-higher derivation on $\mathscr{A}$ if for each $m \in \mathbb{N}$,

$$d_m(p_n(A_1,A_2,\cdots,A_n)) = \sum_{i_1+i_2+\cdots+i_n=m} p_n(d_{i_1}(A_1),d_{i_2}(A_2),\cdots,d_{i_n}(A_n))$$

for all $A_1,A_2,\cdots,A_n \in \mathscr{A}$. Clearly, every nonlinear skew Lie higher derivation is a nonlinear skew Lie 2-higher derivation and every nonlinear skew Lie triple higher derivation is a nonlinear skew Lie 3-higher derivation. These type of nonlinear skew Lie higher derivations, nonlinear skew Lie triple higher derivations and nonlinear skew Lie n-higher derivations are collectively referred to as nonlinear skew Lie type higher derivations.

It is the objective of this article to investigate nonlinear skew Lie type higher derivations. Many authors have paid more attentions on the related topics (for example, see [1, 4, 13, 22, 23, 25]). Zhang et al [25] showed that every nonlinear skew Lie higher derivation on factor von Neumann algebras is linear. Ashraf et al. [1] observed that every nonlinear skew Lie triple higher derivation on standard operator algebra is an additive $*$-higher derivation. Wani et al. [22] proved every nonlinear skew Lie n-higher derivation on standard operator algebra is an additive $*$-higher derivation. Let $\mathscr{A}$ be a unital $*$-algebra over the complex field $\mathbb{C}$. In our current paper, under some mild conditions on $\mathscr{A}$, we prove that every nonlinear skew Lie n-higher derivation on $\mathscr{A}$ is an additve $*$-higher derivation. As applications, nonlinear skew Lie n-higher derivations on prime $*$-algebras, von Neumann algebras with no central summands of type I_1, factor von Neumann algebras and standard operator algebras are characterized.

2. The main result and its proof

The following is our main result in this paper.

THEOREM 1. *Let $\mathscr{A}$ be a unital $*$-algebra with the unit I and P be a nontrivial projection in $\mathscr{A}$. Assume that $\mathscr{A}$ satisfies*

$$(\spadesuit) \quad X\mathscr{A}P = 0 \quad \textit{implies} \quad X = 0$$

and

$$(\clubsuit) \quad X\mathscr{A}(I-P) = 0 \quad \textit{implies} \quad X = 0.$$

Then every nonlinear skew Lie type higher derivation $\mathscr{D} = \{d_m\}_{m\in\mathbb{N}}$ on $\mathscr{A}$ is an additive $*$-higher derivation.

In the following, let $P_1 = P$ and $P_2 = I - P$. Denote $\mathscr{A}_{ij} = P_i \mathscr{A} P_j$, $i, j = 1, 2$. Then $\mathscr{A} = \sum_{i,j=1}^{2} \mathscr{A}_{ij}$. For every $A \in \mathscr{A}$, we may write $A = \sum_{i,j=1}^{2} A_{ij}$. In all that follows, when we write A_{ij}, it indicates that $A_{ij} \in \mathscr{A}_{ij}$. We will complete the proof by several lemmas.

LEMMA 1. *$d_m(0) = 0$ for each $m \in \mathbb{N}$.*

Proof. Processing by induction on $m \in \mathbb{N}$ with $m \geqslant 1$. If $m = 1$, by Lemma 1 in [11], the result is true. Now assume that the result holds true for $k < m$, i.e. $d_k(0) = 0$. Our aim is to prove that $d_m(0) = 0$. Then

$$\begin{aligned} d_m(0) &= d_m(p_n(0,0,\cdots,0)) \\ &= \sum_{i_1+i_2+\cdots+i_n=m} p_n(d_{i_1}(0), d_{i_2}(0), \cdots, d_{i_n}(0)) \\ &= p_n(d_m(0),0,\cdots,0) + p_n(0,d_m(0),\cdots,0) + \cdots + p_n(0,0,\cdots,d_m(0)) \\ &\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n(d_{i_1}(0), d_{i_2}(0), \cdots, d_{i_n}(0)) \\ &= 0. \quad \square \end{aligned}$$

LEMMA 2. *For any $A_{12} \in \mathscr{A}_{12}$ and $B_{21} \in \mathscr{A}_{21}$, we have*

$$d_m(A_{12} + B_{21}) = d_m(A_{12}) + d_m(B_{21}).$$

Proof. Using the induction on $m \in \mathbb{N}$ with $m \geqslant 1$. By Lemma 2 in [11], the result holds true for $m = 1$. Assume that the result holds true for $k < m$, i.e. $d_k(A_{12} + B_{21}) = d_k(A_{12}) + d_k(B_{21})$. Let

$$T = d_m(A_{12} + B_{21}) - d_m(A_{12}) - d_m(B_{21}).$$

Let us now show that $T = 0$. It is easy to see that for each $A \in \mathscr{A}$,

$$p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, A_{12}\Big) = p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, B_{21}\Big) = 0.$$

On one hand, we get

$$\begin{aligned} 0 &= d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, A_{12} + B_{21}\Big)\Big) \\ &= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_1 - P_2, A_{12} + B_{21}\Big) \\ &\quad + p_n\Big(\frac{1}{2}iI, d_m\Big(\frac{1}{2}I\Big), \cdots, P_1 - P_2, A_{12} + B_{21}\Big) \\ &\quad + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_1 - P_2, d_m(A_{12} + B_{21})\Big) \\ &\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}\Big), \cdots, d_{i_{n-1}}(P_1 - P_2), d_{i_n}(A_{12} + B_{21})\Big). \end{aligned}$$

On the other hand, by induction hypothesis, we have

$$
\begin{aligned}
0 &= d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,P_1-P_2,A_{12}+B_{21}\Big)\Big)\\
&= d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1-P_2,A_{12}\Big)\Big)\\
&\quad + d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1-P_2,B_{21}\Big)\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1-P_2,A_{12}\Big)\\
&\quad + p_n\Big(\frac{1}{2}iI,d_m\Big(\frac{1}{2}I\Big),\cdots,P_1-P_2,A_{12}\Big)\\
&\quad +\cdots+ p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1-P_2,d_m(A_{12})\Big)\\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}\Big),\cdots,d_{i_{n-1}}(P_1-P_2),d_{i_n}(A_{12})\Big)\\
&\quad + p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1-P_2,B_{21}\Big) + p_n\Big(\frac{1}{2}iI,d_m\Big(\frac{1}{2}I\Big),\cdots,P_1-P_2,B_{21}\Big)\\
&\quad +\cdots+ p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1-P_2,d_m(B_{21})\Big)\\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}\Big),\cdots,d_{i_{n-1}}(P_1-P_2),d_{i_n}(B_{21})\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1-P_2,A_{12}+B_{21}\Big)\\
&\quad + p_n\Big(\frac{1}{2}iI,d_m\Big(\frac{1}{2}I\Big),\cdots,P_1-P_2,A_{12}+B_{21}\Big)\\
&\quad +\cdots+ p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1-P_2,d_m(A_{12})+d_m(B_{21})\Big)\\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}\Big),\cdots,d_{i_{n-1}}(P_1-P_2),d_{i_n}(A_{12})+d_{i_n}(B_{21})\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1-P_2,A_{12}+B_{21}\Big)\\
&\quad + p_n\Big(\frac{1}{2}iI,d_m\Big(\frac{1}{2}I\Big),\cdots,P_1-P_2,A_{12}+B_{21}\Big)\\
&\quad +\cdots+ p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1-P_2,d_m(A_{12})+d_m(B_{21})\Big)\\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}\Big),\cdots,d_{i_{n-1}}(P_1-P_2),d_{i_n}(A_{12}+B_{21})\Big).
\end{aligned}
$$

Comparing the above two equations, we see that $p_n(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,P_1-P_2,T)=0$. This leads to $T_{11}=T_{22}=0$. Invoking the fact that $p_n(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,A_{12},P_1)=0$ and

Lemma 2.2, we find that

$$
\begin{aligned}
& d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,A_{12}+B_{21},P_1\Big)\Big)\\
&= d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,A_{12},P_1\Big)\Big)+d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,B_{21},P_1\Big)\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,A_{12},P_1\Big)+p_n\Big(\frac{1}{2}iI,d_m\Big(\frac{1}{2}I\Big),\cdots,A_{12},P_1\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,d_m(A_{12}),P_1\Big)+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,A_{12},d_m(P_1)\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}\Big),\cdots,d_{i_{n-1}}(A_{12}),d_{i_n}(P_1)\Big)\\
&\quad +p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,B_{21},P_1\Big)+P_n\Big(\frac{1}{2}iI,d_m\Big(\frac{1}{2}I\Big),\cdots,B_{21},P_1\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,d_m(B_{21}),P_1\Big)+P_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,B_{21},d_m(P_1)\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}\Big),\cdots,d_{i_{n-1}}(B_{21}),d_{i_n}(P_1)\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,A_{12}+B_{21},P_1\Big)+P_n\Big(\frac{1}{2}iI,d_m\Big(\frac{1}{2}I\Big),\cdots,A_{12}+B_{21},P_1\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,d_m(A_{12}+B_{21}),P_1\Big)\\
&\quad +p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,A_{12}+B_{21},d_m(P_1)\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(A_{12})+d_{i_{n-1}}(B_{21}),d_{i_n}(P_1)\Big).
\end{aligned}
$$

On the other hand, by induction hypothesis, we also have

$$
\begin{aligned}
& d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,A_{12}+B_{21},P_1\Big)\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,A_{12}+B_{21},P_1\Big)+p_n\Big(\frac{1}{2}iI,d_m\Big(\frac{1}{2}I\Big),\cdots,A_{12}+B_{21},P_1\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,d_m(A_{12}+B_{21}),P_1\Big)\\
&\quad +p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,A_{12}+B_{21},d_m(P_1)\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(A_{12})+d_{i_{n-1}}(B_{21}),d_{i_n}(P_1)\Big).
\end{aligned}
$$

The above two relations give that $p_n(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, T, P_1) = 0$. Thus $T_{21} = 0$. And then we get $T_{12} = 0$ in the similar way. So $T = 0$, that is

$$d_m(A_{12} + B_{21}) = d_m(A_{12}) + d_m(B_{21}). \quad \square$$

LEMMA 3. *For any $A_{11} \in \mathscr{A}_{11}$, $B_{12} \in \mathscr{A}_{12}$, $C_{21} \in \mathscr{A}_{21}$ and $D_{22} \in \mathscr{A}_{22}$, we have*

$$d_m(A_{11} + B_{12} + C_{21}) = d_m(A_{11}) + d_m(B_{12}) + d_m(C_{21})$$

and

$$d_m(B_{12} + C_{21} + D_{22}) = d_m(B_{12}) + d_m(C_{21}) + d_m(D_{22}).$$

Proof. By Lemma 3 in [11], the result holds true for $m = 1$. Assume that the result holds true for $k < m$, i.e.

$$d_k(A_{11} + B_{12} + C_{21}) = d_k(A_{11}) + d_k(B_{12}) + d_k(C_{21}).$$

Let

$$T = d_m(A_{11} + B_{12} + C_{21}) - d_m(A_{11}) - d_m(B_{12}) - d_m(C_{21}).$$

We need to prove that $T = 0$. Our aim is to show that the result is true for every $m \in \mathbb{N}$. On the one hand, we obtain

$$\begin{aligned}
& d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, A_{11} + B_{12} + C_{21}\Big)\Big) \\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_1 - P_2, A_{11} + B_{12} + C_{21}\Big) \\
&\quad + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_1 - P_2, d_m(A_{11} + B_{12} + C_{21})\Big) \\
&\quad + \sum_{\substack{i_1 + i_2 + \cdots + i_n = m \\ 0 \leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_1 - P_2), d_{i_n}(A_{11} + B_{12} + C_{21})\Big).
\end{aligned}$$

On the other hand, by Lemma 2.3 and invoking the fact

$$p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, B_{12}\Big) = p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, C_{21}\Big) = 0,$$

we know that

$$\begin{aligned}
& d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, A_{11} + B_{12} + C_{21}\Big)\Big) \\
&= d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, A_{11}\Big)\Big) + d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, B_{12}\Big)\Big) \\
&\quad + d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_1 - P_2, C_{21}\Big)\Big) \\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_1 - P_2, A_{11}\Big) + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_1 - P_2, d_m(A_{11})\Big) \\
&\quad + \sum_{\substack{i_1 + i_2 + \cdots + i_n = m \\ 0 \leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_1 - P_2), d_{i_n}(A_{11})\Big)
\end{aligned}$$

$$+p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1-P_2,B_{12}\Big)+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1-P_2,d_m(B_{12})\Big)$$
$$+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_1-P_2),d_{i_n}(B_{12})\Big)$$
$$+p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1-P_2,C_{21}\Big)+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1-P_2,d_m(C_{21})\Big)$$
$$+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_1-P_2),d_{i_n}(C_{21})\Big)$$
$$=p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1-P_2,A_{11}+B_{12}+C_{21}\Big)$$
$$+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1-P_2,d_m(A_{11})+d_m(B_{12})+d_m(C_{21})\Big)$$
$$+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_1-P_2),d_{i_n}(A_{11}+B_{12}+C_{21})\Big).$$

Comparing the above two equations, we arrive at $p_n(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,P_1-P_2,T)=0$, which leads to $T_{11}=T_{22}=0$.

Invoking the fact that $p_n(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,P_2,A_{11})=0$ and using Lemma 2.3, we find that

$$d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,P_2,A_{11}+B_{12}+C_{21}\Big)\Big)$$
$$=d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,A_{11}\Big)\Big)+d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,B_{12}+C_{21}\Big)\Big)$$
$$=d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,A_{11}\Big)\Big)+d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,B_{12}\Big)\Big)$$
$$+d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,C_{21}\Big)\Big)$$
$$=p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_2,A_{11}\Big)+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,d_m(A_{11})\Big)$$
$$+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_2),d_{i_n}(A_{11})\Big)$$
$$+p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_2,B_{12}\Big)+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,d_m(B_{12})\Big)$$
$$+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_2),d_{i_n}(B_{12})\Big)$$
$$+p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_2,C_{21}\Big)+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,d_m(C_{21})\Big)$$
$$+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_2),d_{i_n}(C_{21})\Big)$$

$$
\begin{aligned}
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_2,A_{11}+B_{12}+C_{21}\Big)\\
&\quad+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,d_m(A_{11})+d_m(B_{12})+d_m(C_{21})\Big)\\
&\quad+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_2),d_{i_n}(A_{11})+d_{i_n}(B_{12})+d_{i_n}(C_{21})\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_2,A_{11}+B_{12}+C_{21}\Big)\\
&\quad+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,d_m(A_{11})+d_m(B_{12})+d_m(C_{21})\Big)\\
&\quad+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_2),d_{i_n}(A_{11}+B_{12}+C_{21})\Big).
\end{aligned}
$$

On the other hand, we also obtain

$$
\begin{aligned}
&d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,A_{11}+B_{12}+C_{21}\Big)\Big)\\
&\quad= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_2,A_{11}+B_{12}+C_{21}\Big)\\
&\quad\quad+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_2,d_m(A_{11}+B_{12}+C_{21})\Big)\\
&\quad\quad+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_2),d_{i_n}(A_{11}+B_{12}+C_{21})\Big).
\end{aligned}
$$

Comparing the above two equations, we get $p_n(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,P_2,T)=0$, which further implies that $T_{12}=T_{21}=0$. So $d_m(A_{11}+B_{12}+C_{21})=d_m(A_{11})+d_m(B_{12})+d_m(C_{21})$. Similarly, we can show that $d_m(B_{12}+C_{21}+D_{22})=d_m(B_{12})+d_m(C_{21})+d_m(D_{22})$. ☐

LEMMA 4. *For any $A_{11}\in\mathscr{A}_{11}$, $B_{12}\in\mathscr{A}_{12}$, $C_{21}\in\mathscr{A}_{21}$ and $D_{22}\in\mathscr{A}_{22}$, we have*

$$d_m(A_{11}+B_{12}+C_{21}+D_{22})=d_m(A_{11})+d_m(B_{12})+d_m(C_{21})+d_m(D_{22}).$$

Proof. By Lemma 4 in [11], the result is true for $m=1$. Suppose that the result is true for $k<m$, i.e.,

$$d_k(A_{11}+B_{12}+C_{21}+D_{22})=d_k(A_{11})+d_k(B_{12})+d_k(C_{21})+d_k(D_{22}).$$

Our aim is to show that the result is true for every $m\in\mathbb{N}$. Let

$$T=d_m(A_{11}+B_{12}+C_{21}+D_{22})-d_m(A_{11})-d_m(B_{12})-d_m(C_{21})-d_m(D_{22}).$$

Obviously, we have

$$
\begin{aligned}
&d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,P_1,A_{11}+B_{12}+C_{21}+D_{22}\Big)\Big)\\
&\ = p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1,A_{11}+B_{12}+C_{21}+D_{22}\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1,d_m(A_{11}+B_{12}+C_{21}+D_{22})\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_1),d_{i_n}(A_{11}+B_{12}+C_{21}+D_{22})\Big).
\end{aligned}
$$

Based on the fact that $p_n(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,P_1,D_{22})=0$. By Lemma 2.4, we find that

$$
\begin{aligned}
&d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,P_1,A_{11}+B_{12}+C_{21}+D_{22}\Big)\Big)\\
&\ = d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1,A_{11}+B_{12}+C_{21}\Big)\Big)+d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1,D_{22}\Big)\Big)\\
&\ = p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1,A_{11}+B_{12}+C_{21}\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1,d_m(A_{11}+B_{12}+C_{21})\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_1),d_{i_n}(A_{11}+B_{12}+C_{21})\Big)\\
&\quad +p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1,D_{22}\Big)+\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1,d_m(D_{22})\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_1),d_{i_n}(D_{22})\Big)\\
&\ = p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1,A_{11}+B_{12}+C_{21}+D_{22}\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1,d_m(A_{11}+B_{12}+C_{21})+d_m(D_{22})\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI,d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_1),d_{i_n}(A_{11}+B_{12}+C_{21})+d_{i_n}(D_{22})\Big)\Big)\\
&\ = p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_1,A_{11}+B_{12}+C_{21}+D_{22}\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_1,d_m(A_{11})+d_m(B_{12})+d_m(C_{21})+d_m(D_{22})\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n(d_{i_1}\Big(\frac{1}{2}iI,d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}(P_1),d_{i_n}(A_{11}+B_{12}+C_{21}+D_{22})\Big).
\end{aligned}
$$

Comparing the above two equations yields that $d_m(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_1, T) = 0$, which implies that $T_{11} = T_{12} = T_{21} = 0$. Using the fact that $d_m(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_2, A_{11}) = 0$ and the above similar arguments, we can obtain $d_m(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_2, T) = 0$ which implies $T_{22} = 0$. Hence $T = 0$. So $d_m(A_{11} + B_{12} + C_{21} + D_{22}) = d_m(A_{11}) + d_m(B_{12}) + d_m(C_{21}) + d_m(D_{22})$. □

LEMMA 5. *For any $A_{jk}, B_{jk} \in \mathscr{A}_{jk}, 1 \leqslant j \neq k \leqslant 2$, we have*

$$d_m(A_{jk} + B_{jk}) = d_m(A_{jk}) + d_m(B_{jk}).$$

Proof. Since

$$p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_j + A_{jk}, i(P_k + B_{jk})\Big) = B_{jk} + A_{jk} + A_{jk}^* + B_{jk}A_{jk}^*,$$

we get from Lemma 2.5 that

$$\begin{aligned}
&d_m(A_{jk} + B_{jk}) + d_m(A_{jk}^*) + d_m(B_{jk}A_{jk}^*)\\
&= d_m(B_{jk} + A_{jk} + A_{jk}^* + B_{jk}A_{jk}^*)\\
&= d_m\Big(p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j + A_{jk}, i(P_k + B_{jk})\Big)\Big)\\
&= \sum_{i_1+i_2+\cdots+i_n=m} p_n\Big(d_{i_1}\Big(-\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_j + A_{jk}), d_{i_n}(iP_k + iB_{jk})\Big)\\
&= \sum_{i_1+i_2+\cdots+i_n=m} p_n(d_{i_1}\Big(-\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_j) + d_{i_{n-1}}(A_{jk}), d_{i_n}(iP_k) + d_{i_n}(iB_{jk}))\\
&= \sum_{i_1+i_2+\cdots+i_n=m} p_n\Big(d_{i_1}\Big(-\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_j), d_{i_n}(iP_k)\Big)\\
&\quad + \sum_{i_1+i_2+\cdots+i_n=m} p_n\Big(d_{i_1}\Big(-\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_j), d_{i_n}(iB_{jk})\Big)\\
&\quad + \sum_{i_1+i_2+\cdots+i_n=m} p_n\Big(d_{i_1}\Big(-\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(A_{jk}), d_{i_n}(iP_k)\Big)\\
&\quad + \sum_{i_1+i_2+\cdots+i_n=m} p_n\Big(d_{i_1}\Big(-\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(A_{jk}), d_{i_n}(iB_{jk})\Big)\\
&= d_m\Big(p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, iP_k\Big)\Big) + d_m\Big(p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, iB_{jk}\Big)\Big)\\
&\quad + d_m\Big(p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \cdots, A_{jk}, iP_k\Big)\Big) + d_m\Big(p_n\Big(-\frac{1}{2}iI, \frac{1}{2}I, \cdots, A_{jk}, iB_{jk}\Big)\Big)\\
&= d_m(B_{jk}) + d_m(A_{jk} + A_{jk}^*) + d_m(B_{jk}A_{jk}^*)\\
&= d_m(B_{jk}) + d_m(A_{jk}) + d_m(A_{jk}^*) + d_m(B_{jk}A_{jk}^*).
\end{aligned}$$

Hence $d_m(A_{jk} + B_{jk}) = d_m(A_{jk}) + d_m(B_{jk})$. □

LEMMA 6. *For every $A_{jj}, B_{jj} \in \mathscr{A}_{jj}$, $1 \leqslant j \leqslant 2$, we have*

$$d_m(A_{jj}+B_{jj}) = d_m(A_{jj}) + d_m(B_{jj}).$$

Proof. By Lemma 6 in [11], the result is true for $m=1$. Suppose that the result is true for $k<m$, i.e.,

$$d_k(A_{jj}+B_{jj}) = d_k(A_{jj}) + d_k(B_{jj}).$$

Our aim is to show that the result is true for every $m \in \mathbb{N}$. Let

$$T = d_m(A_{jj}+B_{jj}) - d_m(A_{jj}) - d_m(B_{jj}).$$

For $1 \leqslant l \neq j \leqslant 2$, since $p_n(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, A_{jj}) = p_n(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_l, B_{jj}) = 0$, we obtain

$$\begin{aligned}
& d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, A_{jj}+B_{jj}\Big)\Big) \\
&= d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, A_{jj}\Big)\Big) + d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, B_{jj}\Big)\Big) \\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_l, A_{jj}\Big) + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, d_m(A_{jj})\Big) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0 \leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_l), d_{i_n}(A_{jj})\Big) \\
&\quad + p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_l, B_{jj}\Big) + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, d_m(B_{jj})\Big) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0 \leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_l), d_{i_n}(B_{jj})\Big) \\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_l, A_{jj}+B_{jj}\Big) \\
&\quad + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, d_m(A_{jj}) + d_m(B_{jj})\Big) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0 \leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_l), d_{i_n}(A_{jj}) + d_{i_n}(B_{jj})\Big).
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned}
& d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, A_{jj}+B_{jj}\Big)\Big) \\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_l, A_{jj}+B_{jj}\Big) + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, d_m(A_{jj}+B_{jj})\Big) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0 \leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_l), d_{i_n}(A_{jj}+B_{jj})\Big)
\end{aligned}$$

$$= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_l, A_{jj}+B_{jj}\Big) + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_l, d_m(A_{jj}+B_{jj})\Big)$$
$$+ \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}(P_l), d_{i_n}(A_{jj}) + d_{i_n}(B_{jj})\Big).$$

Comparing the above two equations, we conclude that $p_n(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_l, T) = 0$. Thus $T_{ll} = T_{lj} = T_{jl} = 0$. Now we need to show that $T_{jj} = 0$. For all $X_{jl} \in \mathscr{A}_{jl}$,

$$p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_j, A_{jj}, X_{jl}\Big) \in \mathscr{A}_{jl}$$

and

$$p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_j, B_{jj}, X_{jl}\Big) \in \mathscr{A}_{jl}.$$

By Lemma 2.6, we have

$$d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, P_j, A_{jj}+B_{jj}, X_{jl}\Big)\Big)$$
$$= d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, A_{jj}, X_{jl}\Big)\Big) + d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, B_{jj}, X_{jl}\Big)\Big)$$
$$= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_j, A_{jj}, X_{jl}\Big) + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, d_m(A_{jj}), X_{jl}\Big)$$
$$+ p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, A_{jj}, d_m(X_{jl})\Big)$$
$$+ \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-2}}(P_j), d_{i_{n-1}}(A_{jj}), d_{i_n}(X_{jl})\Big)$$
$$+ p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_j, B_{jj}, X_{jl}\Big) + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, d_m(B_{jj}), X_{jl}\Big)$$
$$+ p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, B_{jj}, d_m(X_{jl})\Big)$$
$$+ \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-2}}(P_j), d_{i_{n-1}}(B_{jj}), d_{i_n}(X_{jl})\Big)$$
$$= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, P_j, A_{jj}+B_{jj}, X_{jl}\Big)$$
$$+ \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, d_m(A_{jj}) + d_m(B_{jj}), X_{jl}\Big)$$
$$+ p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, P_j, A_{jj}+B_{jj}, d_m(X_{jl})\Big)$$
$$+ \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-2}}(P_j), d_{i_{n-1}}(A_{jj}) + d_{i_{n-1}}(B_{jj}), d_{i_n}(X_{jl})\Big).$$

On the other hand, we obtain

$$
\begin{aligned}
& d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_j,A_{jj}+B_{jj},X_{jl}\Big)\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_j,A_{jj}+B_{jj},X_{jl}\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_j,d_m(A_{jj}+B_{jj}),X_{jl}\Big)\\
&\quad +p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_j,A_{jj}+B_{jj},d_m(X_{jl})\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-2}}(P_j),d_{i_{n-1}}(A_{jj}+B_{jj}),d_{i_n}(X_{jl})\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,P_j,A_{jj}+B_{jj},X_{jl}\Big)\\
&\quad +\cdots+p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_j,d_m(A_{jj}+B_{jj}),X_{jl}\Big)\\
&\quad +p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_j,A_{jj}+B_{jj},d_m(X_{jl})\Big)\\
&\quad +\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-2}}(P_j),d_{i_{n-1}}(A_{jj})+d_{i_{n-1}}(B_{jj}),d_{i_n}(X_{jl})\Big).
\end{aligned}
$$

Comparing the above two equations, we conclude that $p_n(\frac{1}{2}iI,\frac{1}{2}I,\cdots,P_j,T,X_{jl})=0$. So $T_{jj}X_{jl}=0$ for all $X_{jl}\in\mathscr{A}_{jl}$. It follows from (♠) and (♣) that $T_{jj}=0$. Hence $d_m(A_{jj}+B_{jj})=d_m(A_{jj})+d_m(B_{jj})$. $\square$

LEMMA 7. *d_m is additive on $\mathscr{A}$.*

Proof. For any $A,B\in\mathscr{A}$, we write $A=\sum_{j,k=1}^{2}A_{jk}$ and $B=\sum_{j,k=1}^{2}B_{jk}$. It follows from Lemmas 2.5–2.7 that

$$
\begin{aligned}
d_m(A+B) &= d_m(\sum_{j,k=1}^{2}(A_{jk}+B_{jk}))=\sum_{j,k=1}^{2}d_m(A_{jk}+B_{jk})\\
&=\sum_{j,k=1}^{2}(d_m(A_{jk})+d_m(B_{jk}))=d_m(\sum_{j,k=1}^{2}A_{jk})+d_m(\sum_{j,k=1}^{2}B_{jk})\\
&=d_m(A)+d_m(B). \quad\square
\end{aligned}
$$

LEMMA 8. *$d_m(\frac{1}{2}I)=0$, $d_m(\frac{1}{2}iI)=0$ for each $m\in\mathbb{N}$ with $m\geqslant 1$ and $d_m(iA)=id_m(A)$ for all $A\in\mathscr{A}$.*

Proof. The result is true for $m=1$ by Claim 2 in [9]. Assume that the result holds true for $k<m$, i.e., $d_k(\frac{1}{2}I)=0$, $d_k(\frac{1}{2}iI)=0$ and $d_k(iA)=id_k(A)$. We prove this lemma by serval steps.

Step 1. $d_m(\frac{1}{2}I) = d_m(\frac{1}{2}I)^*$.
Since $p_n(\frac{1}{2}I, \frac{1}{2}I, \cdots, \frac{1}{2}I) = 0$, we get

$$
\begin{aligned}
0 &= d_m\Big(p_n\Big(\frac{1}{2}I, \frac{1}{2}I, \cdots, \frac{1}{2}I\Big)\Big) \\
&= \sum_{i_1+i_2+\cdots+i_n=m} p_n\Big(d_{i_1}\Big(\frac{1}{2}I\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_n}\Big(\frac{1}{2}I\Big)\Big) \\
&= p_n\Big(d_m\Big(\frac{1}{2}I\Big), \frac{1}{2}I, \cdots, \frac{1}{2}I\Big) + p_n\Big(\frac{1}{2}I, d_m\Big(\frac{1}{2}I\Big), \cdots, \frac{1}{2}I\Big) \\
&\quad + \cdots + p_n\Big(\frac{1}{2}I, \frac{1}{2}I, \cdots, d_m\Big(\frac{1}{2}I\Big)\Big) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}I\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_n}\Big(\frac{1}{2}I\Big)\Big) \\
&= p_n\Big(d_m\Big(\frac{1}{2}I\Big), \frac{1}{2}I, \cdots, \frac{1}{2}I\Big) \\
&= \frac{1}{2}d_m\Big(\frac{1}{2}I\Big) - \frac{1}{2}d_m\Big(\frac{1}{2}I\Big)^*.
\end{aligned}
$$

This leads to

$$d_m\Big(\frac{1}{2}I\Big) = d_m\Big(\frac{1}{2}I\Big)^*.$$

Step 2. $d_m(\frac{1}{2}iI)^* = -d_m(\frac{1}{2}iI)$.
By $p_n(\frac{1}{2}iI, \frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I) = 0$, we have

$$
\begin{aligned}
0 &= d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I\Big)\Big) \\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I\Big) + p_n\Big(\frac{1}{2}iI, d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, \frac{1}{2}I\Big) \\
&\quad + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}iI, \frac{1}{2}I, \cdots, d_m\Big(\frac{1}{2}I\Big)\Big) \\
&\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}iI\Big), d_{i_3}\Big(\frac{1}{2}I\Big), \cdots, d_{i_n}\Big(\frac{1}{2}I\Big)\Big) \\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I\Big) + p_n\Big(\frac{1}{2}iI, d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, \frac{1}{2}I\Big) \\
&= p_n\Big(\frac{1}{2}iI, d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, \frac{1}{2}I\Big) \\
&= \frac{1}{2}id_m\Big(\frac{1}{2}iI\Big) + \frac{1}{2}id_m\Big(\frac{1}{2}iI\Big)^*.
\end{aligned}
$$

So

$$d_m\Big(\frac{1}{2}iI\Big)^* = -d_m\Big(\frac{1}{2}iI\Big).$$

Step 3. $d_m(\frac{1}{2}I)=0$.
Since $p_n(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I)=\frac{1}{2}iI$, by Step 1 and Step 2, we conclude that

$$\begin{aligned}
d_m\Big(\frac{1}{2}iI\Big) &= d_m\Big(p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I\Big)\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,\frac{1}{2}I\Big)+\sum_{k=2}^{n}p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\overset{k}{d_m\Big(\frac{1}{2}I\Big)},\cdots,\frac{1}{2}I\Big)\\
&\quad+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_n}\Big(\frac{1}{2}I\Big)\Big)\\
&= p_n\Big(d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,\frac{1}{2}I\Big)+\sum_{k=2}^{n}p_n\Big(\frac{1}{2}iI,\frac{1}{2}I,\cdots,\overset{k}{d_m\Big(\frac{1}{2}I\Big)},\cdots,\frac{1}{2}I\Big)\\
&= d_m\Big(\frac{1}{2}iI\Big)+(n-1)id_m\Big(\frac{1}{2}I\Big).
\end{aligned}$$

Thus

$$d_m\Big(\frac{1}{2}I\Big)=0.$$

Step 4. $d_m(\frac{1}{2}iI)=0$.
Since $p_n(-\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,\frac{1}{2}iI)=\frac{1}{2}I$, by Lemma 6, Step 2 and Step 3, we get

$$\begin{aligned}
0 &= d_m\Big(\frac{1}{2}I\Big)\\
&= d_m\Big(p_n\Big(-\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,\frac{1}{2}iI\Big)\Big)\\
&= p_n\Big(d_m\Big(-\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,\frac{1}{2}I,\frac{1}{2}iI\Big)+p_n\Big(-\frac{1}{2}iI,d_m\Big(\frac{1}{2}I\Big),\cdots,\frac{1}{2}I,\frac{1}{2}iI\Big)\\
&\quad+\cdots+p_n\Big(-\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,d_m\Big(\frac{1}{2}iI\Big)\Big)\\
&\quad+\sum_{\substack{i_1+i_2+\cdots+i_n=m\\ 0\leqslant i_1,i_2,\cdots,i_n\leqslant m-1}} p_n\Big(d_{i_1}\Big(-\frac{1}{2}iI\Big),d_{i_2}\Big(\frac{1}{2}I\Big),\cdots,d_{i_{n-1}}\Big(\frac{1}{2}I\Big),d_{i_n}\Big(\frac{1}{2}iI\Big)\Big)\\
&= p_n\Big(d_m\Big(-\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,\frac{1}{2}I,\frac{1}{2}iI\Big)+p_n\Big(-\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,d_m\Big(\frac{1}{2}iI\Big)\Big)\\
&= p_n\Big(-d_m\Big(\frac{1}{2}iI\Big),\frac{1}{2}I,\cdots,\frac{1}{2}I,\frac{1}{2}iI\Big)+p_n\Big(-\frac{1}{2}iI,\frac{1}{2}I,\cdots,\frac{1}{2}I,d_m\Big(\frac{1}{2}iI\Big)\Big)\\
&= -2id_m\Big(\frac{1}{2}iI\Big).
\end{aligned}$$

So

$$d_m\Big(\frac{1}{2}iI\Big)=0.$$

Step 5. $d_m(iA) = id_m(A)$ for all $A \in \mathscr{A}$.
For any $A \in \mathscr{A}$, by Step 3 and Step 4, we have

$$\begin{aligned} d_m(iA) &= d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, A\Big)\Big) \\ &= \sum_{i_1+i_2+\cdots+i_n=m} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}\Big(\frac{1}{2}I\Big), d_{i_n}(A)\Big) \\ &= p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, d_m(A)\Big) \\ &= id_m(A). \quad \square \end{aligned}$$

LEMMA 9. *For all $A \in \mathscr{A}$, we have $d_m(A^*) = d_m(A)^*$.*

Proof. By Lemmas 7 and 8, we obtain $d_m(I) = 2d_m(\frac{1}{2}I) = 0$ for each $m \in \mathbb{N}$ with $m \geqslant 1$. For any $A \in \mathscr{A}$, we have

$$\begin{aligned} 2^{n-2}d_m(A) - 2^{n-2}d_m(A^*) &= d_m(2^{n-2}(A - A^*)) \\ &= d_m(p_n(A, I, I, \cdots, I)) \\ &= \sum_{i_1+i_2+\cdots+i_n=m} p_n(d_{i_1}(A), d_{i_2}(I), \cdots, d_{i_n}(I)) \\ &= p_n(d_m(A), I, I, \cdots, I) \\ &= 2^{n-2}d_m(A) - 2^{n-2}d_m(A)^*. \end{aligned}$$

Hence $d_m(A^*) = d_m(A)^*$. $\square$

LEMMA 10. *d_m is an additive $*$-higher derivation on $\mathscr{A}$.*

Proof. Based on the fact that $p_n(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, A, B) = i(AB + BA^*)$ for any $A, B \in \mathscr{A}$, and using Lemma 2.8 and Lemma 2.9, we obtain

$$\begin{aligned} id_m(AB + BA^*) &= d_m\Big(p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, A, B\Big)\Big) \\ &= p_n\Big(d_m\Big(\frac{1}{2}iI\Big), \frac{1}{2}I, \cdots, \frac{1}{2}I, A, B\Big) + p_n\Big(\frac{1}{2}iI, d_m\Big(\frac{1}{2}I\Big), \cdots, \frac{1}{2}I, A, B\Big) \\ &\quad + \cdots + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, d_m(A), B\Big) + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, A, d_m(B)\Big) \\ &\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}\Big(\frac{1}{2}I\Big), d_{i_{n-1}}(A), d_{i_n}(B)\Big) \\ &= p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, d_m(A), B\Big) + p_n\Big(\frac{1}{2}iI, \frac{1}{2}I, \cdots, \frac{1}{2}I, A, d_m(B)\Big) \\ &\quad + \sum_{\substack{i_1+i_2+\cdots+i_n=m \\ 0\leqslant i_1, i_2, \cdots, i_n \leqslant m-1}} p_n\Big(d_{i_1}\Big(\frac{1}{2}iI\Big), d_{i_2}\Big(\frac{1}{2}I\Big), \cdots, d_{i_{n-1}}\Big(\frac{1}{2}I\Big), d_{i_{n-1}}(A), d_{i_n}(B)\Big) \end{aligned}$$

$$
\begin{aligned}
&= p_2(id_m(A),B) + P_2(iA,d_m(B)) + \sum_{\substack{s+t=m\\0<s,t\leqslant m-1}} p_2(id_s(A),d_t(B))\\
&= id_m(A)B + iBd_m(A)^* + iAd_m(B) + id_m(B)A^*\\
&\quad + \sum_{\substack{s+t=m\\0<s,t\leqslant m-1}} (id_s(A)d_t(B) + id_t(B)d_s(A)^*).
\end{aligned}
$$

Hence

$$
\begin{aligned}
d_m(AB+BA^*) &= d_m(A)B + Ad_m(B) + d_m(B)A^* + Bd_m(A)^*\\
&\quad + \sum_{\substack{s+t=m\\0<s,t\leqslant m-1}} (d_s(A)d_t(B) + d_t(B)d_s(A)^*).
\end{aligned}
$$

Replacing A by iA in the above equality, we get

$$
\begin{aligned}
d_m(AB-BA^*) &= d_m(A)B + Ad_m(B) - d_m(B)A^* - Bd_m(A)^*\\
&\quad + \sum_{\substack{s+t=m\\0<s,t\leqslant m-1}} (d_s(A)d_t(B) - d_t(B)d_s(A)^*).
\end{aligned}
$$

Therefore

$$
d_m(AB) = d_m(A)B + Ad_m(B) + \sum_{\substack{s+t=m\\0<s,t\leqslant m-1}} (d_s(A)d_t(B)) = \sum_{\substack{s+t=m\\0\leqslant s,t\leqslant m}} d_s(A)d_t(B).
$$

Together with Lemmas 7 and 9, d_m is an additive $*$-higher derivation on $\mathscr{A}$, which completes the proof. $\square$

3. Corollaries

Recall that an algebra $\mathscr{A}$ is prime if $A\mathscr{A}B = \{0\}$ for $A,B \in \mathscr{A}$ implies either $A = 0$ or $B = 0$. It is easy to see that prime $*$-algebras satisfy (♠) and (♣). Applying Theorem 1 to prime $*$-algebras, we have the following corollary.

COROLLARY 1. *Let $\mathscr{A}$ be a prime $*$-algebra with unit I and P be a nontrivial projection in $\mathscr{A}$. Then every nonlinear skew Lie type higher derivation $\mathscr{D} = \{d_m\}_{m\in\mathbb{N}}$ on $\mathscr{A}$ is an additive $*$-higher derivation.*

Let $B(\mathscr{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ and $\mathscr{F}(\mathscr{H}) \subseteq B(\mathscr{H})$ be the subalgebra of all bounded finite rank operators. A subalgebra $\mathscr{A} \subseteq B(\mathscr{H})$ is called a standard operator algebra if it contains $\mathscr{F}(\mathscr{H})$. Let $\mathscr{A}$ be a standard operator algebra. Note that every additive derivation $d : \mathscr{A} \to B(\mathscr{H})$ is an inner derivation (see [18]). Nowicki [13] proved that if every additive (linear) derivation of $\mathscr{A}$ is inner, then every additive (linear) higher derivation of $\mathscr{A}$ is inner (see also [23]). Since standard operator algebra is prime, by Corollary 3.1, the following corollary is immediate.

COROLLARY 2. *Let $\mathscr{A}$ be a standard operator algebra on an infinite dimensional complex Hilbert space $\mathscr{H}$ containing the identity operator I. Suppose that $\mathscr{A}$ is closed under the adjoint operation. Then every nonlinear skew Lie type higher derivation $\mathscr{D} = \{d_m\}_{m\in\mathbb{N}}$ from $\mathscr{A}$ to $B(\mathscr{H})$ is an additive $*$-higher derivation. Moreover, $\mathscr{D}$ is inner.*

A von Neumann algebra $\mathscr{M}$ is a weakly closed, self-adjoint algebra of operators on a Hilbert space $\mathscr{H}$ containing the identity operator I. $\mathscr{M}$ is a factor von Neumann algebra if its center only contains the scalar operators. It is well known that a factor von Neumann algebra is prime. Now we have the following corollary.

COROLLARY 3. *Let $\mathscr{M}$ be a factor von Neumann algebra with $\dim(\mathscr{M}) \geqslant 2$. Then every nonlinear skew Lie type higher derivation $\mathscr{D} = \{d_m\}_{m\in\mathbb{N}}$ on $\mathscr{M}$ is an additive $*$-higher derivation.*

It is shown in [7] that every von Neumann algebra with no central summands of type I_1 satifies $(\spadesuit)$ and $(\clubsuit)$. Now we have the following corollary.

COROLLARY 4. *Let $\mathscr{M}$ be a von Neumann algebra with no central summands of type I_1. Then every nonlinear skew Lie type higher derivation $\mathscr{D} = \{d_m\}_{m\in\mathbb{N}}$ on $\mathscr{M}$ is an additive $*$-higher derivation.*

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