Operators
and
Matrices
Volume 18, Number 3 (2024), 611–621
doi:10.7153/oam-2024-18-36

A SPECIAL PROPERTY OF RESISTANCE MATRICES

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(*Communicated by S. Fallat*)

Abstract. We deduce a new property exhibited by the resistance matrices of connected graphs. Specifically, we show that if $R = (r_{ij})$ is the resistance matrix of a connected graph on n vertices, then every off-diagonal entry in the Moore-Penrose inverse of

$$\mathrm{Diag}(\sum_{j=1}^{n} r_{1j}, \ldots, \sum_{j=1}^{n} r_{nj}) - R$$

is negative. Thus, we establish that the Moore-Penrose inverse of the resistance Laplacian matrices are $\mathbf{M}$-matrices.

1. Introduction

This paper aims to generalize the following technical result in [1] for an arbitrary connected graph.

THEOREM 1. *Let $D = (d_{ij})$ be the distance matrix of a weighted tree on n vertices. Then, the Moore-Penrose inverse of*

$$\mathrm{Diag}(\sum_{j=1}^{n} d_{1j}, \ldots, \sum_{j=1}^{n} d_{nj}) - D$$

is an $\mathbf{M}$-matrix.

In our context, a positive semidefinite matrix is an $\mathbf{M}$-matrix if all the off-diagonal entries are non-positive. Let $\mathscr{G}$ be a connected graph with n vertices. We assume that each vertex in $\mathscr{G}$ is uniquely labelled from 1 to n and each edge (i,j) is assigned a positive weight w_{ij}. The length of a path $\mathscr{P}$ is computed by adding all the weights on $\mathscr{P}$. The distance between two vertices is then the length of the shortest path connecting them. We denote this by d_{ij} and define the distance matrix by $D(\mathscr{G}) := (d_{ij})$. Resistance distance is another metric used in graphs. The Laplacian matrix is defined by $L(\mathscr{G}) := \mathrm{Diag}(\sum_{j=1}^{n} \alpha_{1j}, \ldots, \sum_{j=1}^{n} \alpha_{nj}) - (\alpha_{ij})$, where

$$\alpha_{ij} := \begin{cases} \frac{1}{w_{ij}} & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{else.} \end{cases}$$

Mathematics subject classification (2020): 05C50.
Keywords and phrases: Resistance matrices, Laplacian matrices, $\mathbf{P}$-matrices, connected graphs, complete graphs, Jacobi identity.
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Let g_{ij} denote the $(i,j)^{\text{th}}$ entry in the Moore-Penrose inverse of $L(\mathscr{G})$. Then the *resistance distance* between i and j is defined by

$$r_{ij} := g_{ii} + g_{jj} - 2g_{ij}.$$

The *resistance* matrix is $R(\mathscr{G}) := (r_{ij})$. Resistance distance has certain advantages over the shortest distance. Its significance and properties are discussed elaborately in [2]. In particular, the resistance distance is always less than or equal to the classical distance, that is, $r_{ij} \leqslant d_{ij}$ for all i,j and the equality holds if and only if $\mathscr{G}$ is a tree. We define the *resistance Laplacian* matrix of a connected graph $\mathscr{G}$ on n vertices by

$$\text{Diag}(\sum_{j=1}^{n} r_{1j}, \ldots, \sum_{j=1}^{n} r_{nj}) - R(\mathscr{G}).$$

In this article, we deduce the following special property of resistance matrices.

THEOREM 2. *Let $\mathscr{G}$ be a connected graph with n vertices. Then, the Moore-Penrose inverse of the resistance Laplacian matrix of $\mathscr{G}$ is an* $\mathbf{M}$*-matrix.*

Since $R(\mathscr{G}) = D(\mathscr{G})$ if and only if $\mathscr{G}$ is a tree, Theorem 1 follows immediately from our result. The proof of Theorem 1 in [1] uses specific arguments that work only for trees. On the other hand, the proof of Theorem 2 is simpler and relies only on techniques from the theory of matrices.

Let $A^{\dagger}$ denote the Moore-Penrose inverse of a matrix A. We shall say that A has the M-property if both A and $A^{\dagger}$ are $\mathbf{M}$-matrices. In connection with this, we note the following questions from the existing literature:

(Q1) Find necessary and sufficient conditions for the M-property.

(Q2) Classify connected graphs whose Laplacian matrices exhibit the M-property.

(Q3) For a connected graph $\mathscr{G}$, determine edge weights that ensure $L(\mathscr{G})$ acquires the M-property.

Deutsch and Neumann [3] have established necessary conditions for the M-property. So far, only a few connected graphs are known to have Laplacian matrices possessing the M-property. A result shown in [4] asserts that if T is a tree, then $L(T)$ has the M-property if and only if T is a star. Further investigations on weighted graphs with Laplacians exhibiting the M-property appear in Styan and Subak-Sharpe [6], Kirkland and Neumann [5] and Kirkland, Neumann and Shader [4].

Theorem 2 finds a rich class of matrices with the M-property. Let K_n be the complete graph on n vertices. The Laplacian of K_n is then a positive multiple of its Moore-Penrose inverse and hence carries the M-property. The conclusion of Theorem 2 extends the M-property to the weighted case: If each edge (i,j) of K_n is assigned the weight r_{ij}, then the resulting weighted Laplacian matrix retains the M-property.

2. Preliminaries

Notations

- If A is a $k\times k$ matrix with columns $a^1,\dots,a^k$, then we write $A = [\![a^1,\dots,a^k]\!]$. The determinant of A is denoted by $\det(A)$.
- The identity matrix is denoted by I and the column vector of all n-ones by $\mathbf{1}$. The symbol J will denote the order n matrix $[\![\mathbf{1},\dots,\mathbf{1}]\!]$. If $k<n$, then we use I_k, $\mathbf{1}_k$ and J_k.
- If Q is an $n\times n$ matrix and Γ is a subset of $\{1,\dots,n\}$, then Q_Γ will denote the principal submatrix of Q obtained by selecting rows and columns indexed from Γ. The submatrix obtained by removing the p^{th} row and q^{th} column will be denoted by $Q(p|q)$.

DEFINITION 1. An $n\times n$ real matrix is called a $\mathbf{P}$-matrix if all the principal minors are positive. If $A+\varepsilon I$ is a $\mathbf{P}$-matrix for any $\varepsilon>0$, then A is called a $\mathbf{P}_0$-matrix. Equivalently, A is a $\mathbf{P}_0$-matrix if and only if all the principal minors of A are non-negative.

3. Main result

We shall prove a more general result from which Theorem 2 will follow immediately.

THEOREM 3. *Let L be an $n\times n$ symmetric matrix. Suppose L is an $\mathbf{M}$-matrix,* $\operatorname{rank}(L)=n-1$ *and $L\mathbf{1}=0$. Let $r_{ij}:=h_{ii}+h_{jj}-2h_{ij}$, where h_{ij} is the $(i,j)^{\text{th}}$ entry in $L^\dagger$. If $R:=(r_{ij})$, then the Moore-Penrose inverse of*

$$\operatorname{Diag}\Big(\sum_{j=1}^n r_{1j},\dots,\sum_{j=1}^n r_{nj}\Big)-R$$

is an $\mathbf{M}$-matrix. Furthermore, in this $\mathbf{M}$-matrix, all the off-diagonal entries are negative numbers.

Proof. Define

$$\theta_i:=\sum_{j=1}^n r_{ij},\quad S:=\operatorname{Diag}(\theta_1,\dots,\theta_n)-R\ \text{ and }\ S^\dagger:=(\xi_{ij})\ \ i,j=1,\dots,n.$$

We need to show that $S^\dagger$ is an $\mathbf{M}$-matrix. If S is 2×2, then the result can be verified directly. In the sequel, we assume $n>2$. Since S is positive semidefinite, $S^\dagger$ is also positive semidefinite. Thus, to complete the proof, we need to show that $\xi_{ij}<0$ for any $i<j$. By a permutation similarity argument, it suffices to show that $\xi:=\xi_{12}<0$. We have the following claim now.

CLAIM 1. Let

$$B := \begin{pmatrix} \beta_{33}+\theta_3 & \beta_{34} & \dots & \beta_{3n} \\ \beta_{43} & \beta_{44}+\theta_4 & \dots & \beta_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n3} & \beta_{n4} & \dots & \beta_{nn}+\theta_n \end{pmatrix},$$

where

$$\beta_{ij} := 2(h_{21}-h_{i1}-h_{j2}+h_{ij}) \quad i,j = 3,\dots,n.$$

Then, $\xi < 0$ if and only if $\det(B) > 0$.

Proof of the claim 1. Let b_{ij} denote the $(i,j)^{\text{th}}$ entry in B. We first express b_{ij} in terms of r_{ij}. By writing

$$\beta_{ij} = -(h_{22}+h_{11}-2h_{21}) + (h_{ii}+h_{11}-2h_{i1}) + (h_{jj}+h_{22}-2h_{2j}) - (h_{ii}+h_{jj}-2h_{ij}),$$

we see that $\beta_{ij} = -r_{21}+r_{i1}+r_{2j}-r_{ij}$. Thus,

$$b_{ij} := \begin{cases} -r_{21}+r_{i1}+r_{2j}-r_{ij} & i \neq j \\ -r_{21}+r_{i1}+r_{2i}+\theta_i & i = j. \end{cases}$$

Now, to prove the claim, we use similar ideas as in [1]. Since $\operatorname{rank}(L^\dagger) = n-1$ and $L^\dagger$ is positive semidefinite, the off-diagonal entries in R are positive numbers. Hence, the off-diagonal entries of S are negative. Furthermore, $S\mathbf{1} = 0$. Thus, $S(1|1)$ is strictly diagonally dominant, and hence $\gamma := \det(S(1|1)) > 0$. This says that $\operatorname{rank}(S) = n-1$. Therefore, $SS^\dagger = I - \frac{J}{n}$. Using this equation, we obtain

$$(S+J)^{-1} = S^\dagger + \frac{J}{n^2}. \tag{3.1}$$

Let τ be the $(1,2)^{\text{th}}$ entry in $(S+J)^{-1}$ and $C := S(1|2)$. By Cramer's rule,

$$\tau = -\frac{\det(C+J(1|2))}{\det(S+J)} = -\frac{\det(C+\mathbf{1}_{n-1}\mathbf{1}'_{n-1})}{\det(S+J)}. \tag{3.2}$$

Since $S\mathbf{1} = 0$, all the cofactors of S are equal and therefore, $\operatorname{adj}(S) = \gamma J$. Hence, $\mathbf{1}'\operatorname{adj}(S)\mathbf{1} = n^2\gamma$ and $\det(C) = -\gamma$. Put $\upsilon := \mathbf{1}'_{n-1}C^{-1}\mathbf{1}_{n-1}$. By the matrix determinant lemma,

$$\det(C+\mathbf{1}_{n-1}\mathbf{1}'_{n-1}) = \det(C)(1+\upsilon) = -\gamma(1+\upsilon);$$
$$\det(S+J) = \det(S) + \mathbf{1}'\operatorname{adj}(S)\mathbf{1} = n^2\gamma.$$

By (3.1) and (3.2), we get $\xi = \tau - \frac{1}{n^2}$ and $\tau = \frac{1}{n^2}(1+\upsilon)$. Eliminating τ, we have $\xi = \frac{\upsilon}{n^2}$. We now express $\det(B)$ in terms of υ. Let

$$F := \begin{pmatrix} 1 & \mathbf{1}'_{n-2} \\ \mathbf{0} & I_{n-2} \end{pmatrix}.$$

Then, $F^{-1} = \begin{pmatrix} 1 & -\mathbf{1}'_{n-2} \\ \mathbf{0} & I_{n-2} \end{pmatrix}$. Denoting the $(i,j)^{\text{th}}$ entry of S by s_{ij}, we see that

$$G := F'^{-1}CF^{-1} = \begin{pmatrix} s_{21} & -s_{21}+s_{23} & \dots & -s_{21}+s_{2n} \\ -s_{21}+s_{31} & s_{21}-s_{31}-s_{23}+s_{33} & \dots & s_{21}-s_{31}-s_{2n}+s_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -s_{21}+s_{n1} & s_{21}-s_{n1}-s_{23}+s_{n3} & \dots & s_{21}-s_{n1}-s_{2n}+s_{nn} \end{pmatrix}.$$

Since

$$s_{ij} = \begin{cases} -r_{ij} & i \neq j \\ \theta_i & i = j \end{cases}$$

it follows that $B = G(1|1)$. As $\det(C) = -\gamma$ and $\det(F) = 1$, $\det(G) = -\gamma$. The $(1,1)^{\text{th}}$ entry of G^{-1} is

$$\frac{1}{\det(G)} \det(G(1|1)) = -\frac{1}{\gamma} \det(B).$$

Furthermore, an easy observation reveals that the $(1,1)^{\text{th}}$ entry in G^{-1} is υ. We thus have

$$\upsilon = -\frac{1}{\gamma} \det(B) = n^2 \xi.$$

Hence, $\xi < 0$ if and only if $\det(B) > 0$. The proof of the claim is complete.

We now proceed to show that $\det(B) > 0$. Define

$$A := \begin{pmatrix} \beta_{33} & \beta_{34} & \dots & \beta_{3n} \\ \beta_{43} & \beta_{44} & \dots & \beta_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n3} & \beta_{n4} & \dots & \beta_{nn} \end{pmatrix} \quad \text{and} \quad E := \mathrm{Diag}(\theta_3, \dots, \theta_n).$$

Now, $B = A + E$. We first establish that A is a $\mathbf{P}_0$-matrix. Let $\widetilde{A}$ be a principal submatrix of A. Without loss of generality, let

$$\widetilde{A} := \begin{pmatrix} \beta_{33} & \beta_{34} & \dots & \beta_{3k} \\ \beta_{43} & \beta_{44} & \dots & \beta_{4k} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k3} & \beta_{k4} & \dots & \beta_{kk} \end{pmatrix}.$$

We shall show that $\det(\widetilde{A}) \geqslant 0$. As usual, let e_k denote the k^{th} vector in the standard basis of $\mathbb{R}^n$. Define

$$P := [\![e_1, e_2 - (e_3 + \ldots + e_k), e_3, \ldots, e_n]\!]$$
$$Q := [\![e_1, e_2, e_3 - e_1, \ldots, e_k - e_1, e_{k+1}, \ldots, e_n]\!].$$

Let $L^{\dagger} = [\![f^1,\dots,f^n]\!]$. Then,

$$L^{\dagger}Q = \left[\!\!\left[f^1, f^2, f^3 - f^1, \dots, f^k - f^1, f^{k+1}, \dots, f^n\right]\!\!\right].$$

Let

$$\Delta := \{3,\dots,k\} \text{ and } \nabla := \{1,\dots,n\} \smallsetminus \Delta.$$

If $i, j \in \Delta$, then the $(i,j)^{\text{th}}$ entry of $PL^{\dagger}Q$ is

$$-e_2' f^j + e_i' f^j + e_2' f^1 - e_i' f^1 = -h_{2j} + h_{ij} + h_{21} - h_{i1} = \frac{\beta_{ij}}{2}.$$

Thus, $\widetilde{A} = 2(PL^{\dagger}Q)_{\Delta}$. By a similar argument, $(PJQ)_{\Delta} = 0$. Therefore, we can write $\widetilde{A}$ as

$$\widetilde{A} = 2(P(L^{\dagger} + \frac{J}{n})Q)_{\Delta}. \tag{3.3}$$

The inverse of P and Q are

$$P^{-1} = [\![e_1, e_2 + \cdots + e_k, e_3, \dots, e_n]\!]$$
$$Q^{-1} = [\![e_1, e_2, e_3 + e_1, \dots, e_k + e_1, e_{k+1}, \dots, e_n]\!].$$

Define $M := Q^{-1}(L+J)$. Let l_{ij}, m_{ij} and α_{ij} denote the $(i,j)^{\text{th}}$ entries of L, M and MP^{-1}. Then,

$$m_{ij} = \begin{cases} l_{ij} + \sum_{r=3}^{k} l_{rj} + k - 1 & i = 1 \\ l_{ij} + 1 & \text{else.} \end{cases} \tag{A}$$

$$\alpha_{ij} = \begin{cases} \sum_{r=2}^{k} m_{ir} & j = 2 \\ m_{ij} & \text{else.} \end{cases} \tag{B}$$

Since $L\mathbf{1} = 0$ and $\operatorname{rank}(L) = n-1$, $\operatorname{adj}(L) = \mu J$, where $\mu > 0$. Put $V := MP^{-1}$.

CLAIM 2. $\det(\widetilde{A}) = \frac{2}{n^2\mu}\det(V_{\nabla})$.

Proof of the claim 2. We know that $\det(P) = \det(Q) = 1$. So,

$$\det(V) = \det(MP^{-1}) = \det(Q^{-1}(L+J)P^{-1}) = \det(L+J).$$

By the matrix determinant lemma,

$$\det(L+J) = n^2\mu.$$

By Jacobi's determinant formula,

$$\det((V^{-1})_{\Delta}) \ \det(V) = \det(V_{\nabla}). \tag{3.4}$$

The assumptions on L imply $L^{\dagger} + \frac{J}{n} = (L+J)^{-1}$. Thus,

$$V^{-1} = P(L^{\dagger} + \frac{1}{n}J)Q. \tag{3.5}$$

Using (3.3), (3.4) and (3.5), we get

$$\det(\widetilde{A}) = \frac{2}{\det(V)}\det(V_{\nabla}) = \frac{2}{n^2\mu}\det(V_{\nabla}).$$

This completes the proof of the claim.

We now compute the column sums of V_{∇}. In view of (B), the $(i,j)^{\text{th}}$ entry of V is given by α_{ij}.

CLAIM 3. If $j \in \nabla$, then

$$\sum_{i\in\nabla} \alpha_{ij} = \begin{cases} n(k-1) & j = 2 \\ n & j \in \nabla \smallsetminus \{2\}. \end{cases}$$

Proof of the claim 3. Let $i \in \nabla$ and $j \in \nabla \smallsetminus \{2\}$. Then, by (B),

$$\sum_{i\in\nabla} \alpha_{ij} = \sum_{i\in\nabla} m_{ij},$$

and by (A),

$$m_{ij} = \begin{cases} l_{1j} + \sum_{\nu=3}^{k} l_{\nu j} + k - 1 & i = 1 \\ l_{ij} + 1 & i = 2, k+1, \dots, n. \end{cases} \tag{3.6}$$

Therefore,

$$\sum_{i\in\nabla} \alpha_{ij} = \sum_{i=1}^{n} l_{ij} + n = n.$$

Let $j = 2$. Then, by (B),

$$\sum_{i\in\nabla} \alpha_{i2} = \sum_{r=2}^{k} m_{1r} + \sum_{r=2}^{k} m_{2r} + \sum_{r=2}^{k} m_{(k+1)r} + \cdots + \sum_{r=2}^{k} m_{nr}.$$

By (3.6), we have

$$\sum_{r=2}^{k} m_{ir} = \begin{cases} \sum_{j=2}^{k} l_{1j} + \sum_{r=3}^{k} (l_{r2} + \cdots + l_{rk}) + (k-1)(k-1) & i = 1 \\ \sum_{j=2}^{k} l_{ij} + (k-1) & i \in \nabla \smallsetminus \{1\}. \end{cases}$$

Therefore,

$$\sum_{i\in\nabla} \alpha_{i2} = \sum_{r=1}^{n} l_{r2} + \cdots + \sum_{r=1}^{n} l_{rk} + n(k-1) = n(k-1).$$

This proves the claim.

The row sums of V_∇ are computed now.

CLAIM 4. Let $i \in \nabla \setminus \{1\}$. Then, $\sum_{j\in\nabla} \alpha_{ij} = n$.

Proof of the claim 4. In view of equations (A) and (B),

$$\sum_{j\in\nabla} \alpha_{ij} = m_{i1} + \sum_{r=2}^{n} m_{ir} = \sum_{j=1}^{n} l_{ij} + n = n.$$

The proof of the claim is complete.

Let $\partial := \det(V_\nabla)$. Utilising the previous claims, we now show that ∂ is non-negative.

CLAIM 5. $\partial \geqslant 0$.

Proof of the claim 5. Define

$$\lambda_j := \sum_{i\in\nabla} \alpha_{ij} \quad j \in \nabla.$$

As the determinant is multilinear in columns,

$$\partial = \det \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_{k+1} & \dots & \lambda_n \\ \alpha_{21} & \alpha_{22} & \alpha_{2(k+1)} & \dots & \alpha_{2n} \\ \alpha_{(k+1)1} & \alpha_{(k+1)2} & \alpha_{(k+1)(k+1)} & \dots & \alpha_{(k+1)n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n(k+1)} & \dots & \alpha_{nn} \end{pmatrix}.$$

By Claims 3 and 4,

$$\partial = n \det \begin{pmatrix} 1 & k-1 & 1 & \dots & 1 \\ \alpha_{21} & \alpha_{22} & \alpha_{2(k+1)} & \dots & \alpha_{2n} \\ \alpha_{(k+1)1} & \alpha_{(k+1)2} & \alpha_{(k+1)(k+1)} & \dots & \alpha_{(k+1)n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n(k+1)} & \dots & \alpha_{nn} \end{pmatrix} \text{ and } \sum_{j\in\nabla} \alpha_{ij} = n \quad i \in \nabla.$$

Thus, by the multilinearity of the determinant,

$$\partial = n^2 \det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ \alpha_{21} & 1 & \alpha_{2(k+1)} & \dots & \alpha_{2n} \\ \alpha_{(k+1)1} & 1 & \alpha_{(k+1)(k+1)} & \dots & \alpha_{(k+1)n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & 1 & \alpha_{n(k+1)} & \dots & \alpha_{nn} \end{pmatrix}.$$

In view of (A) and (B),

$$\partial = n^2 \det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ l_{21}+1 & 1 & l_{2(k+1)}+1 & \dots & l_{2n}+1 \\ l_{(k+1)1}+1 & 1 & l_{(k+1)(k+1)}+1 & \dots & l_{(k+1)n}+1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1}+1 & 1 & l_{n(k+1)}+1 & \dots & l_{nn}+1 \end{pmatrix}.$$

Again, by the multilinearity of the determinant,

$$\partial = n^2 \det \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ l_{21} & 1 & l_{2(k+1)} & \dots & l_{2n} \\ l_{(k+1)1} & 1 & l_{(k+1)(k+1)} & \dots & l_{(k+1)n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & 1 & l_{n(k+1)} & \dots & l_{nn} \end{pmatrix}.$$

Expanding along the first row,

$$\partial = -n^2 \det \begin{pmatrix} l_{21} & l_{2(k+1)} & \dots & l_{2n} \\ l_{(k+1)1} & l_{(k+1)(k+1)} & \dots & l_{(k+1)n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n(k+1)} & \dots & l_{nn} \end{pmatrix}. \tag{3.7}$$

Put

$$\vartheta := l_{21},\ p := (l_{2(k+1)}, \dots, l_{2n}),\ \ q := (l_{(k+1)1}, \dots, l_{n1})' \ \text{ and } \ \Gamma := \{k+1 \dots, n\}.$$

Rewriting (3.7) with these notations,

$$\partial = -n^2 \det \begin{pmatrix} \vartheta & p \\ q & L_\Gamma \end{pmatrix}.$$

As L_Γ is positive definite, by the Schur complement formula,

$$\partial = -n^2 \det(L_\Gamma)(\vartheta - pL_\Gamma^{-1}q).$$

The entries in p and q are non-positive. Since L_Γ is an **M**-matrix, all entries in L_Γ^{-1} are non-negative. Hence, $pL_\Gamma^{-1}q \geqslant 0$. As $\vartheta \leqslant 0$ and $\det(L_\Gamma) > 0$, it follows that $\partial \geqslant 0$. This completes the proof of the claim.

We now complete the proof. By Claim 2, $\det(\widetilde{A}) \geqslant 0$. So, A is a $\mathbf{P}_0$-matrix. Hence, by the multilinear property of the determinants,

$$\det(B) = \det(A+E) = \det(E) + s,$$

where $s \geqslant 0$. Since $\det(E) > 0$, we conclude $\det(B) > 0$. The proof is complete. □

Specializing Theorem 3 to resistance matrices of connected graphs, we have the following result.

COROLLARY 1. *Let $\mathscr{G}$ be a connected graph with n vertices. Then, the Moore-Penrose inverse of the resistance Laplacian matrix of $\mathscr{G}$ is an $\mathbf{M}$-matrix. Furthermore, in this $\mathbf{M}$-matrix, all the off-diagonal entries are negative.*

4. Illustration

We illustrate our result by an example.

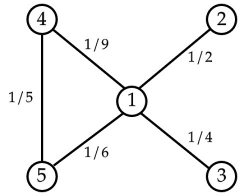

Figure 1: $\mathscr{G}$

EXAMPLE 1. For $\mathscr{G}$,

$$R(\mathscr{G}) = \frac{1}{516}\begin{pmatrix} 0 & 258 & 129 & 44 & 56 \\ 258 & 0 & 387 & 302 & 314 \\ 129 & 387 & 0 & 173 & 185 \\ 44 & 302 & 173 & 0 & 60 \\ 56 & 314 & 185 & 60 & 0 \end{pmatrix}.$$

The Moore-Penrose inverse of $\text{Diag}(\sum_{i=1}^{5} r_{i1}, \ldots, \sum_{i=1}^{5} r_{i5}) - R(\mathscr{G})$ is

$$\frac{1}{17210}\begin{pmatrix} 5707 & -828 & -1288 & -1953 & -1638 \\ -828 & 2222 & -368 & -558 & -468 \\ -1288 & -368 & 3252 & -868 & -728 \\ -1953 & -558 & -868 & 4857 & -1478 \\ -1638 & -468 & -728 & -1478 & 4312 \end{pmatrix}.$$

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(Received February 19, 2024)

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