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THE SPECTRAL RADII ON UNIFORM TRICYCLIC HYPERGRAPHS

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Abstract. A connected k-uniform hypergraph with n vertices and m edges is called tricyclic hypergraphs if $n = m(k-1)-3+1$. Let $\mathbb{T}^m$ be the set of all connected tricyclic k-uniform hypergraphs with m edges, where $m \geqslant 2$. In this paper, the extremal hypergraphs with the first seven largest spectral radius in $\mathbb{T}^m$ are characterized for $m > 20$.

1. Introduction

A hypergraph $H = (V,E)$ is a pair consisting of a vertex set V and a set E of subsets of V, the elements of which are called hyperedges of H. For convenience, let $V = [n]$. If all hyperedges of H have cardinality k, that is, $E \subseteq [n]^k$, then we say that H is k-uniform hypergraph, named by k-graph for short. Obviously, if $k = 2$, H is the ordinary graph. Two vertices contained in one edge are called adjacent to each other and said to be connected by this edge. An edge e that contains a vertex v is called an incident edge of v. If a vertex has exactly one incident edge, then it is called a pendent vertex, otherwise it is called non-pendent. A pendent edge in a k-graph is an edge containing $k-1$ pendent vertices. Denote the order of E by $|E|(= m)$. If some element $e \in E$ or E itself is a multi-set, then H is called a multi-hypergraph. Otherwise, we call H a simple hypergraph. A simple hypergraph is called linear, if each pair of its edges intersects at no more than one vertex, otherwise it is called non-linear. In the sequel, all hypergraphs mentioned are simple uniform hypergraphs, unless otherwise stated.

For a k-graph H, the adjacency tensor $\mathcal{A} = \mathcal{A}(H)$ of order $k(\geqslant 2)$ and dimension n of H refers to a multi-dimensional array with entries $a_{i_1 \cdots i_k}$ such that

$$a_{i_1 \cdots i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, \ldots, i_k\} \text{ is an edge of } H, \\ 0, & \text{otherwise.} \end{cases}$$

where each i_j runs from 1 to n for $j \in [k]$. For a vector $x = (x_1, x_2, \ldots, x_n)^T$ and a positive integer k, let $x^{[k]} = (x_1^k, x_2^k, \ldots, x_n^k)^T$. For a complex number ρ and a vector $x \neq 0$, if

$$\mathcal{A}x = \rho x^{[k]} \tag{1.1}$$

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then ρ is called an eigenvalue of $\mathcal{A}$, and x is an eigenvector of $\mathcal{A}$ corresponding to ρ. The spectrum of H is defined as the multiset of eigenvalues of the tensor $\mathcal{A}$ and the spectral radius of H, denoted by $\rho(H)$, is the maximum modulus among all eigenvalues of $\mathcal{A}$.

Since the initial work of Qi [10] and Lim [6], the research on spectra of hypergraphs via tensors has attracted much attention and interest. For examples, Cooper and Dutle [1] presented a spectral theory of k-graphs that closely parallels spectral graph theory. Li, Shao and Qi [5] determined the unique k-graph with maximum spectral radius among all supertrees. Further in [12], Yuan, Shao and Shan proceeded to order the uniform supertrees with larger spectral radii by their newly introduced edge operation and a relation established by Zhou et al. [13] between spectral radius of an ordinary graph and its kth power. P. Xiao et al [11] determined the supertrees with the first two largest spectral radii among all supertrees. Fan, Tan, Peng and Liu [2] investigated the hypergraphs that attain largest spectral radii among all unicyclic and bicyclic k-graphs and determined the linear hypergraph with maximum spectral radius over all linear unicyclic k-graphs, at the same time, they proposed several candidates for the bicyclic case. Kang et al. [4] proved a conjecture in [2] which lead to the hypergraph maximizing the spectral radius among all linear bicyclic k-graphs. In [8], the authors determined the first five hypergraphs with largest spectral radius among all unicyclic hypergraphs and the first three over all bicyclic hypergraphs.

Let $\mathbb{T}^m$ be the set of all connected tricyclic k-graphs with m edges, where $m \geqslant 2$. Motivating by the preceding work on maximizing and ordering spectral radius, we take into consideration and try to characterize the first few hypergraphs with larger spectral radii among all tricyclic k-graphs.

We first present some edge operations that help investigating k-graphs with larger spectral radius.

DEFINITION 1. [5] Let $r \geqslant 1$ and let $H = (V,E)$ be a k-graph with $u \in V$ and $e_1, \cdots, e_r \in E$ such that $u \notin \cup_{i=1}^{r} e_i$. Suppose that $v_i \in e_i$ and write $e_i' = (e_i \backslash \{v_i\}) \cup u$ for $i \in [r]$. Let $H' = (V,E')$ be the hypergraph with $E' = (E \backslash \{e_i : i \in [r]\}) \cup \{e_i' : i \in [r]\}$. Then we say that H' is obtained from H by moving edges $(e_1, \cdots, e_r)$ from $(v_1, \cdots, v_r)$ to u.

LEMMA 1.1. [8] *Let H be a connected k-graph and $(v_1, \cdots, v_r)$ be some of its vertices for $r \geqslant 2$. Let H_i be a simple hypergraph obtained from H by moving at least one edge from vertices $\{v_j : j \in [r] \setminus \{i\}\}$ to v_i. Then we have*

$$max\{\rho(H_i) : i \in [r]\} > \rho(H).$$

LEMMA 1.2. [8] *Let H be a connected k-graph having two adjacent vertices u_1 and u_2. Let H' be the hypergraph obtained from H by moving all incident edges of u_2 except all common edges shared by u_1, u_2 from u_2 to u_1. If $H' \ncong H$, then*

$$\rho(H) < \rho(H').$$

LEMMA 1.3. [12] *Let* $k \geqslant 3$*,* H *be a connected* k*-graph on* n *vertices having two edges* e *and* f *such that* $|e \cap f| = k - r (2 \leqslant r \leqslant k-1)$*. Let* $V_1 = e \cap f$ *and* $e \setminus V_1 = \{u_1, \cdots, u_r\}$ *and* $f \setminus V_1 = \{v_1, \cdots, v_r\}$ *where* $r \geqslant 2$*,* u_1*,* v_1 *are non-pendent vertices while* $u_2, \cdots, u_r$ *and* $v_2, \cdots, v_r$ *are pendent vertices. Let* $H_{e,f}$ *be the hypergraph obtained from* H *by moving all the edges incident with* v_1 *except* f *from* v_1 *to* u_2*. Then* $\rho(H_{e,f}) > \rho(H)$*.*

2. Preliminaries

In this section, we will compare the spectral radii among some specialized tricyclic k-graphs by the methods on weighted incidence matrix and the definition of eigenvalue and eigenvector of tensor.

2.1. Method on weighted incidence matrix

Let $G = (V, E)$ be a multi-graph containing no loops, i.e. cycles of length 1. Let $G(a,b)$ be a multi-graph obtained from a cycle of length 2 by attaching a and b pendent edges at its two vertices u and v respectively. Denote by $H(a,b)$ the multi-graph obtained from $G(a,b)$ by adding a new edge connecting u and v. Denote by $M(a,b)$ the multi-graph obtained from $G(a,b)$ by adding two new edges connecting u and v.

The adjacency matrix $A(G)$ of a multi-graph G on n vertices without loops is an $n \times n$ matrix whose (ij)-entry is the number of parallel edges connecting i and j if $i \neq j$ and zero otherwise. Denote by $\phi_G(x) = det(xI - A(G))$ the characteristic polynomial of a multi-graph G, where I denotes the unit matrix. By direct calculation, we have

$$\begin{aligned}\phi_{M(a,b)}(x) &= \phi_{M(a,0)} \cdot x^b + \phi_{K_{1,a}} \cdot \phi_{K_{1,b}} - x \cdot \phi_{K_{1,a}} \cdot x^b \\ &= x^{m-6}[x^4 - (m+12)x^2 + ab], \end{aligned} \tag{2.1}$$

thus $\rho(M(m-4,0))^2 = m + 12$.

$$\left(\frac{1 - \frac{m-1}{m+6}}{1 - \frac{m-4}{m+6}}\right)^3 \tag{2.2}$$

The kth power of a multi-graph G is the k-graph G^k obtained from G by blowing up its edges to hyperedges through adding $k-2$ new pendent vertices to each edge of G.

LEMMA 2.1. [13] *If* $\lambda \neq 0$ *is an eigenvalue of a multi-graph* G*, then* $\lambda^{\frac{2}{k}}$ *is an eigenvalue of* G^k*. Moreover,* $\rho(G^k) = \rho(G)^{\frac{2}{k}}$*.*

DEFINITION 2. [7] A weighted incidence matrix B of a hypergraph $H = (V, E)$ is a $|V| \times |E|$ matrix such that for any vertex v and any edge e, the entry $B(v, e) > 0$ if $v \in e$ and $B(v, e) = 0$ if $v \notin e$.

DEFINITION 3. [7] A hypergraph H is called α-subnormal if there exists a weighted incidence matrix B satisfying

(a) $\sum_{e:v\in e} B(v,e) \leqslant 1$, for any $v \in V(H)$;
(b) $\prod_{v\in e} B(v,e) \geqslant \alpha$, for any $e \in E(H)$.

If no strict inequality appears in (a) and (b), then H is α-normal. Otherwise, H is called strictly α-subnormal. If furthermore,

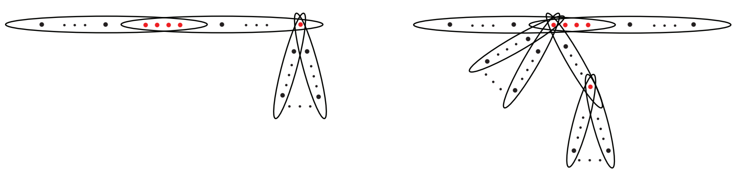
$$\prod_{i=1}^{l} \frac{B(v_i,e_i)}{B(v_{i-1},e_i)} = 1$$

for any cycle $v_0e_1v_1e_2\cdots e_lv_0 (l \geqslant 1)$ in H, then B is consistent and H is called strictly and consistently α-subnormal.

LEMMA 2.2. [7] *Let H be a k-graph. Then*

(i) $\rho(H) = \alpha^{-\frac{1}{k}}$ *if and only if H is consistently α-normal;*
(ii) *if H is strictly and consistently α-subnormal, then $\rho(H) < \alpha^{-\frac{1}{k}}$.*

Let $T_4^1(a,b,c,d)$ be a k-graph with merely two non-pendent edges which intersect at exactly four vertices u,v,w,t, where a,b,c,d are the number of pendent edges attached at u,v,w,t respectively. Let $T_5^1(a,b,c,d,e)$ be the k-graph obtained from $T_4^1(a,b,c,d)$ by attaching e pendent edges at an arbitrary pendent vertex s in a cycle edge. Let $T_5^2(a,b,c,d,e)$ be the k-graph obtained from $T_4^1(a+1,b,c,d)$ by attaching e pendent edges at a pendent vertex s adjacent to u outside the cycle. The k-graphs $T_5^1(a,b,c,d,e)$ and $T_5^2(a,b,c,d,e)$ are presented in Figure 1.



$T_5^1(a,b,c,d,e)$ $T_5^2(a,b,c,d,e)$

Figure 1: *k-graphs of $T_5^1(a,b,c,d,e)$ and $T_5^2(a,b,c,d,e)$*

Denote by $U_2(a,b)$ the kth power of $G(a,b)$. Let $U_3^1(a,b,c)$ be the k-graph obtained from $U_2(a,b)$ by attaching c pendent edges at an arbitrary pendent vertex w in a cycle edge. Let $U_3^2(a,b;c)$ be the k-graph obtained from $U_2(a+1,b)$ by attaching c pendent edges at a pendent vertex w adjacent to u outside the cycle.

Denote by $B_2(a,b)$ the kth power of $H(a,b)$. Denote by $B_3^1(a,b,c)$ the k-graph with merely two non-pendent edges which intersect at exactly three vertices u,v,w, where a,b,c are the number of pendent edges attached at u,v,w respectively. Let

$B_3^2(a,b,c)$ ($B_3^3(a,b,c)$, resp.) be the hypergraph obtained from $U_3^1(a,b;c)$ by adding a new edge containing u,v (v,w resp.) and $k-2$ new pendent vertices. Let $B_3^4(a,b,c)$ ($B_3^5(a,b,c)$ and $B_3^6(a,b,c)$ resp.) be the hypergraph obtained from $U_3^2(a,b;c)$ by adding a new edge containing u,v (u,w and v,w resp.) and $k-2$ new pendent vertices.

Denote by $T_2(a,b)$ the kth power of $M(a,b)$. Let $T_3^1(a,b,c)$ be the k-graph obtained from $T_2(a,b)$ by attaching c pendent edges at an arbitrary pendent vertex w in a cycle edge. Let $T_3^8(a,b,c)$ be the k-graph obtained from $T_2(a+1,b)$ by attaching c pendent edges at an arbitrary pendent vertex w adjacent to u outside the cycle. Let $T_3^2(a,b,c)$ be the hypergraph obtained from $B_3^2(a,b;c)$ by adding a new edge containing v,w and $k-2$ new pendent vertices. Let $T_3^3(a,b,c)$ be the hypergraph obtained from $B_3^3(a,b;c)$ by adding a new edge containing u,w and $k-2$ new pendent vertices. Let $T_3^4(a,b,c)$ be the hypergraph obtained from $B_3^1(a,b;c)$ by adding a new edge containing u,v and $k-2$ new pendent vertices. Let $T_3^5(a,b,c)$ be the hypergraph obtained from $B_3^4(a,b;c)$ by adding a new edge containing v,w and $k-2$ new pendent vertices. Let $T_3^6(a,b,c)$ be the hypergraph obtained from $B_3^5(a,b;c)$ by adding a new edge containing v,w and $k-2$ new pendent vertices. Let $T_3^7(a,b,c)$ be the hypergraph obtained

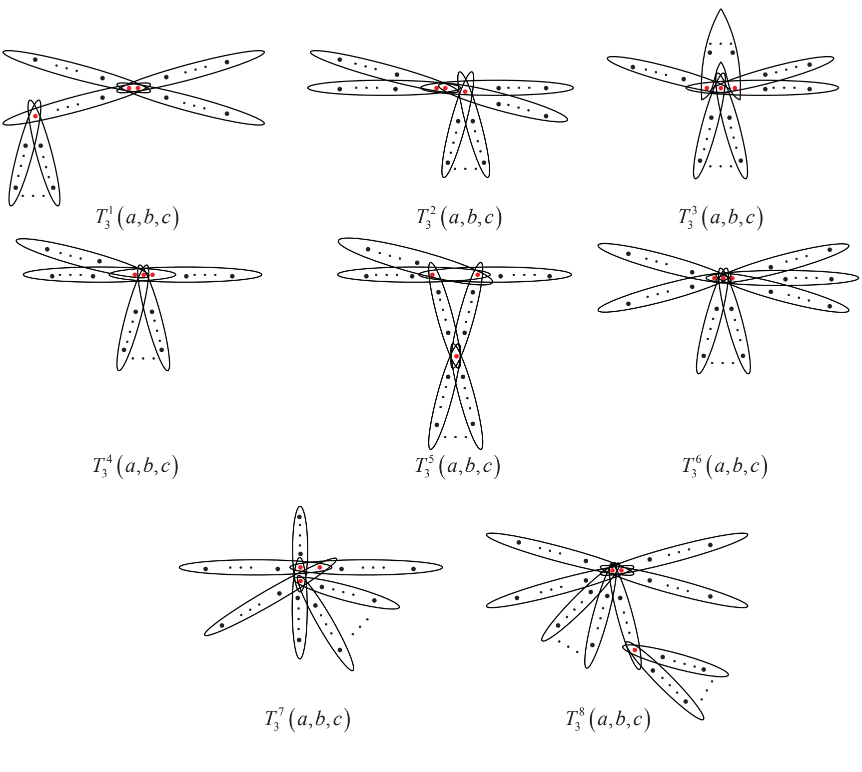


Figure 2: *Some tricyclic k-graphs in* $\mathbb{T}_3^m$

from $B_3^6(a,b;c)$ by adding a new edge containing u,w and $k-2$ new pendent vertices. The k-graph $T_3^i(a,b,c)$ $(i=1,2,\ldots,8)$ is as shown in Figure 2.

LEMMA 2.3. *For* $m \geqslant 6$, $\rho(T_3^2(0,m-4,0)) < \rho(T_2(m-4,0))$.

Proof. Let $\alpha = \rho(M(m-4,0))^{-2}$. Since $T_2(m-4,0)$ is the kth power of $M(m-4,0)$, by Lemma 2.1 we have $\alpha^{-\frac{1}{k}} = \rho(M(m-4,0))^{\frac{2}{k}} = \rho(T_2(m-4,0))$.

We first construct a weighted incidence matrix B for $T_3^2(0,m-4,0)$. Let $B(p,e) = 1$ for every pendent vertex p in edge e and let $B(q,f) = \alpha$ for each non-pendent vertex q in a pendent edge f. Suppose that e_1 and e_3 are the two edges intersecting at u,v, e_2 is one edge contains u,v,w, and e_4 is one edge contains v,w. Write $x_i = B(u,e_i)$ for $i=1,2,3$, $y_i = B(v,e_i)$ for $i=1,2,3,4$ and $z_i = B(w,e_i)$ for $i=2,4$. Let

$$\begin{cases} x_1+x_2+x_3 = 1, \\ y_1+y_2+y_3+y_4 = 1-(m-4)\alpha, \\ z_2+z_4 = 1, \\ x_1y_1 = \alpha, \\ x_3y_3 = \alpha, \\ y_4z_4 = \alpha, \end{cases}$$

and let $A = \frac{x_2}{x_1} = \frac{y_2}{y_1} > 0$, $B = \frac{y_2}{y_4} = \frac{z_2}{z_4} > 0$, so $\frac{y_1}{y_4} = \frac{B}{A} > 0$.

Since B is consistent for all cycles in $T_3^2(0,m-4,0)$, and B is consistent according to Definition 3. It is easy to verify that all equalities hold for (a) and (b) of Definition 3 except on the edge e_2.

Now we compare $x_2y_2z_2$ with α. Note that

$$(A+2)\left(A+2+\frac{A}{B}\right) = \frac{1-(m-4)\alpha}{\alpha} = m+12-(m-4) = 16.$$

By a direct calculation, we have $B = \frac{2A}{16-(A+2)^2}$ and $AB^2+2B^2+2AB+2B-15A = 0$. Thus

$$14A^5+124A^4-104A^3-1480A^2+2112A = 0,$$

Further by $A>0$, $B>0$, we have $A = 1.7741$, $B = 2.0204$. Then

$$\prod_{t\in e_2} B(t,e_2) = x_2y_2z_2 = A^2\frac{B}{1+B}x_1y_1 = 2.1057\alpha > \alpha.$$

Thus $T_3^2(0,m-4,0)$ is strictly and consistently α-subnormal by Definition 3. By Lemma 2.2 (ii), we have $\rho(T_3^2(0,m-4,0)) < \alpha^{-\frac{1}{k}} = \rho(T_2(m-4,0))$. □

LEMMA 2.4. *For* $m \geqslant 6$, $\rho(T_3^3(0,m-4,0)) < \rho(T_2(m-4,0))$.

Proof. Let $\alpha=\rho(M(m-4,0))^{-2}$. Since $T_2(m-4,0)$ is the kth power of $M(m-4,0)$, by Lemma 2.1 we have $\alpha^{-\frac{1}{k}}=\rho(M(m-4,0))^{\frac{2}{k}}=\rho(T_2(m-4,0))$.

We first construct a weighted incidence matrix B for $T_3^3(0,m-4,0)$. Let $B(p,e)=1$ for every pendent vertex p in edge e and let $B(q,f)=\alpha$ for each non-pendent vertex q in a pendent edge f. Suppose that e_1 is the one edge contains u,v,w, e_2 is one edge contains u,w, e_3 is one edge contains u,v, and e_4 is one edge contains v,w. Write $x_i=B(u,e_i)$ for $i=1,2,3$, $y_i=B(v,e_i)$ for $i=1,3,4$ and $z_i=B(w,e_i)$ for $i=1,2,4$. Let

$$\begin{cases} x_1+x_2+x_3=1, \\ y_1+y_3+y_4=1-(m-4)\alpha, \\ z_1+z_2+z_4=1, \\ y_4z_4=\alpha, \\ x_3y_3=\alpha, \\ x_2z_2=\alpha, \end{cases}$$

and let $A=\frac{x_3}{x_1}=\frac{y_3}{y_1}=\frac{z_4}{z_1}=\frac{y_4}{y_1}>0$, $B=\frac{z_2}{z_1}=\frac{z_2}{z_1}>0$.

Since B is consistent for all cycles in $T_3^3(0,m-4,0)$, and B is consistent according to Definition 3. It is easy to verify that all equalities hold for (a) and (b) of Definition 3 except on the edge e_1.

Now we compare $x_1y_1z_1$ with α. Note that

$$\frac{(1+A+B)(1+2A)}{A^2}=\frac{1-(m-4)\alpha}{\alpha}=16.$$

By direct calculation, we have $B=\frac{14A^2-3A-1}{1+2A}$ and

$$\left(\frac{B}{1+A+B}\right)^2=\alpha=\frac{1}{m+12}<\frac{1}{16}.$$

Then $3B<1+A$, and $3\frac{14A^2-3A-1}{1+2A}<1+A$, that is, $10A^2-3A-1<0$. Hence $0<A<0.5$ and $B<0.5$. Thus

$$\prod_{t\in e_1}B(t,e_1)=x_1y_1z_1=\frac{1}{A^2}\frac{1}{1+A+B}x_3y_3>4\times\frac{1}{2}\alpha=2\alpha>\alpha.$$

Thus $T_3^3(0,m-4,0)$ is strictly and consistently α-subnormal by Definition 3. By Lemma 2.2 (ii), we have $\rho(T_3^3(0,m-4,0))<\alpha^{-\frac{1}{k}}=\rho(T_2(m-4,0))$. $\square$

LEMMA 2.5. *For* $m \geqslant 6$, $\rho(T_3^4(0,m-3,0)) < \rho(T_2(m-4,0))$.

Proof. Let $\alpha = \rho(M(m-4,0))^{-2}$. Since $T_2(m-4,0)$ is the kth power of $M(m-4,0)$, by Lemma 2.1 we have $\alpha^{-\frac{1}{k}} = \rho(M(m-4,0))^{\frac{2}{k}} = \rho(T_2(m-4,0))$.

We first construct a weighted incidence matrix B for $T_3^4(0,m-3,0)$. Let $B(p,e) = 1$ for every pendent vertex p in edge e and let $B(q,f) = \alpha$ for each non-pendent vertex q in a pendent edge f. Suppose that e_1 and e_2 are the two edges intersecting at u,v,w, e_3 is one edge contains u,v. Write $x_i = B(u,e_i)$, $y_i = B(v,e_i)$ for $i = 1,2,3$ and $z_i = B(w,e_i)$ for $i = 1,2$. Let

$$\begin{cases} x_1 + x_2 + x_3 = 1, \\ y_1 + y_2 + y_3 = 1 - (m-3)\alpha, \\ z_1 + z_2 = 1, \\ x_2 y_2 z_2 = \alpha, \\ x_3 y_3 = \alpha, \end{cases}$$

and let $A = \frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2} > 0$, $B = \frac{x_1}{x_3} = \frac{y_1}{y_3} > 0$.

Since $x_1 y_2 = x_2 y_1$, $x_1 z_2 = x_2 z_1$, $y_1 z_2 = y_2 z_1$ and $x_1 y_3 = x_3 y_1$ for all four cycles, B is consistent according to Definition 3. It is easy to verify that all equalities hold for (a) and (b) of Definition 3 except on the edge e_1.

Now we compare $x_1 y_1 z_1$ with α. Note that

$$\frac{(AB+A+B)^2}{A^2} = \frac{1-(m-3)\alpha}{\alpha},$$
$$\frac{(A+1)(AB+A+B)^2}{B^2} = \frac{1-(m-3)\alpha}{\alpha}.$$

By direct calculation, we have $B = A\sqrt{A+1}$. Then

$$\frac{(A^2\sqrt{A+1} + A + A\sqrt{A+1})^2}{A^2} = \frac{1-(m-3)\alpha}{\alpha} = m + 12 - (m-3) = 15$$

and $A = \sqrt[3]{(15-1)^2} - 1 = 1.020948$. Thus

$$\prod_{t \in e_1} B(t,e_1) = x_1 y_1 z_1 = A^3 x_2 y_2 z_2 = A^3 \alpha > \alpha.$$

Thus $T_3^4(0,m-3,0)$ is strictly and consistently α-subnormal by Definition 3. By Lemma 2.2 (ii), we have $\rho(T_3^4(0,m-3,0)) < \alpha^{-\frac{1}{k}} = \rho(T_2(m-4,0))$. □

2.2. Method on the definition of eigenvalue and eigenvector of tensor

Let $T_2(m-4,0)$, $T_4^1(m-3,1,0,0)$, $T_4^1(m-4,2,0,0)$, $T_4^1(m-4,1,1,0)$, $T_5^1(m-3,0,0,0,1)$, $T_5^2(m-4,0,0,0,1)$, $T_5^2(m-5,0,0,0,2)$ be k-graphs as shown in Figure 3. For simplicity, let $X = [\lambda \cdot (k-1)!]^k$ and $\underbrace{a \ldots a}_{b} = a^b$.

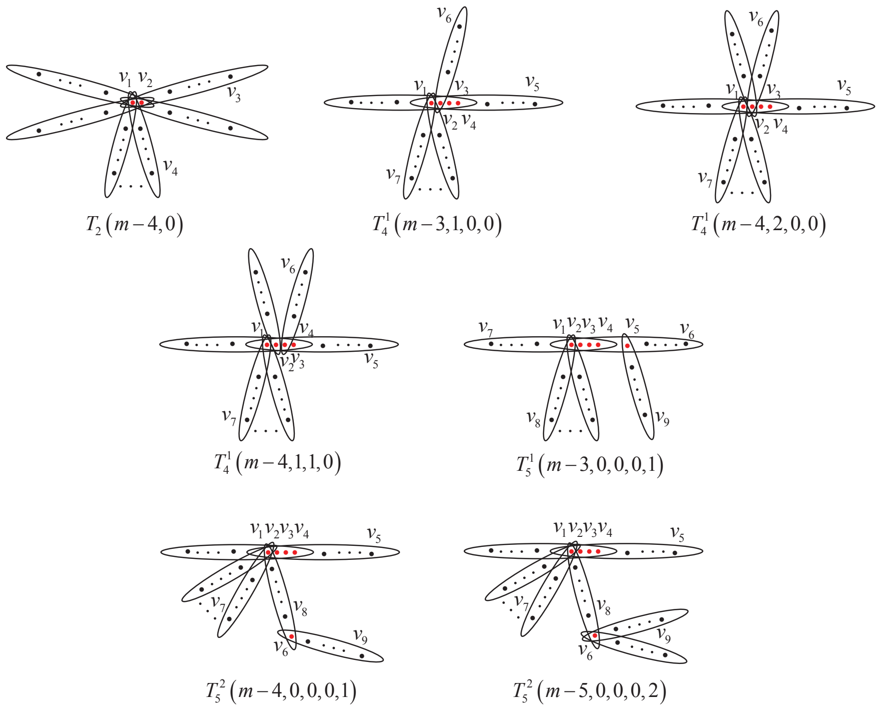


Figure 3: *Some tricyclic k-graphs in T_i^m for $i=4,5$.*

LEMMA 2.6. *For $m \geqslant 6$, $\rho(T_4^1(m-4,1,1,0)) > \rho(T_5^2(m-5,0,0,0,2))$.*

Proof. Let λ and $x=(x_1,x_2,\ldots,x_n)$ be the eigenvalue and eigenvector of the adjacency tensor $\mathcal{A}$ of $T_4^1(m-4,1,1,0)$, respectively. By (1.1) and symmetry of vertices of $T_4^1(m-4,1,1,0)$, we have

$$\begin{cases} 2a_{12345^{k-4}}x_2x_3x_4x_5^{k-4}+(m-4)a_{17^{k-1}}x_7^{k-1}=\lambda x_1^{k-1}, \\ 2a_{12345^{k-4}}x_1x_3x_4x_5^{k-4}+a_{26^{k-1}}x_6^{k-1}=\lambda x_2^{k-1}, \\ 2a_{12345^{k-4}}x_1x_2x_4x_5^{k-4}+a_{36^{k-1}}x_6^{k-1}=\lambda x_3^{k-1}, \\ 2a_{12345^{k-4}}x_1x_2x_3x_5^{k-4}=\lambda x_4^{k-1}, \\ a_{12345^{k-4}}x_1x_2x_3x_4x_5^{k-5}=\lambda x_5^{k-1}, \\ a_{26^{k-1}}x_2x_6^{k-2}=\lambda x_6^{k-1}, \\ a_{17^{k-1}}x_1x_7^{k-2}=\lambda x_7^{k-1}, \end{cases}$$

By direct calculation, we have $X^3-(m+14)X^2+(2m-7)X-(m-4)=0$.

Similarly, for k-graph $T_5^2(m-5,0,0,0,2)$, we have

$$\begin{cases} 2a_{12345^{k-4}}x_2x_3x_4x_5^{k-4}+(m-5)a_{17^{k-1}}x_7^{k-1}+a_{168^{k-2}}x_6x_8^{k-2}=\lambda x_1^{k-1},\\ 2a_{12345^{k-4}}x_1x_3x_4x_5^{k-4}=\lambda x_2^{k-1},\\ 2a_{12345^{k-4}}x_1x_2x_4x_5^{k-4}=\lambda x_3^{k-1},\\ 2a_{12345^{k-4}}x_1x_2x_3x_5^{k-4}=\lambda x_4^{k-1},\\ a_{12345^{k-4}}x_1x_2x_3x_4x_5^{k-5}=\lambda x_5^{k-1},\\ a_{168^{k-1}}x_1x_8^{k-2}+2a_{69^{k-1}}x_9^{k-1}=\lambda x_6^{k-1},\\ a_{12347^{k-4}}x_1x_2x_3x_4x_7^{k-5}=\lambda x_7^{k-1},\\ a_{168^{k-1}}x_1x_6x_8^{k-3}=\lambda x_8^{k-1},\\ a_{69^{k-1}}x_6x_9^{k-2}=\lambda x_9^{k-1}. \end{cases}$$

By direct calculation, we have $X^2-(m+14)X+2(m+11)=0$

Let $Y=[\rho(T_5^2(m-5,0,0,0,2))(k-1)!]^k$, then $Y^2=(m+14)Y-2(m+11)$. Let $g(X)=X^3-(m+14)X^2+(2m-7)X-(m-4)$. Then

$$\begin{aligned} g(Y) &= Y^3-(m+14)Y^2+(2m-7)Y-(m-4)\\ &= Y[(m+14)Y-2(m+11)]-(m+14)Y^2+(2m-7)Y-(m-4)\\ &= -29Y-m+4<0. \end{aligned}$$

Note that $[\rho(T_4^1(m-4,1,1,0))(k-1)!]^k$ is the largest zero point of $g(X)$, thus it is strictly larger than $Y=[\rho(T_5^2(m-5,0,0,0,2))(k-1)!]^k$, then $\rho(T_4^1(m-4,1,1,0))>\rho(T_5^2(m-5,0,0,0,2))$. □

LEMMA 2.7. *Let $m>20$, then $\rho(T_5^2(m-4,0,0,0,1))>\rho(T_4^1(m-4,2,0,0))$.*

Proof. Let λ and $x=(x_1,x_2,\ldots,x_n)$ be the eigenvalue and eigenvector of the adjacency tensor $\mathcal{A}$ of $T_4^1(m-4,2,0,0)$, respectively. By (1.1) and symmetry of vertices of $T_4^1(m-4,2,0,0)$, we have

$$\begin{cases} 2a_{12345^{k-4}}x_2x_3x_4x_5^{k-4}+(m-4)a_{17^{k-1}}x_7^{k-1}=\lambda x_1^{k-1},\\ 2a_{12345^{k-4}}x_1x_3x_4x_5^{k-4}+2a_{26^{k-1}}x_6^{k-1}=\lambda x_2^{k-1},\\ 2a_{12345^{k-4}}x_1x_2x_4x_5^{k-4}=\lambda x_3^{k-1},\\ 2a_{12345^{k-4}}x_1x_2x_3x_5^{k-4}=\lambda x_4^{k-1},\\ a_{12345^{k-4}}x_1x_2x_3x_4x_5^{k-5}=\lambda x_5^{k-1},\\ a_{26^{k-1}}x_2x_6^{k-2}=\lambda x_6^{k-1},\\ a_{17^{k-1}}x_1x_7^{k-2}=\lambda x_7^{k-1}, \end{cases}$$

By direct calculation, we have $X^2-(m+14)X+2(m-4)=0$.

Similarly, for k-graph $T_5^2(m-4,0,0,0,1)$, we have

$$\begin{cases} 2a_{12345^{k-4}}x_2x_3x_4x_5^{k-4}+(m-4)a_{17^{k-1}}x_7^{k-1}+a_{168^{k-2}}x_6x_8^{k-2}=\lambda x_1^{k-1}, \\ 2a_{12345^{k-4}}x_1x_3x_4x_5^{k-4}=\lambda x_2^{k-1}, \\ 2a_{12345^{k-4}}x_1x_2x_4x_5^{k-4}=\lambda x_3^{k-1}, \\ 2a_{12345^{k-4}}x_1x_2x_3x_5^{k-4}=\lambda x_4^{k-1}, \\ a_{12345^{k-4}}x_1x_2x_3x_4x_5^{k-5}=\lambda x_5^{k-1}, \\ a_{168^{k-1}}x_1x_8^{k-2}+a_{19^{k-1}}x_9^{k-1}=\lambda x_6^{k-1}, \\ a_{12347^{k-4}}x_1x_2x_3x_4x_7^{k-5}=\lambda x_7^{k-1}, \\ a_{168^{k-1}}x_1x_6x_8^{k-3}=\lambda x_8^{k-1}, \\ a_{69^{k-1}}x_6x_9^{k-2}=\lambda x_9^{k-1}. \end{cases}$$

and $X^2-(m+14)X+(m+12)=0$.

Let $Y=[\rho(T_4^1(m-4,2,0,0))(k-1)!]^k$, then $Y^2=(m+14)Y-2(m-4)$. Let $g(X)=X^2-(m+14)X+(m+12)$. Then

$$\begin{aligned} g(Y) &= Y^2-(m+14)Y+(m+12) \\ &= -m+20<0. \end{aligned}$$

Note that $[\rho(T_5^2(m-4,0,0,0,1))(k-1)!]^k$ is the largest zero point of $g(X)$, thus it is strictly larger than $Y=[\rho(T_4^1(m-4,2,0,0))(k-1)!]^k$, then $\rho(T_5^2(m-4,0,0,0,1))>\rho(T_4^1(m-4,2,0,0))$. $\square$

LEMMA 2.8. *For $m\geqslant 6$, $\rho(T_4^1(m-3,1,0,0))>\rho(T_5^2(m-4,0,0,0,1))$.*

Proof. Let λ and $x=(x_1,x_2,\dots,x_n)$ be the eigenvalue and eigenvector of the adjacency tensor $\mathcal{A}$ of $T_4^1(m-3,1,0,0)$, respectively. By (1.1) and symmetry of vertices of $T_4^1(m-3,1,0,0)$, we have

$$\begin{cases} 2a_{12345^{k-4}}x_2x_3x_4x_5^{k-4}+(m-3)a_{17^{k-1}}x_7^{k-1}=\lambda x_1^{k-1}, \\ 2a_{12345^{k-4}}x_1x_3x_4x_5^{k-4}+a_{16^{k-1}}x_6^{k-1}=\lambda x_2^{k-1}, \\ 2a_{12345^{k-4}}x_1x_2x_4x_5^{k-4}=\lambda x_3^{k-1}, \\ 2a_{12345^{k-4}}x_1x_2x_3x_5^{k-4}=\lambda x_4^{k-1}, \\ a_{12345^{k-4}}x_1x_2x_3x_4x_5^{k-5}=\lambda x_5^{k-1}, \\ a_{26^{k-1}}x_2x_6^{k-2}=\lambda x_6^{k-1}, \\ a_{17^{k-1}}x_1x_7^{k-2}=\lambda x_7^{k-1}, \end{cases}$$

By direct calculation, we have $X^2-(m+14)X+(m-3)=0$.

Let $Y=[\rho(T_5^2(m-4,0,0,0,1))(k-1)!]^k$, then $Y=(m+14)Y-(m+12)$. Let $g(X)=X^2-(m+14)X+(m-3)$. Then

$$\begin{aligned} g(Y) &= Y^2-(m+14)Y+(m-3) \\ &= -15<0. \end{aligned}$$

Note that $[\rho(T_4^1(m-3,1,0,0))(k-1)!]^k$ is the largest zero point of $g(X)$, thus it is strictly larger than $Y=[\rho(T_5^2(m-4,0,0,0,1))(k-1)!]^k$, then $\rho(T_4^1(m-3,1,0,0)) > \rho(T_5^2(m-4,0,0,0,1))$. $\square$

By lemmas 1.2 and 2.8–10, we have the following results.

COROLLARY 2.9. *For $m > 20$, $\rho(T_2(m-4,0)) < \rho(T_5^2(m-5,1,0,0,1)) < \rho(T_5^2(m-5,0,0,0,2)) < \rho(T_4^1(m-4,1,1,0)) < \rho(T_4^1(m-4,2,0,0)) < \rho(T_5^2(m-4,0,0,0,1)) < \rho(T_4^1(m-3,1,0,0)) < \rho(T_4^1(m-2,0,0,0))$.*

LEMMA 2.10. *For $m \geqslant 6$, $\rho(T_2(m-4,0)) > \rho(T_5^1(m-3,0,0,0,1))$.*

Proof. Let λ and $x=(x_1,x_2,\ldots,x_n)$ be the eigenvalue and eigenvector of the adjacency tensor $\mathcal{A}$ of $T_5^1(m-3,0,0,0,1)$, respectively. By (1.1) and symmetry of vertices of $T_5^1(m-3,0,0,0,1)$, we have

$$\begin{cases} a_{123456^{k-5}}x_2x_3x_4x_5x_6^{k-5}+a_{12347^{k-4}}x_2x_3x_4x_7^{k-4}+(m-3)a_{18^{k-1}}x_8^{k-1}=\lambda x_1^{k-1}, \\ a_{123456^{k-5}}x_1x_3x_4x_5x_6^{k-5}+a_{12347^{k-4}}x_1x_3x_4x_7^{k-4}=\lambda x_2^{k-1}, \\ a_{123456^{k-5}}x_1x_2x_4x_5x_6^{k-5}+a_{12347^{k-4}}x_1x_2x_4x_7^{k-4}=\lambda x_3^{k-1}, \\ a_{123456^{k-5}}x_1x_2x_3x_5x_6^{k-5}+a_{12347^{k-4}}x_1x_2x_3x_7^{k-4}=\lambda x_4^{k-1}, \\ a_{123456^{k-5}}x_2x_3x_4x_5x_6^{k-5}+a_{59^{k-1}}x_9^{k-1}=\lambda x_5^{k-1}, \\ a_{123456^{k-5}}x_1x_2x_3x_4x_5x_6^{k-6}=\lambda x_6^{k-1}, \\ a_{12347^{k-4}}x_1x_2x_3x_4x_7^{k-5}=\lambda x_7^{k-1}, \\ a_{18^{k-1}}x_1x_8^{k-2}=\lambda x_8^{k-1}, \\ a_{59^{k-1}}x_5x_9^{k-2}=\lambda x_9^{k-1}, \end{cases}$$

By direct calculation, we have $X^2-mX+(m-2)=0$.

Similarly, for k-graph $T_2(m-4,0)$, we have

$$\begin{cases} 4a_{123^{k-2}}x_2x_3^{k-2}+(m-4)a_{14^{k-1}}x_4^{k-1}=\lambda x_1^{k-1}, \\ 4a_{123^{k-2}}x_1x_3^{k-2}=\lambda x_2^{k-1}, \\ a_{123^{k-2}}x_1x_2x_3^{k-3}=\lambda x_3^{k-1}, \\ a_{14^{k-1}}x_1x_4^{k-2}=\lambda x_4^{k-1}, \end{cases}$$

and $X-m-12=0$.

Let $g(X)=X^2-mX+(m-2)$. Then $g'(X)=2X-m>0$ and $g(X)$ is increasing for $X \geqslant m+12$. Note that

$$\begin{aligned} g(m+12) &= (m+12)^2-m(m+12)+(m-2) \\ &= 13m+142>0. \end{aligned}$$

Hence all zero points of $g(X)$ are less than $m+12$. Let $[\rho(T_5^1(m-3,0,0,0,1))(k-1)!]^k$ be the largest zero point of $g(X)$, thus it is strictly smaller than $m+12=[\rho(T_2(m-4,0))(k-1)!]^k$, then $\rho(T_2(m-4,0)) > \rho(T_5^1(m-3,0,0,0,1))$. $\square$

3. The extremal tricyclic k-graphs with larger spectral radius in $\mathbb{T}^m$

For a k-graph $H=(V,E)$, if $E' \subset E$ and $V' = \bigcup_{e\in E'} e \subset V$, then $H'=(V',E')$ is called a sub-k-graph induced by E'. A path in H refers to an alternative sequence of distinct vertices and distinct edges such that two consecutive vertices are contained in the edge between them in this sequence. If every two vertices in H appear in at least one path, then H is called a connected k-graph. A cycle in H is formed from a path and another edge in H containing the two end vertices of that path. The number of edges in this cycle is called its length. An edge contained in a cycle is called a cycle edge. A k-graph on n vertices and m edges is called r-cyclic if $m(k-1)-n+l=r$, where l is the number of its connected components.

LEMMA 3.1. [8] *Let $H=(V,E)$ be a simple connected r-cyclic k-graph with n vertices and m edges. Let $H_1=(V_1,E_1)$ be a connected subgraph of H. If H_1 is r_1-cyclic, then $r_1 \leqslant r$.*

PROPOSITION 3.2. *Let H be a k-graphs in $\mathbb{T}^m$. Then*

(i) *every two vertices in H share at most four common edges;*
(ii) *every four vertices in H have at most two common edge.*

Proof. If there exist two vertices in H having five common edges, or there are five vertices sharing two common edges, then the subgraph in H induced by those common edges is 4-cyclic sub-k-graph, which contradicts Lemma 3.1. □

Denote by $\mathbb{T}_i^m$ the set of hypergraphs in $\mathbb{T}^m$ with exactly i non-pendent vertices where $i \geqslant 2$. Let H be a k-graph in $\mathbb{T}^m$. We first consider that H is in $\mathbb{T}_2^m$. Let u,v be the non-pendent vertices in H. Since H is tricyclic, u,v have at least four common edges. By Proposition 3.2, there are exactly four edges sharing u,v. As the remaining edges of H (if there exists any) are pendent edges, $H \cong T_2(a,b)$ for some integers a,b. Thus k-graphs in $\mathbb{T}_2^m$ are in the form of $T_2(a,b)$ with $a,b \in N$.

LEMMA 3.3. *Let $a \geqslant b \geqslant 1$ and $a+b=m-3$. Then*

$$\rho(T_2(a,b)) < \rho(T_2(a+1,b-1)) \leqslant \rho(T_2(m-4,0))$$

Proof. Note that $\rho(T_2(a+1,b-1))$ can be obtained from $\rho(T_2(a,b))$ by moving one pendent edge from v to u, or by moving $a-b+1$ pendent edges from u to v. By Lemma 1.2, $\rho(T_2(a,b)) < \rho(T_2(a+1,b-1))$. Then by induction, $\rho(T_2(a+1,b-1)) \leqslant \rho(T_2(m-4,0))$ with equality if and only if $b=1$. □

Now we investigate $H \in \mathbb{T}_3^m$. Let u,v,w be the three non-pendent vertices of H. We may discuss by the number of common edges u,v,w have. By Proposition 3.2, u,v,w share at most two common edges.

If u,v,w have only one common edge e_1, then there are three edges must contain two of them, and these three edges only contain two non-pendent vertices, otherwise there is a 4-cyclic sub-k-graph induced by three non-pendent edges in H, which contradicts Lemma 3.1. If $e_2 \bigcap e_3 \bigcap e_4 = \{u,v\}$, then $H \cong T_3^1(a,b,c)$. If $e_2 \bigcap e_3 = \{u,v\}$ and e_4 contains v,w, then $H \cong T_3^2(a,b,c)$. If e_2 contains u,v, e_3 contains v,w and e_4 contains u,w then $H \cong T_3^3(a,b,c)$.

If u,v,w have exactly two common edges e_1,e_2, then there are one edge must contain two of them, and these three edges only contain two non-pendent vertices, otherwise there is a 4-cyclic sub-k-graph induced by three non-pendent edges in H, which contradicts Lemma 3.1. If e_3 contains u,v, then $H \cong T_3^4(a,b,c)$.

Suppose that u,v,w do not have common edge. Since H is connected, there is a path connecting u,v,w, say ve_1ue_2w. As H is bicyclic, there are exactly three more non-pendent edges, say e_3,e_4 and e_5, that each contains two of u,v,w, and these three edges only contain two non-pendent vertices. Otherwise H is not tricyclic hypergraphs. If $e_3 \bigcap e_4$ is $\{u,v\}$ and e_5 contains v,w, then $H \cong T_3^5(a,b,c)$ for some a,b,c. If $e_3 \bigcap e_4$ is $\{u,v\}$ and e_5 contains u,w, then $H \cong T_3^6(a,b,c)$ for some a,b,c. If e_3 contains u,v, e_4 contains u,w and e_5 contains v,w, then $H \cong T_3^7(a,b,c)$ for some a,b,c. If all of e_3,e_4,e_5 contain u,v, and each of them don't have other non-pendent vertex, then $H \cong T_3^8(a,b,c)$ for some a,b,c.

Therefore, k-graphs in $\mathbb{T}_3^m$ have eight forms $T_3^j(a,b,c)$, $j = 1,\cdots,8$.

LEMMA 3.4. *Let H be a k-graph in $\mathbb{T}_3^m$. If $m \geqslant 6$, then $\rho(H) < \rho(T_2(m-4,0))$.*

Proof. We discuss by the formation of H.

Case 1. $H \cong T_3^2(a,b,c)$.

Note that $H \cong T_3^2(0,a+b+c,0)$ can be obtained from $H \cong T_3^2(a,b,c)$ by moving a pendent edges from u to v and moving c pendent edges from w to v, or by moving c pendent edges from w to v and moving a pendent edges from u to v. By Lemma 1.2, $\rho(T_3^2(a,b,c)) \leqslant \rho(T_3^2(0,a+b+c,0))$ with equality if and only if $a,c=0$. By Lemma 2.3,

$$\rho(H) \leqslant \rho(T_3^2(0,m-4,0)) < \rho(T_2(m-4,0)).$$

Case 2. $H \cong T_3^1(a,b,c)$ with $c \geqslant 1$.

Suppose that $a \geqslant b$ within this case. If $a \geqslant 1$, then by Lemma 1.2, take $u_1 = v$ and $u_2 = w$, we have for $b+c \geqslant 1$ that

$$\rho(H) < \rho(T_2(a,b+c)) \leqslant \rho(T_2(m-4,0))$$

Suppose that $a = b = 0$. By removing one pendent edge from w to u, we obtain $T_3^1(1,0,m-5)$. Besides, by removing a non-pendent edge not containing w from u to w, we obtain $T_3^2(0,0,m-4)$ from H. Then by Lemma 1.3 and the discussion in case 1,2,

$$\rho(H) < max\{\rho(T_3^1(1,0,m-5)),\rho(T_3^2(0,0,m-4))\} < \rho(T_2(m-4,0)).$$

Case 3. $H \cong T_3^3(a,b,c)$.

Note that $H \cong T_3^3(0,a+b+c,0)$ can be obtained from $H \cong T_3^3(a,b,c)$ by moving a pendent edges from u to v and moving c pendent edges from w to v, or by moving c pendent edges from w to v and moving a pendent edges from u to v. By Lemma 1.2, $\rho(T_3^3(a,b,c)) \leqslant \rho(T_3^3(0,a+b+c,0))$ with equality if and only if $a,c=0$. By Lemma 2.4,

$$\rho(H) \leqslant \rho(T_3^3(0,m-4,0)) < \rho(T_2(m-4,0)).$$

Case 4. $H \cong T_3^4(a,b,c)$.

Suppose that $b \geqslant a$ within this case. If $a=c=0$, then by Lemma 2.5, $\rho(H) = \rho(T_3^4(0,m-3,0)) < \rho(T_2(m-4,0))$. If $a \geqslant 1$, then by Lemma 1.2,

$$\rho(H) < \rho(T_3^4(0,a+b,c)) \leqslant \rho(T_3^4(0,a+b+c,0)) < \rho(T_2(m-4,0)).$$

Case 5. $H \cong T_3^j(a,b,c)$, $j=5,6,7,8$.

If $H \cong T_3^5(a,b,c)$.Then by moving c pendent edges and one edge containing u,w from w to an arbitrary pendent vertex in an edge containing u,v, we obtain $T_3^2(a+1,b,c)$. By Lemma 1.3 and the discussion in case 1,

$$\rho(H) < \rho(T_3^2(a+1,b,c)) < \rho(T_2(m-4,0)).$$

If $H \cong T_3^6(a,b,c)$.Then by moving c pendent edges and one edge containing u,w from w to an arbitrary pendent vertex in an edge containing u,v, we obtain $T_3^2(a,b+1,c)$. By Lemma 1.3 and the discussion in case 1,

$$\rho(H) < \rho(T_3^2(a,b+1,c)) < \rho(T_2(m-4,0)).$$

If $H \cong T_3^7(a,b,c)$.Then by moving c pendent edges and one edge containing v,w from w to an arbitrary pendent vertex in an edge containing u,v, we obtain $T_3^3(a+1,b,c)$. By Lemma 1.3 and the discussion in case 3,

$$\rho(H) < \rho(T_3^3(a+1,b,c)) < \rho(T_2(m-4,0)).$$

If $H \cong T_3^8(a,b,c)$.Then by moving c pendent edges and one edge containing v,w from w to an arbitrary pendent vertex in an edge containing u,v, we obtain $T_3^1(a+1,b,c)$. By Lemma 1.3 and the discussion in case 2,

$$\rho(H) < \rho(T_3^1(a+1,b,c)) < \rho(T_2(m-4,0)). \quad \square$$

LEMMA 3.5. *Let $i \geqslant 4$ and H be a k-graph in $\mathbb{T}_i^m \setminus \{F : F$ has the subgraph $T_4^1(a,b,c,d)\}$ with $m \geqslant 7$. Then $\rho(H) < \max\{\rho(F) : F$ is a k-graph in $\mathbb{T}_{i-1}^m \setminus \{F : F$ has the subgraph $T_4^1(a,b,c,d)\}\}$.*

Proof. If all non-pendent vertices in H are in one edge say f, then we can find two non-pendent vertices v_1, v_2 that do not have other common edge. Otherwise every two non-pendent vertices shares two common edges, then H contains a 4-cyclic subgraph, a contradiction. Denote by H' the k-graph obtained from H by moving all edges incident with v_2 except f from v_2 to v_1. It is obvious that $H' \in \mathbb{T}_{i-1}^m$. By Lemma 1.2, we have $\rho(H) < \rho(H')$.

Suppose there exists two non-pendent vertices v_1, v_2 in H that do not have common edge. Let $P = v_1 e_1 \cdots e_s v_2$ be a shortest path connecting v_1 and v_2 where $s \geqslant 2$. Denote H_1 be the k-graph obtained from H by moving all edges incident with v_1 except e_1 from v_1 to v_2, and denote H_2 be the k-graph obtained from H by moving all edges incident with v_2 except e_s from v_2 to v_1. Then H_1 and H_2 are in $\mathbb{T}_{i-1}^m$. By Lemma 1.1, we have $\rho(H) < \max\{\rho(H_1), \rho(H_2)\}$.

This completes the proof. □

Repeatedly by Lemma 3.4 and Lemma 3.5, we have the following result.

COROLLARY 3.6. *Let H be a k-graph in $\mathbb{T}_i^m \setminus \{F : F$ has the subgraph $T_4^1(a,b,c,d)\}$ with $m \geqslant 7$, if $i \geqslant 3$ and $m > 9$, then $\rho(H) < \rho(T_2(m-4,0))$.*

LEMMA 3.7. *Let $M = \{F : F$ has the subgraph $T_4^1(a,b,c,d)\}$, $N = \{T_4^1(m-2,0,0,0), T_4^1(m-3,1,0,0), T_4^1(m-4,2,0,0), T_4^1(m-4,1,1,0), T_5^2(m-4,0,0,0,1), T_5^2(m-5,0,0,0,2), T_5^2(m-5,1,0,0,1)\}$. If $H \in M \setminus N$, then $\rho(H) < \rho(T_5^2(m-5,1,0,0,1))$ for $m > 6$.*

Proof. For a k-graph $H \in M$, it can be obtained from $T_4^1(a,b,c,d)$ by attaching some k-trees at the vertices of $T_4^1(a,b,c,d)$. Let the non-pendent vertices of $T_4^1(a,b,c,d)$ be v_1, v_2, v_3, v_4, and v_5 be a pendent vertex of $T_4^1(a,b,c,d)$ on a cycle edge, and v_6 be a pendent vertex of $T_4^1(a,b,c,d)$ on a pendent edge which is adjacent to v_1. Repeatedly by moving edge operation, we can attain a k-graph H_1 which is obtained from $T_4^1(a,b,c,d)$ by attaching some k-stars at the vertices $v_1, v_2, v_3, v_4, v_5, v_6$, respectively. By Lemma 1.2, we have $\rho(H) \leqslant \rho(H_1)$. The equality holds if and only if $H \cong H_1$. Let $l_1, l_2, l_3, l_4, l_5, l_6$ be the number of pendent edges at $v_1, v_2, v_3, v_4, v_5, v_6$, respectively.

Case 1. If $l_1, l_2, l_3, l_4, l_5, l_6 \geqslant 1$, we can obtain a k-graph $T_5^2(m-5,1,0,0,1)$ by the moving edge operation, we have $\rho(H_1) < \rho(T_5^2(m-5,1,0,0,1))$.

Case 2. $l_5 = 0$, $l_6 \neq 0$.

Subcase 2.1. If $l_1, l_2, l_3, l_4, l_6 \geqslant 1$, $l_5 = 0$, we can obtain a k-graph $T_5^2(m-5,1,0,0,1)$ and $\rho(H_1) \leqslant \rho(T_5^2(m-5,1,0,0,1))$. The equality holds if and only if $H_1 \cong T_5^2(m-5,1,0,0,1)$.

Subcase 2.2. If $l_1 \geqslant 1$, $l_6 \geqslant 2$, $l_2 = l_3 = l_4 = l_5 = 0$, we can obtain a k-graph $T_5^2(m-5,0,0,0,2)$ and $\rho(H_1) \leqslant \rho(T_5^2(m-5,0,0,0,2))$. The equality holds if and only if $H_1 \cong T_5^2(m-5,0,0,0,2)$.

Subcase 2.3. If $l_1 \geqslant 1$, $l_6 = 1$, $l_2 = l_3 = l_4 = l_5 = 0$, we can obtain a k-graph $T_5^2(m-4,0,0,0,1)$ and $\rho(H_1) \leqslant \rho(T_5^2(m-4,0,0,0,1))$. The equality holds if and only if $H_1 \cong T_5^2(m-4,0,0,0,1)$.

Case 3. If $l_5 \neq 0$, $l_6 = 0$, we can obtain a k-graph $T_5^1(m-3,0,0,0,1)$. By Lemma 1.2 and Lemma 2.14, $\rho(H_1) \leqslant \rho(T_5^1(m-3,0,0,0,1)) < \rho(T_2(m-4,0))$. The equality holds if and only if $H_1 \cong T_5^1(m-3,0,0,0,1)$.

Case 4. $l_5 = l_6 = 0$. We can find a k-graph $H_2 \in \{T_4^1(m-2,0,0,0), T_4^1(m-3,1,0,0), T_4^1(m-4,2,0,0), T_4^1(m-4,1,1,0)\}$ such that $\rho(H_1) < \rho(H_2)$.

By corollary 2.9, we have our desirable results. □

By corollaries 2.9 and 3.6 and Lemma 3.7, we have our main results:

THEOREM 3.8. *Let H be a k-graph in $\mathbb{T}^m$ with $m > 20$. Then*

(i) $\rho(T_2(m-4,0)) < \rho(T_5^2(m-5,1,0,0,1)) < \rho(T_5^2(m-5,0,0,0,2)) < \rho(T_4^1(m-4,1,1,0)) < \rho(T_4^1(m-4,2,0,0)) < \rho(T_5^2(m-4,0,0,0,1)) < \rho(T_4^1(m-3,1,0,0)) < \rho(T_4^1(m-2,0,0,0))$;

(ii) *If* $H \notin \{T_4^1(m-2,0,0,0), T_4^1(m-3,1,0,0), T_4^1(m-4,2,0,0), T_4^1(m-4,1,1,0), T_5^2(m-4,0,0,0,1), T_5^2(m-5,0,0,0,2), T_5^2(m-5,1,0,0,1)\}$, *then* $\rho(H) < \rho(T_5^2(m-5,1,0,0,1))$.

4. Concluding remarks

The concept of tensor is a natural extension of matrix. It is similar to the relationship between matrix and graph, there is one-to-one correspondence between tensor and hypergraph. It is a interesting problem to characterize extremal structure of graph and hypergraph with respect to their spectral radius, respectively. In this paper, the extremal hypergraphs with the first seven largest spectral radius in $\mathbb{T}^m$ for $m > 20$ are characterized. It is natural that we can continue to consider extremal hypergraphs in $\mathbb{T}^m$ given some parameters, such as diameter, matching number, independent number, and so on.

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