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ON NUMBER THEORETIC PROPERTIES OF THE KDV FREQUENCIES

THOMAS KAPPELER AND JÜRG KRAMER*

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Abstract. In this paper we investigate some number theoretic properties of the frequencies of the Korteweg–de Vries equation on the torus, relevant for the stability of finite gap solutions.

1. Introduction

The Korteweg–de Vries (KdV) equation

$$\partial_t u = -\partial_x^3 u + 6u\partial_x u, \qquad u(t,x) \in \mathbb{R}, \quad t,x \in \mathbb{R}, \tag{1.1}$$

is a widely used model equation for describing dispersive phenomena. It is named after the two Dutch mathematician Korteweg and de Vries [29] (see also Boussinesq [12], Raleigh [38]). The pioneering numerical experiments by Kruskal and Zabusky (see [31]) on special solutions of (1.1), referred to as solitons, the seminal discovery by Gardner, Greene, Kruskal, and Miura that (1.1) admits infinitely many conservation laws ([23], [36]), and the invention of the concept of what nowadays is referred to as a Lax pair representation of evolution equations such as (1.1) by Lax [35] led to the modern theory of integrable systems of finite and infinite dimension (see, e.g., [17], [20], and references therein). As one of the most prominent examples among dispersive equations, equation (1.1) has been extensively studied and played a major role in the development of the theory of dispersive PDEs to which many of the leading analysts of our times contributed. In particular, the (globally in time) well-posedness of (1.1) has been established in various setups in great detail – see [15].

In the sequel we consider (1.1) on the torus $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. We record that for any integer $s \geqslant 0$, equation (1.1) is globally (in time) C^0-well-posed in the Sobolev space $H^s \equiv H^s(\mathbb{T},\mathbb{R})$, consisting of functions q, whose Fourier expansions $q(x) = \sum_{n\in\mathbb{Z}} \widehat{q}(n)e^{inx}$ satisfy

$$\widehat{q}_{-n} = \overline{\widehat{q}(n)}, \qquad \forall n \in \mathbb{Z}, \qquad \|q\|_s := \left(\sum_{n\in\mathbb{Z}} \langle n\rangle^{2s} |\widehat{q}(n)|^2\right)^{1/2} < \infty,$$

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* Corresponding author.

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where $\langle n\rangle := \max\{1,|n|\}$. A distinguished feature of equation (1.1) is that all its solutions in H^s with $s \geqslant 0$, are almost periodic (in time) and that they can be approximated by quasi-periodic (in time) solutions, referred to as finite gap solutions, which densely fill finite dimensional invariant tori (see, e.g., [27]). Due to the importance of finite gap solutions, their stability, in particular their structural stability, is of great interest. It encompasses two major issues:

(1) The persistence of quasi-periodic solutions (and of the finite dimensional invariant tori on which they evolve) under (small) perturbations of (1.1).

(2) The long time asymptotics of solutions of perturbations of (1.1) with initial data close to the orbit of a finite gap solution and hence close to the corresponding finite dimensional invariant torus.

In the last thirty years, the persistence of quasi-periodic solutions of integrable PDEs such as (1.1) has been extensively studied. KAM type methods, which were pioneered by Kolmogorov, Arnold, and Moser to treat perturbations of finite dimensional integrable system, were developed for (Hamiltonian) PDEs which allowed to prove that a large portion of these quasi-periodic solutions persist – see [32], [33], [34], [39], [10], [27], [1], [8], and references therein.

Concerning item (2), for Hamiltonian perturbations of linear integrable PDEs on $\mathbb{T}$, which satisfy non-resonance conditions, a normal form method has been developed allowing to prove the stability of the *equilibrium solution* $u \equiv 0$ of (Hamiltonian) perturbations for large time intervals – see, e.g., [2], [3], [4], [7], [11], [13], [14], [22], and references therein. More recently, these techniques have been refined so that in specific cases, such results can also be proved for Hamiltonian perturbations of resonant linear integrable PDEs by approximating the perturbed equation by nonlinear integrable systems, satisfying non-resonance conditions – see [11], [5], [6]. In contrast, first results on the long time asymptotics of solutions of perturbations of integrable PDEs such as (1.1) with possibly *large* initial data close to an invariant finite dimensional torus were obtained only very recently. In [26], such results are obtained for the KdV equation.

It turns out that the time of stability of solutions of perturbations of (1.1) with initial data close to finite gap solutions of (1.1) is closely related to resonances or almost resonances of the KdV frequencies. The goal of this paper is to discuss number theoretic properties of these frequencies which are relevant for the time of stability of the solutions mentioned above.

2. Stability of finite gap solutions of the KdV equation

In this section we describe the results in [26] on the stability of solutions of perturbations of (1.1) with initial data close to finite gap solutions of (1.1) in more detail and then state the main result of this paper, which concerns number theoretic properties of the KdV frequencies.

We begin with some preliminary considerations. It is well known that (1.1) is a Hamiltonian PDE with Poisson structure ∂_x,

$$\partial_t u = \partial_x \nabla H, \qquad H(u) = \int_0^1 \left(\frac{1}{2}(\partial_x u)^2 + u^3 \right) dx, \tag{2.1}$$

where ∇H denotes the L^2-gradient of H. We consider semilinear Hamiltonian perturbations of the form

$$\partial_t u = \partial_x \nabla H + \varepsilon \partial_x \nabla F, \qquad F(u) = \int_0^1 f(x, u(x))\, dx, \tag{2.2}$$

where ε is a small perturbation parameter and the density $f\colon \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ of F is C^∞-smooth and might depend explicitly on x. Note that $\widehat{u}(t,0) = \int_0^1 u(t,x)\, dx$ (mass) of any solution of (2.1) or of (2.2) is conserved and hence the subspace

$$H_0^s := \left\{ q \in H^s \,\middle|\, \widehat{q}(0) = \int_0^1 q(x)\, dx = 0 \right\}$$

of H^s is left invariant by (2.1) as well as by (2.2) (see [27], Section 13). It means that for any initial data $u_0 \in H_0^s$ with $s \geqslant 0$, the solution u of (2.1) in H_0^s with initial data u_0 evolves in H_0^s as

$$u\colon \mathbb{R} \longrightarrow H_0^s,$$

given by the assignment $t \mapsto u(t) = u(t,\cdot)$ with initial condition $u(0) = u_0$. To simplify the exposition, we choose H_0^s as our phase space. The corresponding space of Fourier coefficients is denoted by h_0^s, i.e.,

$$h_0^s := \left\{ (w_n)_{n \in \mathbb{Z}_0} \in h_{0,c}^s \,\middle|\, w_{-n} = \overline{w}_n,\ \forall n \geqslant 1 \right\},$$

where

$$\mathbb{Z}_0 := \mathbb{N} \cup (-\mathbb{N}) \quad \text{with} \quad \mathbb{N} := \{1, 2, \dots\}$$

and $h_{0,c}^s \equiv h^s(\mathbb{Z}_0, \mathbb{C})$ is defined to be the sequence space

$$h_{0,c}^s := \left\{ w = (w_n)_{n \in \mathbb{Z}_0} \,\middle|\, w_n \in \mathbb{C},\ \forall n \neq 0,\ \|w\|_s < \infty \right\}, \quad \|w\|_s := \left(\sum_{n \neq 0} |n|^{2s} |w_n|^2 \right)^{1/2}.$$

It turns out that (2.1) admits a nonlinear Fourier transform. It is described by the following theorem in a somewhat informal way. For a precise statement and its proof see [27].

THEOREM 2.1. ([27]) *There exists a map*

$$\Phi\colon L_0^2 \equiv H_0^0 \longrightarrow \ell_0^2 \equiv h_0^0, \qquad q \mapsto w(q) := (w_n(q))_{n \in \mathbb{Z}_0},$$

so that $w_n(q)$, $n \neq 0$, *are nonlinear Fourier coefficients for* (2.1) *with*

$$I_n(q) := \frac{1}{2\pi n} w_n(q) w_{-n}(q) \geqslant 0, \qquad \forall n \geqslant 1,$$

being action variables and hence prime integrals of (2.1). *More precisely, the following holds:*

(1) For any integer $s \geqslant 0$, *the map* $\Phi|_{H_0^s} \colon H_0^s \longrightarrow h_0^s$ *is a real analytic diffeomorphism.*

(2) $H \circ \Phi^{-1}$ *is a real analytic functional* $\mathscr{H}$ *of the actions* $I := (I_n)_{n \in \mathbb{N}}$ *alone.*

(3) Equation (2.1), *when expressed in the nonlinear Fourier coefficients, takes the form*

$$\partial_t w_n(t) = i\,\omega_n w_n(t), \qquad \forall n \in \mathbb{Z}_0, \tag{2.3}$$

where $\omega_n \equiv \omega_n(I)$, $n \neq 0$, *denote the KdV frequencies,*

$$\omega_n(I) := \partial_{I_n} \mathscr{H}(I), \qquad \omega_{-n}(I) := -\omega_n(I), \qquad \forall n \in \mathbb{N}. \tag{2.4}$$

Since the actions I_n, $n \geqslant 1$, *are prime integrals of* (2.1), *so are the frequencies. As a consequence,* (2.3) *can be solved by quadrature,*

$$w_n(t) = w_n(0) e^{i\omega_n t}, \qquad \forall t \in \mathbb{R}, \forall n \in \mathbb{Z}_0.$$

We are now in a position to introduce in more precise terms the notion of finite gap solutions of (2.1) and the invariant tori, on which they evolve. For any *finite* subset $S_+ \subseteq \mathbb{N} := \{1, 2, \ldots\}$, let

$$S := S_+ \cup (-S_+), \qquad S^\perp := \mathbb{Z}_0 \setminus S.$$

DEFINITION 2.1. An element $q \in L_0^2$ is said to be an S-gap potential if

$$w_n(q) \neq 0, \quad \forall n \in S, \qquad w_n(q) = 0, \quad \forall n \in S^\perp.$$

We denote by M_S the set of all S-gap potentials of L_0^2. By Theorem 2.1, M_S is contained in $\bigcap_{s \geqslant 0} H_0^s$ and invariant under the flow of (2.1). We say that $u(t,x)$ is a finite gap solution of (2.1) if $u(t,x)$ is in M_S for some S with $S_+ \subset \mathbb{N}$ finite.

Due to the presence of small divisors, the stability result in [26] imposes non-resonance conditions on the KdV frequencies (2.4). To describe them, let us introduce for any given finite subset S_+ of $\mathbb{N}$ the action to frequency map,

$$\omega \colon \mathbb{R}^{S_+}_{>0} \longrightarrow \mathbb{R}^{S_+}, \qquad I_S = (I_n)_{n \in S_+} \mapsto (\omega_n(I_S, 0))_{n \in S_+} \in \mathbb{R}^{S_+}, \tag{2.5}$$

where for notational convenience, we write $(I_S, 0)$ for the sequence of actions $(I_n)_{n \geqslant 1}$ with $I_n = 0$ for any $n \in \mathbb{N} \setminus S_+$. We remark that the action to frequency map ω is real analytic. The following lemma is due to Krichever and its proof has been worked out in [9].

LEMMA 2.1. *For any finite subset $S_+ \subset \mathbb{N}$, the map $\omega\colon \mathbb{R}^{S_+}_{>0} \to \mathbb{R}^{S_+}$ is a local diffeomorphism.*

It turns out to be convenient to locally parametrize the invariant tori of S-gap potentials by ω. More precisely, let $\Xi \subset \mathbb{R}^{S_+}_{>0}$ be a closed ball so that $\omega\colon \Xi \to \Pi := \omega(\Xi)$ is a diffeomorphism onto Π. Denote by μ its inverse,

$$\mu : \Pi \longrightarrow \Xi, \qquad \omega \mapsto \mu(\omega),$$

and define $\mathfrak{T}_{\mu(\omega)}$ to be the torus of S-gap potentials with actions $\mu_n(\omega)$ with $n \in S_+$, given by

$$\mathfrak{T}_{\mu(\omega)} := \Phi^{-1}\big(\big\{\big((w_n)_{n\in S}, 0\big) \in h^0_0 \,\big|\, |w_n|^2 = 2\pi n \mu_n(\omega), \forall n \in S_+\big\}\big).$$

Note that the torus $\mathfrak{T}_{\mu(\omega)}$ has dimension $|S_+|$, is invariant under (2.1), and is Lyapunov stable in H^s_0 for any $s \geqslant 0$, meaning that for any $\varepsilon > 0$ there exists $\delta > 0$, depending on s, so that for any initial data $u_0 \in H^s_0$ with

$$\mathrm{dist}_{H^s}\big(u_0, \mathfrak{T}_{\mu(\omega)}\big) \leqslant \delta, \qquad \mathrm{dist}_{H^s}\big(u_0, \mathfrak{T}_{\mu(\omega)}\big) := \inf_{q \in \mathfrak{T}_{\mu(\omega)}} \|u_0 - q\|_s, \tag{2.6}$$

the solution $u(t,\cdot)$ of (2.1) with $u(0,\cdot) = u_0$ satisfies

$$\mathrm{dist}_{H^s}\big(u(t,\cdot), \mathfrak{T}_{\mu(\omega)}\big) \leqslant \varepsilon, \qquad \forall t \in \mathbb{R}.$$

Finally, we introduce the so called normal frequencies,

$$\Omega_j(\omega) := \omega_j(\mu(\omega), 0), \qquad \forall j \in S^\perp, \forall \omega \in \Pi, \tag{2.7}$$

which have the following properties:

LEMMA 2.2. ([25, Lemma C.7]) *For any $\omega \in \Pi$, the normal frequencies admit an asymptotic expansion of the form*

$$\Omega_j(\omega) = (2\pi j)^3 + \alpha \frac{1}{j} + O\left(\frac{1}{j^3}\right), \qquad \text{as } j \to \pm\infty, \tag{2.8}$$

where the error term is uniform in ω and real analytic on Π. The coefficient $\alpha\colon \Pi \to \mathbb{R}$ is real analytic and conserved by the flow of (2.1).

REMARK 2.1. It can be shown that the coefficient α in the asymptotic expansion (2.8) does not vanish identically. See Appendix A for a proof.

Note that $(2\pi j)^3$ with $j \in \mathbb{Z}$, are the frequencies of the Airy equation, $\partial_t v = -\partial_x^3 v$, which can be viewed as the linearization of (1.1) at the stationary solution $u \equiv 0$. For the following definition of non-resonance conditions, it is convenient to define for any vector $\ell = (\ell_n)_{n\in S_+}$ in $\mathbb{Z}^{S_+}$,

$$\langle \ell \rangle := \max\Big\{1, \Big(\sum_{n\in S_+} |\ell_n|^2\Big)^{1/2}\Big\}.$$

DEFINITION 2.2. (Non-resonance conditions) For any $0<\gamma<1$ and for any $\tau>|S_+|$, introduce the following subsets $\Pi_\gamma^{(i)} \equiv \Pi_{\gamma,\tau}^{(i)}$, $0 \leqslant i \leqslant 3$, of Π,

$$
\begin{aligned}
\Pi_\gamma^{(0)} &:= \Big\{\omega \in \Pi \,\Big|\, |\omega\cdot\ell| \geqslant \frac{\gamma}{\langle\ell\rangle^\tau}, \forall \ell \in \mathbb{Z}^{S_+}\setminus\{0\}\Big\},\\
\Pi_\gamma^{(1)} &:= \Big\{\omega \in \Pi \,\Big|\, |\omega\cdot\ell+\Omega_j(\omega)| \geqslant \frac{\gamma}{\langle\ell\rangle^\tau}, \forall (\ell,j) \in \mathbb{Z}^{S_+}\times S^\perp\Big\},\\
\Pi_\gamma^{(2)} &:= \Big\{\omega \in \Pi \,\Big|\, |\omega\cdot\ell+\Omega_{j_1}(\omega)+\Omega_{j_2}(\omega)| \geqslant \frac{\gamma}{\langle\ell\rangle^\tau},\\
&\qquad \forall (\ell,j_1,j_2) \in \mathbb{Z}^{S_+}\times S^\perp\times S^\perp \text{ with } (\ell,j_1,j_2)\neq(0,j_1,-j_1)\Big\},\\
\Pi_\gamma^{(3)} &:= \Big\{\omega \in \Pi \,\Big|\, |\omega\cdot\ell+\Omega_{j_1}(\omega)+\Omega_{j_2}(\omega)+\Omega_{j_3}(\omega)| \geqslant \frac{\gamma}{\langle\ell\rangle^\tau (j_1j_2j_3)^2},\\
&\qquad \forall (\ell,j_1,j_2,j_3) \in \mathbb{Z}^{S_+}\times S^\perp\times S^\perp\times S^\perp \text{ with } j_k+j_l\neq 0, \forall 1\leqslant k,l\leqslant 3\Big\}.
\end{aligned}
\tag{2.9}
$$

We refer to $\Pi_\gamma^{(i)}$, $0 \leqslant i \leqslant 3$, as the *$i$-th Melnikov conditions*.

Note that the term $1/(j_1j_2j_3)^2$ in the third Melnikov conditions can be viewed as a loss of derivatives in space. Such a loss needs to be admitted in order to prove the following measure estimate.

PROPOSITION 2.1. ([26, Proposition 8.1]) *For any $\tau>|S_+|$, we have*

$$\lim_{\gamma\to 0} meas(\Pi\setminus\Pi_\gamma^{(i)}) = 0, \qquad \forall 0\leqslant i\leqslant 3.$$

REMARK 2.2. To prove that $meas(\Pi\setminus\Pi_\gamma^{(3)})$ converges to 0 as $\gamma\to 0$, one uses that by Fermat's last theorem for the special case of cubic powers, proved by Euler [19], one has

$$\Big|\sum_{k=1}^{3} j_k^3\Big| \geqslant 1, \qquad \forall (j_1,j_2,j_3)\in\mathbb{Z}^3 \quad \text{with} \quad j_k+j_l\neq 0, \quad \forall 1\leqslant k,l\leqslant 3. \tag{2.10}$$

We refer to Section 4 for a general discussion on cubic diophantine equations, relevant in the context of the KdV frequencies.

To state the main result of [26], we need to introduce one additional notation. Let X be a Banach space with norm $\|\cdot\|_X$, $k\geqslant 0$ an integer, and $J\subset\mathbb{R}$ an interval. We then denote by $C^k(J,X)$ the Banach space of k times continuously differentiable functions $f\colon J\to X$, endowed with the supremum norm,

$$\|f\|_{C^k} := \sup\big\{\|\partial_t^j f(t)\|_X \,\big|\, t\in J, 0\leqslant j\leqslant k\big\}.$$

THEOREM 2.2. ([26, Theorem 1.1]) *Let $f\colon\mathbb{T}\to\mathbb{R}$ be C^∞-smooth, S_+ a finite subset of $\mathbb{N}$, and $\tau>|S_+|$. Then for any integer s sufficiently large and any $0<\gamma<1$,* *there exists* $0 < \varepsilon_0 \equiv \varepsilon_0(s,\gamma) < 1$ *with the following properties: for any* $0 < \varepsilon \leqslant \varepsilon_0$, *any* $\omega \in \bigcap_{0 \leqslant i \leqslant 3} \Pi_\gamma^{(i)}$, *and any initial data* $u_0 \in H_0^s$, *satisfying*

$$\mathrm{dist}_{H^s}\big(u_0, \mathfrak{T}_{\mu(\omega)}\big) \leqslant \varepsilon, \tag{2.11}$$

equation (2.2) *admits a unique solution* $t \mapsto u(t,\cdot)$ *in* $C^0([-T,T],H_0^s) \cap C^1([-T,T],H_0^{s-3})$ *with initial data* $u(0,x) = u_0(x)$ *and* $T = K_{s,\gamma}\varepsilon^{-2}$. *Moreover,* u *satisfies the estimate*

$$\mathrm{dist}_{H^s}\big(u(t,\cdot), \mathfrak{T}_{\mu(\omega)}\big) \leqslant M_{s,\gamma}\varepsilon, \qquad \forall -T \leqslant t \leqslant T,$$

where the distance function dist_{H^s} *is defined in* (2.6). *For notational convenience, the dependence of the constants* $K_{s,\gamma} > 0$ *and* $M_{s,\gamma} > 0$ *on* f, S_+, *and* τ *is not indicated.*

In informal terms, Theorem 2.2 can be stated as follows: For any smooth density $f\colon \mathbb{T}\times\mathbb{R} \to \mathbb{R}$, s sufficiently large, $\varepsilon > 0$ sufficiently small, and for most of the finite gap solutions $q\colon t \mapsto q(t,\cdot)$ of (1.1), the following holds: for any initial data $u_0 \in H_0^s$, which is ε-close in H_0^s to the orbit $\mathscr{O}_q := \{q(t,\cdot) \,|\, t \in \mathbb{R}\}$ of q, the perturbed equation (2.1) admits a unique solution $t \mapsto u(t,\cdot)$ in H_0^s with initial data $u(0,\cdot) = u_0$ and life span at least $[-T,T]$ with $T = O(\varepsilon^{-2})$. The solution $u(t,\cdot)$ stays ε-close in H_0^s to the orbit $\mathscr{O}_q$.

The proof of Theorem 2.2 is based on a normal form procedure and a refined nonlinear Fourier transform, which admits an expansion in terms of pseudodifferential operators [25]. This method is quite general and it is to be expected that for any integrable PDE, admitting coordinates of the type constructed in [25], a corresponding version of Theorem 2.2 holds, up to the measure estimates of Proposition 2.1 related to the non-resonance conditions for the frequencies of the integrable PDE considered. These estimates might require specific arithmetic properties of the frequencies. For further comments on Theorem 2.2, we refer the reader to [26].

It is an open question whether the time interval $[-T,T]$ of stability of the solutions of the perturbed KdV equation (2.2), considered in Theorem 2.2, can be proved to be larger. A first result of this paper concerns number theoretic properties of the KdV frequencies, which would be needed to prove by the method of normal forms that the length of this time interval is indeed larger – see Theorem 2.3 in paragraph (A) below.

In the case where the perturbation Hamiltonian in (2.2) is of the form $F(u) = \int_0^1 f(u(x))\,dx$, some of the resonances of the KdV frequencies can be ignored and it is expected that one can prove a result, similar to the one stated in Theorem 2.2, but with a longer time interval of stability. Note that such perturbations are invariant under translation, i.e., for any $\tau \in \mathbb{R}$, we have $F(u_\tau) = F(u)$, where u_τ denotes the translate of u by τ and $u_\tau(\cdot) = u(\cdot + \tau)$. In paragraph (B) below we discuss number theoretic properties of the KdV frequencies, which would be needed for proving such a result for F of the form $F(u) = \int_0^1 f(u(x))\,dx$.

(A) To show by a normal form procedure that the size of T in Theorem 2.2 is at least of the order of $O(\varepsilon^{-3})$ requires to impose in addition the 4th Melnikov conditions on the frequencies $\omega \in \Pi$ considered, i.e., for any $\ell \in \mathbb{Z}^{S_+}$ and any $(j_k)_{1 \leqslant k \leqslant 4} \in (S^\perp)^4$

satisfying $j_k + j_l \neq 0$ with $1 \leqslant k,l \leqslant 4$,

$$\left|\omega \cdot \ell + \sum_{k=1}^{4} \Omega_{j_k}(\omega)\right| \geqslant \frac{\gamma}{\langle \ell \rangle^{\tau} (\prod_{k=1}^{4} j_k)^2}.$$

We denote by $\Pi_\gamma^{(4)}$ the frequencies $\omega \in \Pi$, satisfying the 4th Melnikov conditions. Following the line of arguments above (see Proposition 2.1), one needs to prove (among other results) that $\lim_{\gamma \to 0} meas(\Pi \setminus \Pi_\gamma^{(4)}) = 0$. A first difficulty in proving such a result arises due the fact that the analogue of equation (2.10) no longer holds, i.e., there exist integer vectors $(j_k)_{1 \leqslant k \leqslant 4} \in \mathbb{Z}_0^4$ with $j_1 \leqslant j_2 \leqslant j_3 \leqslant j_4$ so that

$$\sum_{k=1}^{4} j_k^3 = 0, \qquad j_k + j_l \neq 0, \qquad \forall 1 \leqslant k,l \leqslant 4. \tag{2.12}$$

Well known solutions of (2.12) are (see [18])

$$(-3,-4,-5,6), \qquad (-10,-9,1,12), \qquad (-9,1,6,8), \tag{2.13}$$

and their non-zero integer multiples. In particular, it follows that (2.12) has infinitely many solutions. In fact, according to [18], there are many more solutions of (2.12). Closely related to (2.12) is the Fermat cubic, defined in $\mathbb{P}^3(\mathbb{C})$ by the equation $x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$, which has been extensively studied in algebraic geometry. We refer to Section 4 for a further discussion of (2.12).

To overcome the difficulty caused by the infinitely many solutions of (2.12), one can try to use the asymptotics of the normal frequencies, stated in Proposition 2.2. The following theorem might suffice for proving that $\lim_{\gamma \to 0} meas(\Pi \setminus \Pi_\gamma^{(4)}) = 0$.

THEOREM 2.3. *The system of equations*

$$\sum_{k=1}^{4} j_k^3 = 0, \qquad \sum_{k=1}^{4} \frac{1}{j_k} = 0, \tag{2.14}$$

has no solutions $(j_k)_{1 \leqslant k \leqslant 4} \in \mathbb{Z}_0^4$ *satisfying*

$$j_k + j_l \neq 0, \qquad \forall 1 \leqslant k,l \leqslant 4. \tag{2.15}$$

Proof. The proof will be given in Section 3. □

(B) If the density f of the perturbation $F(u) = \int_0^1 f(x,u(x))\,dx$ (cf. (2.2)) does not explicitly depend on x, i.e., $F(u) = \int_0^1 f(u(x))\,dx$, then the momentum $M(u) = \int_0^1 u(x)^2\,dx$, which is conserved by (2.1), is also a prime integral of equation (2.2). In this case one expects that the length of the time interval $[-T,T]$ of stability in Theorem 2.2 is at least of the order of ε^{-3}. One of the ingredients for proving such a result is that the system of equations

$$\sum_{k=1}^{4} j_k^3 = 0, \qquad \sum_{k=1}^{4} j_k = 0, \tag{2.16}$$

has no solutions $(j_k)_{1\leqslant k\leqslant 4} \in \mathbb{Z}_0^4$ satisfying

$$j_k + j_l \neq 0, \qquad \forall 1 \leqslant k,l \leqslant 4. \tag{2.17}$$

This can be established by elementary means. Indeed, by substituting $j_4 = -j_1 - j_2 - j_3$ into the first equation of (2.16), one gets

$$j_1^3 + j_2^3 + j_3^3 - (j_1 + j_2 + j_3)^3 = -3(j_1 + j_2)(j_1 + j_3)(j_2 + j_3),$$

which shows that (2.16) has no integer solutions satisfying (2.17).

Going one step further, one might try to prove that the time interval $[-T,T]$ of stability of the solutions of (2.2) in Theorem 2.2 for perturbations of the form $F(u) = \int_0^1 f(u(x))\,dx$ is at least of the order of ε^{-4}. Similarly as in item (A) above, a first difficulty in proving such a result arises from the fact that the system of equations

$$\sum_{k=1}^{5} j_k^3 = 0, \qquad \sum_{k=1}^{5} j_k = 0, \tag{2.18}$$

has infinitely many solutions $(j_k)_{1\leqslant k\leqslant 5} \in \mathbb{Z}_0^5$ satisfying

$$j_k + j_l \neq 0, \qquad \forall 1 \leqslant k,l \leqslant 5. \tag{2.19}$$

The infinitude of the set of integral solutions of the system of equations (2.18) satisfying (2.19) will be discussed in Section 4.

In analogy to Theorem 2.3, one might try to overcome the difficulty for proving the corresponding measure estimate, caused by the infinitely many integral solutions of the system of equations (2.18) satisfying (2.19), in case the following question has an affirmative answer.

QUESTION 2.1. Does the system of equations

$$\sum_{k=1}^{5} j_k^3 = 0, \qquad \sum_{k=1}^{5} j_k = 0, \qquad \sum_{k=1}^{5} \frac{1}{j_k} = 0, \tag{2.20}$$

subject to the constraints

$$j_k + j_l \neq 0, \qquad \forall 1 \leqslant k,l \leqslant 5, \tag{2.21}$$

have no solutions $(j_k)_{1\leqslant k\leqslant 5} \in \mathbb{Z}_0^5$?

The likeliness for an affirmative answer of Question 2.1 will be discussed in Section 3 below.

3. Proofs and discussions

In this section, we prove Theorem 2.3 and then discuss Question 2.1, both stated in Section 2. We also outline some general results from number theory which might be useful for establishing results of the type of Theorem 2.3.

Proof of Theorem 2.3. Let us start by pointing out that in the end, the proof of Theorem 2.3 will be reduced to an elementary problem, which can be solved quite easily. In order to arrive at this reduction, we reformulate the claim of Theorem 2.3 as follows.

Let us introduce the 3-dimensional complex projective space $\mathbb{P}^3(\mathbb{C})$ with the homogenous coordinates $(x_1 : x_2 : x_3 : x_4)$. Then, the cubic equation

$$\sum_{k=1}^{4} x_k^3 = 0$$

defines a cubic surface S_1 in $\mathbb{P}^3(\mathbb{C})$. Similarly, the fractional equation

$$\sum_{k=1}^{4} \frac{1}{x_k} = 0$$

defines also a cubic surface S_2 in $\mathbb{P}^3(\mathbb{C})$. Therefore, the system of equations

$$\sum_{k=1}^{4} x_k^3 = 0, \qquad \sum_{k=1}^{4} \frac{1}{x_k} = 0 \tag{3.1}$$

describes the intersection $S_1 \cap S_2$ of the two cubic surfaces S_1 and S_2, which is an algebraic curve C of degree 9 embedded in $\mathbb{P}^3(\mathbb{C})$. Since it is easily checked that the set of non-zero integral solutions of the system of equations (3.1) equals the set of non-zero rational solutions of the system of equations (3.1), the search of solutions $(j_k)_{1\leqslant k\leqslant 4} \in \mathbb{Z}_0^4$ of the system of equations (2.14) amounts to the search of points on the curve C having non-zero rational coordinates. Thus, it will be useful to compute the polynomial equation describing the curve C explicitly, which will be done below by eliminating the variable x_4.

Before doing so, it is helpful to also interpret the constraints (2.15), which the solutions $(j_k)_{1\leqslant k\leqslant 4} \in \mathbb{Z}_0^4$ of (2.14) have to satisfy, in the present algebraic geometric context. For this, we observe that the system of equations (3.1) admits solutions satisfying the linear relations

$$x_k + x_l = 0, \qquad \forall\, 1 \leqslant k,l \leqslant 4, \tag{3.2}$$

which we call trivial solutions. Therefore, in order to prove Theorem 2.3, we have to show that the curve C contains only points having rational coordinates of which at least one is zero.

As promised, we are now going to compute the polynomial equation describing the curve C in $\mathbb{P}^2(\mathbb{C})$ with the homogeneous coordinates $(x_1 : x_2 : x_3)$ by eliminating the variable x_4. For this we use the second equation in (3.1) to obtain

$$\begin{aligned}\frac{1}{x_4} &= -\frac{1}{x_1} - \frac{1}{x_2} - \frac{1}{x_3}\\ &= -\frac{x_1x_2 + x_1x_3 + x_2x_3}{x_1x_2x_3},\end{aligned}$$

which we rewrite as

$$x_4 = -\frac{x_1x_2x_3}{x_1x_2 + x_1x_3 + x_2x_3}. \tag{3.3}$$

Substituting (3.3) into the first equation of (3.1) and multiplying by $(x_1x_2 + x_1x_3 + x_2x_3)^3$, yields

$$(x_1x_2 + x_1x_3 + x_2x_3)^3(x_1^3 + x_2^3 + x_3^3) - (x_1x_2x_3)^3 = 0, \tag{3.4}$$

which is the polynomial equation describing the curve C in $\mathbb{P}^2(\mathbb{C})$.

We note that a rational solution $(x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^3(\mathbb{Q})$ of the system of equations (3.1) with $x_1x_2x_3x_4 \neq 0$ gives rise to a solution $(x_1 : x_2 : x_3) \in \mathbb{P}^3(\mathbb{Q})$ of (3.4) with $x_1x_2x_3 \neq 0$; conversely, a solution $(x_1 : x_2 : x_3) \in \mathbb{P}^3(\mathbb{Q})$ of (3.4) with $x_1x_2x_3 \neq 0$ gives rise to a solution $(x_1 : x_2 : x_3 : x_4) \in \mathbb{P}^3(\mathbb{Q})$ of the system of equations (3.1) with x_4 being determined by (3.3) and such that $x_1x_2x_3x_4 \neq 0$. This confirms indeed that the solutions $(j_k)_{1\leqslant k\leqslant 4} \in \mathbb{Z}_0^4$ of the system of equations (2.14) correspond to the points on the curve C having non-zero rational coordinates.

MATHEMATICA allows us to fully expand equation (3.4), which leads to the following homogeneous polynomial equation of degree 9 describing the curve C in $\mathbb{P}^2(\mathbb{C})$

$$\begin{aligned} &x_1^6x_2^3 + 3x_1^6x_2^2x_3 + 3x_1^6x_2x_3^2 + x_1^6x_3^3 + 3x_1^5x_2^3x_3 + 6x_1^5x_2^2x_3^2 + 3x_1^5x_2x_3^3 \\ &\quad + 3x_1^4x_2^3x_3^2 + 3x_1^4x_2^2x_3^3 + x_1^3x_2^6 + 3x_1^3x_2^5x_3 + 3x_1^3x_2^4x_3^2 + 2x_1^3x_2^3x_3^3 \\ &\quad + 3x_1^3x_2^2x_3^4 + 3x_1^3x_2x_3^5 + x_1^3x_3^6 + 3x_1^2x_2^6x_3 + 6x_1^2x_2^5x_3^2 + 3x_1^2x_2^4x_3^3 \\ &\quad + 3x_1^2x_2^3x_3^4 + 6x_1^2x_2^2x_3^5 + 3x_1^2x_2x_3^6 + 3x_1x_2^6x_3^2 + 3x_1x_2^5x_3^3 + 3x_1x_2^3x_3^5 \\ &\quad + 3x_1x_2^2x_3^6 + x_2^6x_3^3 + x_2^3x_3^6 = 0. \end{aligned} \tag{3.5}$$

Applying the MATHEMATICA command `Factor` to the polynomial given in equation (3.5), we obtain the factorization into irreducible polynomials

$$(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)f(x_1, x_2, x_3) = 0,$$

where $f(x_1, x_2, x_3)$ is the irreducible homogeneous polynomial of degree 6 given by

$$\begin{aligned} f(x_1, x_2, x_3) := {}& x_1^4x_2^2 + 2x_1^4x_2x_3 + x_1^4x_3^2 - x_1^3x_2^3 - x_1^3x_3^3 + x_1^2x_2^4 + x_1^2x_2^2x_3^2 \\ & + x_1^2x_3^4 + 2x_1x_2^4x_3 + 2x_1x_2x_3^4 + x_2^4x_3^2 - x_2^3x_3^3 + x_2^2x_3^4. \end{aligned} \tag{3.6}$$

Geometrically, this factorization implies that the curve C is reducible and consists of four irreducible components, three of which are given by the projective lines

$$x_k + x_l = 0, \qquad \forall\, 1 \leqslant k, l \leqslant 3, \tag{3.7}$$

while the fourth irreducible component is an irreducible curve C' of degree 6 in $\mathbb{P}^2(\mathbb{C})$, described by the polynomial equation

$$f(x_1, x_2, x_3) = 0. \tag{3.8}$$

Since the points on the projective lines (3.7) give rise to the trivial solutions (3.2), namely

$$(x_1 : -x_1 : x_3), \quad (x_1 : x_2 : -x_1), \quad (x_1 : x_2 : -x_2),$$

which are not of interest to us, it remains to investigate the rational solutions of equation (3.8), i.e., the rational points on the curve C'. The curve C' has the obvious three rational points

$$P_1 = (1:0:0), \quad P_2 = (0:1:0), \quad P_3 = (0:0:1),$$

which in the end will not be relevant in our further discussion, since some of their coordinates vanish.

We next consider the affine curve C'_{aff} in the affine x_1,x_2-plane obtained from C' by dehomogenization, i.e., by setting $x_3 = 1$. In order to ease notation for the subsequent calculations, we set $x = x_1$ and $y = x_2$. Thus, C'_{aff} is given by the equation $f_{\text{aff}}(x,y) = 0$, where

$$\begin{aligned} f_{\text{aff}}(x,y) &:= f(x,y,1) \\ &= x^4y^2 - x^3y^3 + x^2y^4 + 2x^4y + 2xy^4 + x^4 + x^2y^2 + y^4 - x^3 - y^3 + x^2 + 2xy + y^2. \end{aligned}$$

In this affine picture, the points P_1 and P_2 are the two points at infinity of the curve C'_{aff} and the point P_3 reflects that the curve C'_{aff} contains the origin of the x,y-plane. We now claim that the only *real* point on the affine curve C'_{aff} is the origin, which in turn shows that the only real points on the projective curve C' are the points P_1, P_2, P_3.

In order to show this, we observe that we can make the implicit equation $f_{\text{aff}}(x,y) = 0$ explicit by considering it as a polynomial equation of degree four in y and then solve it using Ferrari's formulae. Defining the quantities

$$\begin{aligned} A(x) &:= \frac{x^2 - x + 1}{4(x+1)}, \\ B(x) &:= -\frac{3(x^4 + 2x^3 - x^2 + 2x + 1)}{4(x+1)^2}, \\ C(x) &:= -\frac{x^4 + x^2 + 1}{2(x+1)^2}, \\ D(x) &:= -\frac{3(x^2 + 3x + 1)^2(x^2 - x + 1)}{(x+1)^3}, \end{aligned}$$

MATHEMATICA provides the following four solutions (branches),

$$y_{1,2}(x) = A(x) - \frac{1}{2}\sqrt{B(x)} \pm \frac{1}{2}\sqrt{C(x) - \frac{D(x)}{4\sqrt{B(x)}}},$$

$$y_{3,4}(x) = A(x) + \frac{1}{2}\sqrt{B(x)} \pm \frac{1}{2}\sqrt{C(x) + \frac{D(x)}{4\sqrt{B(x)}}}.$$

One easily checks that for real x, the rational function $B(x)$ has a pole at $x=-1$ and possesses the two real zeros

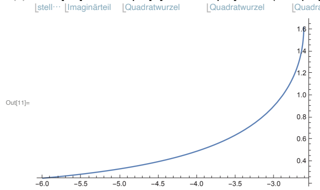
$$x_1 = \frac{-3-\sqrt{5}}{2} = -2.61803\ldots, \qquad x_2 = \frac{-3+\sqrt{5}}{2} = -0.38196\ldots,$$

from which we conclude that for $x < x_1$ and for $x > x_2$, the quantity $B(x)$ is negative and thus $\sqrt{B(x)}$ is purely imaginary. More specifically, a careful analysis (either using MATHEMATICA or a direct inspection by hand) shows that for $x < x_1$ and $x > x_2$, the solutions $y_j(x)$ with $1 \leqslant j \leqslant 4$ are always purely complex unless $x = 0$, which leads to

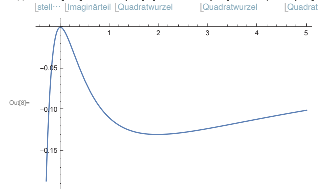
$$y_2(0) = y_3(0) = 0, \quad y_1(0) = \frac{1-\sqrt{-3}}{2}, \quad y_4(0) = \frac{1+\sqrt{-3}}{2},$$

and shows that for $x < x_1$ and $x > x_2$, the origin $(x,y) = (0,0)$ of the affine x,y-plane is the only real point on C'_{aff}. As an example, we present the plots of the imaginary part of $y_1(x)$ for $-6 < x < x_1$ and of the imaginary part of $y_4(x)$ for $x_2 < x < 5$; similar plots can be obtained for the imaginary parts of $y_2(x)$ and $y_3(x)$ in the range under consideration:

```
In[1]:= a := (x^2 - x + 1) / (4 * (x + 1))
In[2]:= b := -3 * (x^4 + 2 * x^3 - x^2 + 2 * x + 1) / (4 * (x + 1)^2)
In[3]:= c := - (x^4 + x^2 + 1) / (2 * (x + 1)^2)
In[4]:= d := -3 * ((x^2 + 3 * x + 1)^2) * (x^2 - x + 1) / ((x + 1)^3)
In[11]:= Plot[Im[-1 / 2 * Sqrt[b] + 1 / 2 * Sqrt[c - d / (4 * Sqrt[b])]], {x, -6, -2.618}]
```




```
In[1]:= a := (x^2 - x + 1) / (4 * (x + 1))
In[2]:= b := -3 * (x^4 + 2 * x^3 - x^2 + 2 * x + 1) / (4 * (x + 1)^2)
In[3]:= c := - (x^4 + x^2 + 1) / (2 * (x + 1)^2)
In[4]:= d := -3 * ((x^2 + 3 * x + 1)^2) * (x^2 - x + 1) / ((x + 1)^3)
In[8]:= Plot[Im[1 / 2 * Sqrt[b] - 1 / 2 * Sqrt[c + d / (4 * Sqrt[b])]], {x, -0.381, 5}]
```

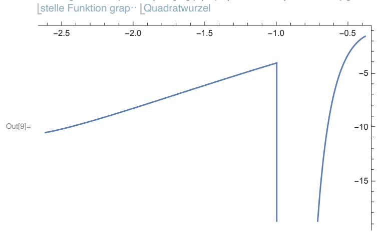


For $x_1 \leqslant x \leqslant x_2$ with $x \neq -1$, the quantities $A(x) \pm \sqrt{B(x)}/2$ are real, however the functions

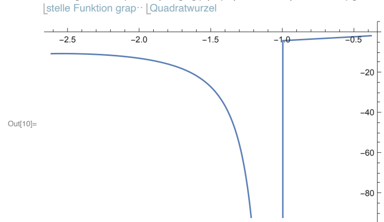
$$C(x) \pm \frac{D(x)}{4\sqrt{B(x)}}$$

are strictly negative, as the following two plots show:

```
In[1]:= a := (x^2 - x + 1) / (4 * (x + 1))
In[2]:= b := -3 * (x^4 + 2 * x^3 - x^2 + 2 * x + 1) / (4 * (x + 1)^2)
In[3]:= c := - (x^4 + x^2 + 1) / (2 * (x + 1)^2)
In[4]:= d := -3 * ((x^2 + 3 * x + 1)^2) * (x^2 - x + 1) / ((x + 1)^3)
In[9]:= Plot[c + d / (4 * Sqrt[b]), {x, -2.618, -0.381}]
```




```
In[1]:= a := (x^2 - x + 1) / (4 * (x + 1))
In[2]:= b := -3 * (x^4 + 2 * x^3 - x^2 + 2 * x + 1) / (4 * (x + 1)^2)
In[3]:= c := - (x^4 + x^2 + 1) / (2 * (x + 1)^2)
In[4]:= d := -3 * ((x^2 + 3 * x + 1)^2) * (x^2 - x + 1) / ((x + 1)^3)
In[10]:= Plot[c - d / (4 * Sqrt[b]), {x, -2.618, -0.381}]
```



Therefore, the solutions $y_j(x)$ with $1 \leqslant j \leqslant 4$ are strictly complex and cannot be real. In case $x = -1$, we find the two solutions $y_{1,2} = \pm i$. Altogether, this shows that the origin $(x,y) = (0,0)$ of the affine x,y-plane is the only real point on the curve C'_{aff}.

This allows us to conclude the proof of the theorem: we have shown that the only non-trivial rational points on the curve C are the points P_1, P_2, P_3. However, since some of their coordinates vanish, these points have to be ignored. Thus, by our introductory reformulations, it follows that the system of equations (2.14) has no solutions $(j_k)_{1\leqslant k\leqslant 4} \in \mathbb{Z}_0^4$ satisfying (2.15). $\square$

REMARK 3.1. In the preceding proof of Theorem 2.3, the crucial point was to show that the irreducible curve C' in $\mathbb{P}^2(\mathbb{C})$, defined by the equation

$$f(x_1,x_2,x_3) = 0,$$

where $f(x_1,x_2,x_3)$ is the irreducible homogeneous polynomial of degree 6 given by (3.6), has

$$P_1 = (1:0:0), \quad P_2 = (0:1:0), \quad P_3 = (0:0:1)$$

as its only rational points. In order to approach such a claim in general, one shows in a first step that the curve C', which is defined over $\mathbb{Q}$, has only *finitely many* rational points. Such a finiteness result then implies that the system (2.14) has only *finitely many primitive* solutions $(j_k)_{1\leqslant k\leqslant 4} \in \mathbb{Z}_0^4$ satisfying (2.15), by which we mean solutions $(j_k)_{1\leqslant k\leqslant 4} \in \mathbb{Z}_0^4$ with greatest common divisor equal to 1.

In order to prove the finiteness of the set of rational points on the curve C', one is tempted to employ Faltings's Theorem (formerly the Mordell Conjecture, see [21]), which states that this set is indeed finite if the genus $g_{C'}$ of C' is bigger than 1. (For an elementary introduction to this circle of problems, we refer to Appendix C in [30] and the references therein.) Hence, we need to compute the genus $g_{C'}$, which requires to determine the singular points of C'. Either by a direct computation or alternatively using the command *singularities*(f,x_1,x_2,x_3) of the software package *with(algcurves)* of MAPLE, one concludes that the three singular points are the points P_1, P_2, P_3 mentioned above, having the following multiplicities and delta-invariants

$$\begin{aligned} m_1 &= 2 \quad \text{and} \quad \delta_1 = 2, \\ m_2 &= 2 \quad \text{and} \quad \delta_2 = 2, \\ m_3 &= 2 \quad \text{and} \quad \delta_3 = 2, \end{aligned}$$

respectively. We note that the delta-invariants being bigger than 1 implies that the singularities in question are not ordinary. With these data at hand, the genus $g_{C'}$ of the curve C' of degree $d = 6$ is given by

$$g_{C'} = \frac{(d-1)(d-2)}{2} - \sum_{j=1}^{3} \delta_j = \frac{5 \cdot 4}{2} - (2+2+2) = 4.$$

The normalization $\widetilde{C'}$ of the curve C' is then an irreducible, smooth, projective curve of genus $g_{\widetilde{C'}} = 4$. The normalization $\widetilde{C'}$ admits a surjective morphism $\pi\colon \widetilde{C'} \longrightarrow C'$, which is an isomorphism away from the three singular points, i.e., we have

$$\widetilde{C'} \setminus \pi^{-1}(\{P_1,P_2,P_3\}) \cong C' \setminus \{P_1,P_2,P_3\}, \tag{3.9}$$

and for any $1 \leqslant j \leqslant 3$, the fiber $\pi^{-1}(P_j)$ over the singular point P_j consists of finitely many points. Finally, since the curve C' is defined over $\mathbb{Q}$, its normalization $\widetilde{C'}$ is also defined over $\mathbb{Q}$.

In summary, $\widetilde{C'}$ is an irreducible, smooth, projective curve of genus $g_{\widetilde{C'}} = 4$, which is moreover defined over $\mathbb{Q}$. By Faltings's Theorem we then conclude that $\widetilde{C'}$ has only finitely many rational points. Because of the isomorphism (3.9) and the fact that the fibers over the three singular points are finite, the curve C' also has only finitely many rational points. This concludes our remark demonstrating that our main result can be reduced to a finite problem in quite general terms.

Let us now turn to the promised discussion of Question 2.1.

Discussion of Question 2.1. In analogy to the proof of Theorem 2.3, in order to approach Question 2.1, we introduce the homogeneous coordinates $(x_1 : x_2 : x_3 : x_4 : x_5)$ in projective space $\mathbb{P}^4(\mathbb{C})$ and first investigate the algebraic geometric objects defined by the system of equations

$$\sum_{k=1}^{5} x_k^3 = 0, \qquad \sum_{k=1}^{5} x_k = 0, \qquad \sum_{k=1}^{5} \frac{1}{x_k} = 0. \tag{3.10}$$

The first equation of (3.10) defines a cubic threefold T_1, the second equation of (3.10) defines a hyperplane T_2, while the third equation of (3.10) defines a quartic threefold T_3 in $\mathbb{P}^4(\mathbb{C})$, given by the homogeneous polynomial equation of degree 4,

$$x_1x_2x_3x_4 + x_1x_2x_3x_5 + x_1x_2x_4x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5 = 0.$$

We are interested in the locus given by the intersection of the three threefolds T_j $(1 \leqslant j \leqslant 3)$ in $\mathbb{P}^4(\mathbb{C})$.

Intersecting the cubic threefold T_1 with the hyperplane T_2, leads to the irreducible cubic surface S_1 in $\mathbb{P}^3(\mathbb{C})$, given by the equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 - (x_1 + x_2 + x_3 + x_4)^3 = 0,$$

and known as the *Clebsch cubic surface*. The set of rational points on S_1 turns out be infinite; it will be discussed in more detail in Section 4. On the other hand, intersecting the hyperplane T_2 with the quartic threefold T_3 gives rise to the irreducible quartic surface S_2 in $\mathbb{P}^3(\mathbb{C})$, given by the equation

$$x_1x_2x_3x_4 - (x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)(x_1 + x_2 + x_3 + x_4) = 0,$$

and known as the *Hessian surface*. We do not know if the set of rational points on S_2 is infinite, however it is likely that this indeed the case since the resolution of the singularities of S_2 leads to a $K3$-surface (see Section 9.4.2 in [16]) which is, of course, not of general type and thus might allow infinitely many rational points projecting down to S_2.

Ultimately, we are interested in the set of rational points on the intersection $T_1 \cap T_2 \cap T_3 = S_1 \cap S_2$, which is a curve D of degree 12 in $\mathbb{P}^3(\mathbb{C})$ defined over $\mathbb{Q}$. Using the algebraic geometry software package MACAULAY, it can be shown that the curve D is irreducible and has genus 19. Arguing as in Remark 3.1 above, it then follows again from Faltings's Theorem that the system of equations (2.20) can have at most *finitely many primitive* solutions $(j_k)_{1\leqslant k\leqslant 5} \in \mathbb{Z}_0^5$ satisfying (2.21). To fully answer Question 2.1, it thus remains to show that for any primitive solution $(j_k)_{1\leqslant k\leqslant 5} \in \mathbb{Z}^5$, there is always an index $k \in \{1,\ldots,5\}$ such that $j_k = 0$, which then implies that the remaining four indices can be grouped into two pairs of indices (l,l') and (m,m') such that $j_l + j_{l'} = 0$ and $j_m + j_{m'} = 0$. It is likely that this can be proved, however this seems not to be an easy task.

4. On cubic diophantine equations

The aim of this section is to give a broad overview about results on the set of rational solutions of cubic equations which are relevant in the context of the KdV frequencies. We start with a discussion of results on the set of rational points on Fermat's cubic in 3, 4, and 5 variables and then proceed by adding to the equation of Fermat's cubic a linear and, subsequently, a fractional constraint. For each of these systems of equations, we study the set of common rational solutions.

4.1. The Fermat cubic in several variables

We start by considering the Fermat cubic curve in $\mathbb{P}^2(\mathbb{C})$, given by

$$F_3\colon x_1^3 + x_2^3 + x_3^3 = 0.$$

We are interested in the set of rational points of F_3, which do not lie on the lines in $\mathbb{P}^2(\mathbb{C})$, defined by

$$x_k + x_l = 0, \qquad \forall\, 1 \leqslant k,l \leqslant 3. \tag{4.1}$$

As Fermat's Last Theorem tells us, there are no such rational points on F_3.

Next, we consider the Fermat cubic surface in $\mathbb{P}^3(\mathbb{C})$, given by

$$F_4\colon x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

We are interested in the set of rational points of F_4, which do not lie on the planes in $\mathbb{P}^3(\mathbb{C})$, defined by

$$x_k + x_l = 0, \qquad \forall\, 1 \leqslant k,l \leqslant 4. \tag{4.2}$$

In contrast to the preceding case, it turns out that there are infinitely many such rational points on F_4. In fact, the set of such rational points can be shown to be Zariski dense in $F_4(\mathbb{Q})$. In the next subsection, we will indicate how such rational points can be constructed. More generally, there is a parametrization of them, given by N. Elkies in [18]. As an aside, we remark that this problem is related to the question of representing an integer in two different ways as a sum of two cubes, the celebrated "cab number problem", e.g.,

$$1^3+12^3+(-9)^3+(-10)^3=0.$$

Finally, turning to 5 variables, we consider the Fermat cubic threefold in $\mathbb{P}^4(\mathbb{C})$, given by

$$F_5\colon x_1^3+x_2^3+x_3^3+x_4^3+x_5^3=0.$$

We are interested in the set of rational points of F_5, which do not lie on the hyperplanes in $\mathbb{P}^4(\mathbb{C})$, defined by

$$x_k+x_l=0, \qquad \forall\, 1\leqslant k,l\leqslant 5. \tag{4.3}$$

Given our preceding discussion, it is not surprising that there is an infinitude of such rational points on F_5. As an example, we mention

$$5^3+7^3+9^3+10^3+(-13)^3=0.$$

4.2. The Fermat cubics with a linear constraint

First consider the system of equations in $\mathbb{P}^2(\mathbb{C})$, given by

$$F_3\colon x_1^3+x_2^3+x_3^3=0, \qquad L_3\colon x_1+x_2+x_3=0.$$

We are interested in the set of rational points of $F_3\cap L_3$, which do not lie on the lines (4.1). Since there are no rational points on F_3 away from the lines (4.1), we have that $F_3(\mathbb{Q})\cap L_3(\mathbb{Q})=\emptyset$.

Next, we consider the system of equations in $\mathbb{P}^3(\mathbb{C})$, given by

$$F_4\colon x_1^3+x_2^3+x_3^3+x_4^3=0, \qquad L_4\colon x_1+x_2+x_3+x_4=0.$$

We are interested in the set of rational points of $F_4\cap L_4$, which do not lie on the planes (4.2). It turns out that $F_4(\mathbb{Q})\cap L_4(\mathbb{Q})=\emptyset$, as discussed on the top of page 649.

Continuing in this way, we next consider the system of equations in $\mathbb{P}^4(\mathbb{C})$, given by

$$F_5\colon x_1^3+x_2^3+x_3^3+x_4^3+x_5^3=0, \qquad L_5\colon x_1+x_2+x_3+x_4+x_5=0.$$

We are interested in the set of rational points of $F_5\cap L_5$, which do not lie on the hyperplanes (4.3). The intersection $F_5\cap L_5$ is known as the *Clebsch cubic surface* S, for which we know that there are infinitely many rational points on $F_5\cap L_5$ away from the hyperplanes (4.3). In fact, it is known that the set of rational points on $F_5\cap L_5$ is Zariski dense.

Let us *sketch* how such rational points can be systematically computed on S: Using the relation $x_5 = -x_1 - x_2 - x_3 - x_4$, the Clebsch cubic surface S can be described equivalently as the cubic surface in $\mathbb{P}^3(\mathbb{C})$, given by the equation

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 - (x_1 + x_2 + x_3 + x_4)^3 = 0.$$

Given a non-zero rational number a, the Clebsch cubic surface S obviously contains the rational point $P = (-a, a, 1, -1) \in S(\mathbb{Q})$. The aim now is to construct further rational points on S, starting with the rational point P. For this, we first construct the tangent plane T of S at the point P, which is easily computed as

$$T\colon\ a^2x_1 + a^2x_2 + x_3 + x_4 = 0.$$

Next, we compute the intersection of the Clebsch cubic surface S with the tangent plane T, which leads to the cubic curve

$$C\colon\ x_1^3 + x_2^3 + x_3^3 - (a^2x_1 + a^2x_2 + x_3)^3 - ((1-a^2)x_1 + (1-a^2)x_2)^3 = 0. \tag{4.4}$$

By construction, the cubic curve C contains the line given by $x_1 + x_2 = 0$. Hence the left-hand side of (4.4) has to be divisible by $(x_1 + x_2)$. Performing this polynomial division leads to the quadric

$$Q\colon\ x_1x_2 + a^2(a^2-1)(x_1+x_2)^2 + a^4(x_1+x_2)x_3 + a^2x_3^2 = 0.$$

Obviously, the point P is also a rational point on the quadric Q. Therefore, we obtain all the rational points P' on Q, by intersecting Q with any line L passing through P and having rational slope, i.e., by intersecting Q with

$$L\colon\ b(x_1 + ax_3) - c(x_2 - ax_3) = 0,$$

where $(b,c) \in \mathbb{Q}^2 \setminus \{(0,0)\}$. Since the quadric Q is by construction contained in the Clebsch cubic surface S, by varying the pair (b,c) through $\mathbb{Q}^2 \setminus \{(0,0)\}$, one obtains infinitely many rational points P' on S. Assuming that $b \neq 0$, a straightforward computation yields the following coordinates for $P' \in S(\mathbb{Q})$:

$$x_{1,P'} = -\frac{a}{b^2}\big(a^2(a^2-1)(b+c)^2 + a^3c(b+c) + c^2\big),$$

$$x_{2,P'} = \frac{a}{b^2}\big(a^2(a^2-1)(b+c)^2 - a^3b(b+c) + b^2\big),$$

$$x_{3,P'} = \frac{1}{b^2}\big(a^2(a^2-1)(b+c)^2 + bc\big),$$

$$x_{4,P'} = \frac{1}{b^2}\big(a^4(a^2-1)(b+c)^2 + a^2(b+c)(-ab+ac+b+c) - bc\big),$$

$$x_{5,P'} = -\frac{1}{b^2}\big(a^4(a^2-1)(b+c)^2 - a(a^2-1)(b^2-c^2)\big).$$

We can simplify the above formulas by multiplying them by b^2/a and then setting $b=1$ to obtain

$$\begin{aligned}
x_1 &= -a^2(a^2-1)(c+1)^2 - a^3c(c+1) - c^2,\\
x_2 &= a^2(a^2-1)(c+1)^2 - a^3(c+1) + 1,\\
x_3 &= a(a^2-1)(c+1)^2 + c/a,\\
x_4 &= a^3(a^2-1)(c+1)^2 + a(c+1)(ac-a+c+1) - c/a,\\
x_5 &= -a^3(a^2-1)(c+1)^2 - (a^2-1)(c^2-1).
\end{aligned}$$

Finally, choosing for example $a=2$ and $c=0$, we arrive at

$$x_1 = -12, \quad x_2 = 5, \quad x_3 = 6, \quad x_4 = 22, \quad x_5 = -21,$$

which constitutes indeed a non-trivial rational point on the Clebsch cubic surface S.

4.3. The Fermat cubics with a fractional constraint

Consider the system of equations in $\mathbb{P}^2(\mathbb{C})$, given by

$$F_3\colon x_1^3+x_2^3+x_3^3=0, \qquad R_3\colon \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}=0.$$

We are interested in the set of rational points of $F_3 \cap R_3$, which do not lie on the lines (4.1). Since there are no rational points on F_3 away from the lines (4.1), we have that $F_3(\mathbb{Q}) \cap R_3(\mathbb{Q}) = \emptyset$.

Next, we consider the system of equations in $\mathbb{P}^3(\mathbb{C})$, given by

$$F_4\colon x_1^3+x_2^3+x_3^3+x_4^3=0, \qquad R_4\colon \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}=0.$$

We are interested in the set of rational points of $F_4 \cap R_4$, which do not lie on the planes (4.2). Theorem 2.3 shows that there are no such rational points.

Continuing in this way, we next consider the system of equations in $\mathbb{P}^4(\mathbb{C})$, given by

$$F_5\colon x_1^3+x_2^3+x_3^3+x_4^3+x_5^3=0, \qquad R_5\colon \frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\frac{1}{x_4}+\frac{1}{x_5}=0.$$

We are interested in the set of rational points of $F_5 \cap R_5$, which do not lie on the hyperplanes (4.3). In analogy to the linear equation, one expects that there are such rational points. However, we have not studied this problem.

REMARK 4.1. The preceding three subsections lead to the following pattern: In the case of three variables, the Fermat cubic F_3 has no rational points away from the lines (4.1). Increasing the number of variables by one, we find that the Fermat cubic

F_4 subject to the additional constraint L_4 or R_4 has no rational points away from the planes (4.2). Continuing and increasing the number of variables again by one, we end up with the conjecture that the Fermat cubic F_5 subject to the two constraints L_5 and R_5 has no rational points away from the hyperplanes (4.3). Therefore, it is to be expected that the Fermat cubic F_n in $n > 5$ variables subject to the constraints L_n and R_n will always have infinitely many rational points away from the hyperplanes $x_k + x_l = 0$ for $0 \leqslant k, l \leqslant n$.

A. Asymptotic expansion of KdV frequencies

In this appendix we prove Remark 2.1, stating that the coefficient α in the expansion of the normal frequencies Ω_j does not vanish identically. First we note that, when viewed as a function of the potential q, it is straightforward to show that the coefficient α analytically extends to the closure $\overline{M}_S$ of M_S (cf. Definition 2.1), consisting of potentials $q \in L_0^2$, satisfying

$$w_n(q) = 0, \qquad \forall n \in S^\perp.$$

Note that the zero potential is in $\overline{M}_S$ as well as any potential $q \in L_0^2$, for which there exists $k \in S_+$, so that $w_n(q) = 0$ for any $n \geqslant 1$ with $n \neq k$. We now compute α for such potentials. Without further reference, we use the notation introduced in [27]. According to [27, Theorem F.4 and Remark 2],

$$\Omega_j = 8j\pi(\tau_j - r_j), \qquad r_j = \sum_{m \geqslant 1} (\sigma_m^j - \dot{\lambda}_m).$$

(For notational convenience, σ_j^j is defined as $\sigma_j^j := \tau_j$ (cf. [27, (D.1) page 212]).) For a one gap potential q as above, $\sigma_m^j = \dot{\lambda}_m (= \tau_m)$ for any $m \geqslant 1$ with $m \neq k$. Hence

$$\Omega_j = 8j\pi\tau_j + 8j\pi(\dot{\lambda}_k - \sigma_k^j).$$

We need to compute

$$\alpha = \lim_{j \to \infty} j(\Omega_j - 8j^3\pi^3) = \lim_{j \to \infty} j\big(8j\pi\tau_j - 8j^3\pi^3 + 8j\pi(\dot{\lambda}_k - \sigma_k^j)\big). \tag{A.1}$$

It is well known that τ_j admits the asymptotic expansion (cf. e.g. Theorem 1.3, Theorem 1.4 in [28] and [37, page 39]),

$$\tau_j = j^2\pi^2 + c_2\frac{1}{j^2\pi^2} + O\Big(\frac{1}{j^4}\Big), \qquad c_2 = \frac{1}{4}\int_0^1 q(x)^2 dx, \tag{A.2}$$

implying that

$$8j\pi\tau_j - 8j^3\pi^3 = 8c_2\frac{1}{j\pi} + O\Big(\frac{1}{j^3}\Big).$$

It thus follows that

$$\lim_{j \to \infty} j(8j\pi\tau_j - 8j^3\pi^3) = \frac{8}{\pi}c_2 = \frac{2}{\pi}\int_0^1 q(x)^2 dx. \tag{A.3}$$

It remains to compute

$$\lim_{j\to\infty} j\cdot 8j\pi(\dot{\lambda}_k - \sigma_k^j).$$

Following [27, Section D], one sees that for $j > k$, the contour integral in the identity $\frac{1}{2\pi}\int_{\Gamma_j} \frac{\psi_j(\lambda)}{\sqrt[c]{\Delta(\lambda)^2-4}}\, d\lambda = 1$ can be computed by Cauchy's theorem, yielding

$$\frac{j\pi}{\sqrt[+]{\tau_j - \lambda_0}} \frac{\tau_j - \sigma_k^j}{\sqrt[+]{(\tau_j-\tau_k)^2 - \gamma_k^2/4}} = 1,$$

or

$$\tau_j - \sigma_k^j = \frac{\sqrt[+]{\tau_j-\lambda_0}}{j\pi}\sqrt[+]{(\tau_j-\tau_k)^2-\gamma_k^2/4}. \tag{A.4}$$

According to [27, page 229, Remark 2], $\tau_k - \dot{\lambda}_k = -\lambda_0/2$. Hence the left hand side of (A.4) can be written as

$$\tau_j - \sigma_k^j = (\tau_j - \tau_k) - \lambda_0/2 + (\dot{\lambda}_k - \sigma_k^j). \tag{A.5}$$

Using the Taylor expansion $\sqrt[+]{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + O(x^3)$ and (A.2), the right hand side of (A.4) can be expanded as

$$\sqrt[+]{\tau_j - \lambda_0} = j\pi - \frac{\lambda_0}{2}\frac{1}{j\pi} + \left(\frac{c_2}{2} - \frac{\lambda_0^2}{8}\right)\frac{1}{j^3\pi^3} + O\Big(\frac{1}{j^5}\Big),$$

$$\sqrt[+]{(\tau_j-\tau_k)^2-\gamma_k^2/4} = (\tau_j - \tau_k) - \frac{\gamma_k^2}{8}\frac{1}{\tau_j-\tau_k} + O\Big(\frac{1}{j^6}\Big),$$

yielding

$$\begin{aligned}
&\frac{\sqrt[+]{\tau_j-\lambda_0}}{j\pi}\sqrt[+]{(\tau_j-\tau_k)^2-\gamma_k^2/4}\\
&\quad= \left(1 - \frac{\lambda_0}{2}\frac{1}{j^2\pi^2} + \left(\frac{c_2}{2}-\frac{\lambda_0^2}{8}\right)\frac{1}{j^4\pi^4}\right)\left((\tau_j-\tau_k) - \frac{\gamma_k^2}{8}\frac{1}{\tau_j-\tau_k}\right) + O\Big(\frac{1}{j^4}\Big)\\
&\quad= (\tau_j - \tau_k) - \frac{\lambda_0}{2}\frac{\tau_j-\tau_k}{j^2\pi^2} + \left(\frac{c_2}{2}-\frac{\lambda_0^2}{8}\right)\frac{\tau_j-\tau_k}{j^4\pi^4} - \frac{\gamma_k^2}{8}\frac{1}{\tau_j-\tau_k} + O\Big(\frac{1}{j^4}\Big).
\end{aligned}$$

Using that $\frac{\tau_j-\tau_k}{j^2\pi^2} = 1 - \frac{\tau_k}{j^2\pi^2} + O(\frac{1}{j^4})$, we finally get

$$\begin{aligned}
&\frac{\sqrt[+]{\tau_j-\lambda_0}}{j\pi}\sqrt[+]{(\tau_j-\tau_k)^2-\gamma_k^2/4}\\
&\quad= (\tau_j-\tau_k) - \frac{\lambda_0}{2} + \left(\frac{\lambda_0}{2}\tau_k + \frac{c_2}{2} - \frac{\lambda_0^2}{8} - \frac{\gamma_k^2}{8}\right)\frac{1}{j^2\pi^2} + O\Big(\frac{1}{j^4}\Big).
\end{aligned} \tag{A.6}$$

Combining (A.5) and (A.6), we obtain

$$\lim_{j\to\infty}\frac{8}{\pi}j^2\pi^2(\dot{\lambda}_k - \sigma_k^j) = \frac{8}{\pi}\left(\frac{\lambda_0}{2}\tau_k + \frac{c_2}{2} - \frac{\lambda_0^2}{8} - \frac{\gamma_k^2}{8}\right),$$

which together with (A.1) and (A.3) then yields for any $q \in \overline{M}_S$ with $w_n(q) = 0$ for any $n \neq k$,

$$\alpha = \frac{8}{\pi}\Big(\frac{\lambda_0}{2}\tau_k + \frac{3c_2}{2} - \frac{\lambda_0^2}{8} - \frac{\gamma_k^2}{8}\Big).$$

It remains to remark that the L^2-gradient of α at $q = 0$ does not vanish. Indeed, since $\lambda_0|_{q=0} = 0$, $\tau_k|_{q=0} = k^2\pi^2$, $\nabla\lambda_0|_{q=0} = f_0^2|_{q=0} \equiv 1$, and $\nabla c_2|_{q=0} = 0$, one has

$$\nabla\Big(\frac{\lambda_0}{2}\tau_k + \frac{3c_2}{2} - \frac{\lambda_0^2}{8} - \frac{\gamma_k^2}{8}\Big)|_{q=0} = \frac{\tau_k}{2}\nabla\lambda_0|_{q=0} - \frac{1}{\pi}\nabla\gamma_k^2|_{q=0} = \frac{k^2\pi^2}{2}, \tag{A.7}$$

where for the latter identity we used that $\nabla\gamma_k^2|_{q=0} = 0$. To see that $\nabla\gamma_k^2|_{q=0}$ vanishes, first note that by [27, Theorem 7.3], $\gamma_k^2 = \frac{1}{\xi_k^2}8I_k$ and $\frac{1}{\xi_k^2}|_{q=0} = k\pi$. By [27, Theorem 7.3] and the fact that $\nabla\Delta(\lambda,\cdot)|_{q=0} \equiv 0$ (cf. [27, Proposition B.3]), it then follows that $\nabla I_k|_{q=0} = 0$ and hence $\nabla\gamma_k^2 = k\pi\nabla I_k = 0$.

The computation (A.7) shows that α does not vanish identically.

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Thomas Kappeler
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190, CH-8057 Zürich
e-mail: thomas.kappeler@math.uzh.ch

Jürg Kramer
Institut für Mathematik
Humboldt-Universität zu Berlin
Unter den Linden 6, 10099 Berlin
e-mail: kramer@math.hu-berlin.de