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THE COMPLETE NEVANLINNA–PICK PROPERTY FOR BEURLING TYPE QUOTIENT MODULES OVER THE BIDISK

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Abstract. In this paper, we show that for the inner function $\theta(z_1,z_2)\in H^2(\mathbb{D}^2)$, the Beurling type quotient module $\mathscr{K}_\theta = H^2(\mathbb{D}^2)\ominus\theta H^2(\mathbb{D}^2)$ over the bidisk has complete Nevanlinna-Pick property if and only if θ is a one-variable Möbius map.

1. Introduction

Let X be a set and let $k : X\times X\to\mathbb{C}$ be a function of two variables. We call k a positive semi-definite function on X if k is self-adjoint ($k(z,w)=\overline{k(w,z)}$), and for any finite set $\{\lambda_1,\lambda_2,\cdots,\lambda_N\}\subseteq X$, the matrix $[k(\lambda_i,\lambda_j)]_{i,j=1}^N$ is positive semi-definite, i.e. for every $\alpha_1,\alpha_2,\cdots,\alpha_N\in\mathbb{C}$, we have that

$$\sum_{i,j=1}^N \alpha_i\bar{\alpha}_j k(\lambda_i,\lambda_j)\geqslant 0.$$

We will use the notation $k\succeq 0$ to denote that k is positive semi-definite.

A reproducing kernel Hilbert space $\mathscr{H}$ on X is a Hilbert space of complex valued functions on X such that every point evaluation is a continuous linear functional. Thus by the Riesz representation theorem, for every $w\in X$, there exists an element $k_w\in\mathscr{H}$ such that for each $f\in\mathscr{H}$,

$$\langle f,k_w\rangle_{\mathscr{H}} = f(w).$$

Since $k_w(z)=\langle k_w,k_z\rangle_{\mathscr{H}}$, k can be regarded as a function on $X\times X$ and we write $k(z,w)=k_w(z)$. Let $\mathscr{H}(k)$ be a reproducing kernel Hilbert space on X with reproducing kernel k. See [6, 13] for background on reproducing kernel Hilbert spaces.

Complete Nevanlinna-Pick kernels are related to the solution of Nevanlinna-Pick interpolation problems. If for a set X, whenever $\{\lambda_1,\cdots,\lambda_N\}\subseteq X$, and $W_1,\cdots,W_N$ are s-by-t matrices such that

$$(I-W_iW_j^*)k(\lambda_i,\lambda_j)\geqslant 0,$$

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we can find Φ in the closed unit ball of

$$\begin{aligned}\operatorname{Mult}(\mathscr{H}(k)\otimes\mathbb{C}^t,\mathscr{H}(k)\otimes\mathbb{C}^s)\\ =\{\Phi : X\to M_{s,t} : \Phi f\in\mathscr{H}(k)\otimes\mathbb{C}^s \text{ for every } f\in\mathscr{H}(k)\otimes\mathbb{C}^t\}\end{aligned}$$

such that $\Phi(\lambda_i)=W_i$, $i=1,2,\cdots,N$, we say k has $s\times t$-Pick property. If k has $s\times t$-Pick property for all positive integers s and t, then we say k has the complete Nevanlinna-Pick property.

Complete Nevanlinna-Pick spaces share many properties with the Hardy space $H^2(\mathbb{D})$, and they have been studied extensively in the literature, see e.g. [2, 3, 4, 5, 6, 11]. Examples of complete Nevanlinna-Pick spaces include the Hardy space $H^2(\mathbb{D})$, the Drury-Arveson spaces H_d^2 [6], the classical Dirichlet space $\mathscr{D}$ [1], all Dirichlet spaces with superharmonic weights $\mathscr{D}_w$ [16], the Sobolev space $W_1^2(0,1)$ [14]. A natural question is to decide which reproducing kernel Hilbert spaces have complete Nevanlinna-Pick property.

In 2020, Chu [8] determined which de Branges-Rovnyak spaces (sub-Hardy spaces) have complete Nevanlinna-Pick property. In 2023, Luo and Zhu [12] determined which sub-Bergman spaces have complete Nevanlinna-Pick property. In this paper, we will characterize which Beurling type quotient modules of the Hardy space over bidisk have complete Nevanlinna-Pick property.

Let $\mathbb{D}^2=\{(z_1,z_2):|z_1|<1,|z_2|<1\}$ be the unit bidisk in $\mathbb{C}^2$, and $\mathbb{T}^2=\{(z_1,z_2):|z_1|=1,|z_2|=1\}$ be the distinguished boundary of $\mathbb{D}^2$. The Hardy space over the bidisk $H^2(\mathbb{D}^2)$ is the closure of all polynomial in $L^2(\mathbb{T}^2,\frac{1}{4\pi^2}d\theta_1 d\theta_2)$. The Szegö kernel

$$k(z,w)=\frac{1}{(1-\bar{w}_1z_1)(1-\bar{w}_2z_2)}$$

on $\mathbb{D}^2$ is the kernel for the Hardy space $H^2(\mathbb{D}^2)$.

A function $\theta\in H^2(\mathbb{D}^2)$ is called an inner function if $|\theta(e^{i\theta_1},e^{i\theta_2})|=1$ almost everywhere on $\mathbb{T}^2$. We say that a submodule M_θ of $H^2(\mathbb{D}^2)$ is of Beurling type if M_θ is generated by an inner function $\theta\in H^2(\mathbb{D}^2)$, that is, $M_\theta=\theta H^2(\mathbb{D}^2)$. To every inner function $\theta(z_1,z_2)$ on $\mathbb{D}^2$, the associated quotient space is defined by

$$\mathscr{K}_\theta=H^2(\mathbb{D}^2)\ominus\theta H^2(\mathbb{D}^2).$$

$\mathscr{K}_\theta$ is called the Beurling type quotient module. The space $\mathscr{K}_\theta$ seems to be the most natural generalization of the one variable space $H^2(\mathbb{D})\ominus\theta H^2(\mathbb{D})$. It is well known that the reproducing kernel of $\mathscr{K}_\theta$ is

$$k^\theta(z,w)=\frac{1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)}{(1-\bar{w}_1z_1)(1-\bar{w}_2z_2)}.$$

A classic example of a complete Nevanlinna-Pick kernel is the Szegö kernel

$$s(z_1,w_1)=\frac{1}{1-\bar{w}_1z_1}$$

on $\mathbb{D}$, however, we will see that the Szegö kernel on $\mathbb{D}^2$ does not have complete Nevanlinna-Pick property. This inspires us to consider which Beurling type quotient module $\mathscr{K}_\theta$ have complete Nevanlinna-Pick property. For $a \in \mathbb{D}$, let $h_a(z_1) = \frac{a-z_1}{1-\bar{a}z_1}$ be the one-variable Möbius map. We now state the main result of this paper.

THEOREM 1.1. (Main Theorem) *Suppose $\theta(z_1,z_2) \in H^2(\mathbb{D}^2)$ is an inner function. Then the Beurling type quotient module $\mathscr{K}_\theta$ has complete Nevanlinna-Pick property if and only if $\theta(z_1,z_2) = \beta h_a(z_1)$ or $\theta(z_1,z_2) = \beta h_a(z_2)$, for some $|a| < 1$, $|\beta| = 1$.*

2. Proof of the main theorem

In this section, we will prove the main Theorem 1.1, which consists of several steps. Firstly, we need some preliminaries.

We will introduce the equivalent definition of complete Nevanlinna-Pick kernel, which will be used in this paper.

Let k be the reproducing kernel of an $\mathscr{H}(k)$ on X. In [5], if $k(z,w) \neq 0$ for all $z,w \in X$, it was shown that the k is a complete Nevanlinna-Pick kernel and if and only if

$$1 - \frac{k(z,z_0)k(z_0,w)}{k(z_0,z_0)k(z,w)} \succeq 0$$

for some $z_0 \in X$. A kernel k is normalized at $z_0 \in X$ if $k(z,z_0) = 1$ for all $z \in X$. If k is normalized, then k is a complete Nevanlinna-Pick kernel if and only if

$$1 - \frac{1}{k(z,w)} \succeq 0.$$

Let $\mathscr{H}_1$, $\mathscr{H}_2$ be reproducing kernel Hilbert spaces on X. A multiplier of $\mathscr{H}_1$ into $\mathscr{H}_2$ is a function $h : X \to \mathbb{C}$ such that whenever $f \in \mathscr{H}_1$, $hf \in \mathscr{H}_2$. We let

$$\mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2) = \{h : X \to \mathbb{C} : hf \in \mathscr{H}_2 \text{ for all } f \in \mathscr{H}_1\}$$

denote the set of all multipliers of $\mathscr{H}_1$ into $\mathscr{H}_2$. Given a multiplier $h \in \mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)$, we let $M_h : \mathscr{H}_1 \to \mathscr{H}_2$ denote the linear map $M_h(f) = hf$. By the closed graph theorem, M_h is a bounded linear operator on $\mathscr{H}_1$ into $\mathscr{H}_2$, and we set $||h||_{\mathrm{Mult}(\mathscr{H}_1,\mathscr{H}_2)} = ||M_h||$.

For two positive kernels k_1, k_2, we write $k_1 \succeq k_2$ if $k_1 - k_2 \succeq 0$. The following lemmas can be obtained from [13], which will be used in our proof.

LEMMA 2.1. ([13], Theorem 5.1) *Let $\mathscr{H}(k_1)$ and $\mathscr{H}(k_2)$ be reproducing kernel Hilbert spaces on X. Then*

$$\mathscr{H}(k_1) \subseteq \mathscr{H}(k_2)$$

if and only if there exists a constant $c > 0$ such that

$$k_1 \preceq c^2 k_2.$$

Moreover, $||f||_{\mathscr{H}(k_2)} \leqslant c||f||_{\mathscr{H}(k_1)}$ for all $f \in \mathscr{H}(k_1)$.

LEMMA 2.2. ([13], Theorem 5.4) *Let $\mathscr{H}(k_1)$ and $\mathscr{H}(k_2)$ be reproducing kernel Hilbert spaces on X and let $k = k_1 + k_2$. Then*

$$\mathscr{H}(k) = \{f = f_1 + f_2 : f_1 \in \mathscr{H}(k_1), f_2 \in \mathscr{H}(k_2)\}.$$

For every $f \in \mathscr{H}(k)$,

$$||f||^2_{\mathscr{H}(k)} = \min\{||f_1||^2_{\mathscr{H}(k_1)} + ||f_2||^2_{\mathscr{H}(k_2)} : f = f_1 + f_2, f_1 \in \mathscr{H}(k_1), f_2 \in \mathscr{H}(k_2)\}.$$

Let $\varphi : S \to X$ be a function. If k is a positive kernel on X, then $k \circ \varphi$ is also a kernel function on S (see e.g. [13, Proposition 5.6]).

LEMMA 2.3. ([13], Theorem 5.7) *Let $\varphi : S \to X$ be a function and let $k : X \times X \to \mathbb{C}$ be a positive kernel. Then*

$$\mathscr{H}(k \circ \varphi) = \{f \circ \varphi : f \in \mathscr{H}(k)\}.$$

Moreover, for $u \in \mathscr{H}(k \circ \varphi)$,

$$||u||_{\mathscr{H}(k\circ\varphi)} = \min\{||f||_{\mathscr{H}(k)} : u = f \circ \varphi\}.$$

LEMMA 2.4. ([13], Theorem 6.28) *Let $\mathscr{H}(k_1)$ and $\mathscr{H}(k_2)$ be reproducing kernel Hilbert spaces on X, let $\mathscr{F}$ be a Hilbert space and $\Phi : X \to \mathscr{B}(\mathscr{F}, \mathbb{C})$. The following are equivalent:*

1. *Φ is a contractive multiplier from $\mathscr{H}(k_1) \otimes \mathscr{F}$ to $\mathscr{H}(k_2)$;*
2. *$k_2(z,w) - k_1(z,w)\Phi(z)\Phi(w)^*$ is positive semi-definite.*

For the general inner function $\theta \in H^2(\mathbb{D}^2)$, it is well known that there exist two subsets $E_1, E_2 \subseteq \mathbb{T}$ of positive measure such that for each $\eta \in E_1$, $\theta(z_1, \eta)$ is a one-variable inner function, and for each $\zeta \in E_2$, $\theta(\zeta, z_2)$ is also one-variable inner function (see [15] for instance).

LEMMA 2.5. *Suppose $\theta(z_1, z_2) \in H^2(\mathbb{D}^2)$ is a nonconstant inner function, which is not one-variable inner function. A necessary condition for $\mathscr{K}_\theta$ to have complete Nevanlinna-Pick property is that there exist two subsets $E_1, E_2 \subseteq \mathbb{T}$ of positive measure such that for each $\eta \in \mathbb{D} \bigcup E_1$, $\theta(\cdot, \eta)$ is injective on $\mathbb{D}$, and for each $\zeta \in \mathbb{D} \bigcup E_2$, $\theta(\zeta, \cdot)$ is injective on $\mathbb{D}$.*

Proof. For the general inner function $\theta \in H^2(\mathbb{D}^2)$, if $\theta(0,0) = a \neq 0$, we can let $\Theta(z_1, z_2) = h_a(\theta(z_1, z_2))$, then $\Theta(0,0) = h_a(\theta(0,0)) = 0$, and

$$k^\Theta(z,w) = \overline{g(w_1, w_2)} g(z_1, z_2) k^\theta(z,w),$$

where $g(z_1, z_2) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}\theta(z_1,z_2)}$, so $k^\theta(z,w)$ is a complete Nevanlinna-Pick kernel if and only if $k^\Theta(z,w)$ is a complete Nevanlinna-Pick kernel. So we can assume that $\theta(0,0) = 0$.

Suppose $\theta(0,0)=0$, and $k^{\theta}(z,w)$ is a complete Nevanlinna-Pick kernel, we have $1-\frac{1}{k^{\theta}(z,w)}\succeq 0$, then

$$1-\frac{(1-\bar{w}_1z_1)(1-\bar{w}_2z_2)}{1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)}\succeq 0. \tag{2.1}$$

Rewrite (2.1) as

$$\frac{\bar{w}_1z_1+\bar{w}_2z_2-\bar{w}_1z_1\bar{w}_2z_2-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)}{1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)}\succeq 0. \tag{2.2}$$

Let

$$k_1(z,w)=\frac{\bar{w}_1z_1}{1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)},\quad k_2(z,w)=\frac{\bar{w}_2z_2}{1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)},$$
$$k_{12}(z,w)=\frac{\bar{w}_1z_1\bar{w}_2z_2}{1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)},\quad k_{\theta}(z,w)=\frac{\overline{\theta(w_1,w_2)}\theta(z_1,z_2)}{1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)},$$

then (2.1) holds if and only if $\mathscr{H}(k_1+k_2)\supseteq\mathscr{H}(k_{12}+k_{\theta})$, $||f||_{\mathscr{H}(k_1+k_2)}\leqslant||f||_{\mathscr{H}(k_{12}+k_{\theta})}$ for all $f\in\mathscr{H}(k_{12}+k_{\theta})$.

Next we identify the corresponding reproducing kernel Hilbert spaces. By Lemma 2.3, we notice that the corresponding reproducing kernel Hilbert space of the reproducing kernel

$$\frac{1}{1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)}$$

is

$$\begin{aligned}\mathscr{H}(s\circ\theta)&=\{f\circ\theta : f\in H^2(\mathbb{D})\}\\&=\left\{h(z_1,z_2)=\sum_{n=0}^{\infty}a_n(\theta(z_1,z_2))^n,\ \sum_{n=0}^{\infty}|a_n|^2<+\infty\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\mathscr{H}(k_i)=z_i\mathscr{H}(s\circ\theta),\ \ i=1,2,\\&\mathscr{H}(k_{12})=z_1z_2\mathscr{H}(s\circ\theta),\\&\mathscr{H}(k_{\theta})=\theta(z_1,z_2)\mathscr{H}(s\circ\theta).\end{aligned}$$

Then for every $f\in H^2(\mathbb{D})$, there exists $g_1,g_2\in H^2(\mathbb{D})$ such that

$$(z_1z_2+\theta(z_1,z_2))f(\theta(z_1,z_2))=z_1g_1(\theta(z_1,z_2))+z_2g_2(\theta(z_1,z_2)). \tag{2.3}$$

By taking $f=1$, then we also have $g_1,g_2\in H^2(\mathbb{D})$ such that

$$z_1z_2+\theta(z_1,z_2)=z_1g_1(\theta(z_1,z_2))+z_2g_2(\theta(z_1,z_2)). \tag{2.4}$$

Let $E_1\subseteq\mathbb{T}$ be a measurable subset of positive measure such that for each $\eta\in E_1$, $\theta(z_1,\eta)$ is a one-variable inner function. We will show that for each $\eta\in\mathbb{D}\bigcup E_1$,

$\theta(\cdot,\eta)$ is injective. Suppose that $\theta(\lambda^1,\eta)=\theta(\lambda^2,\eta)$ for some pair $\lambda^1\neq\lambda^2$, $\lambda^1,\lambda^2\in\mathbb{D}$. Putting $z_1=\lambda^1$, $z_2=\eta$ and $z_1=\lambda^2$, $z_2=\eta$ in (2.4), respectively, we obtain that

$$\begin{cases} \lambda^1\eta+\theta(\lambda^1,\eta)=\lambda^1 g_1(\theta(\lambda^1,\eta))+\eta g_2(\theta(\lambda^1,\eta)), \\ \lambda^2\eta+\theta(\lambda^2,\eta)=\lambda^2 g_1(\theta(\lambda^2,\eta))+\eta g_2(\theta(\lambda^2,\eta)). \end{cases}$$

Then we have

$$(\lambda^1-\lambda^2)\eta=(\lambda^1-\lambda^2)g_1(\theta(\lambda^1,\eta)).$$

Since $\lambda^1\neq\lambda^2$, we see that

$$g_1(\theta(\lambda^1,\eta))=\eta.$$

It follows from $\theta(\lambda^1,\eta)=\theta(\lambda^2,\eta)$ that for each $\tilde{\lambda}^1$ near to λ^1, there exists $\tilde{\lambda}^2$ near to λ^2 such that

$$\theta(\tilde{\lambda}^1,\eta)=\theta(\tilde{\lambda}^2,\eta),$$

and hence $g_1(\theta(\cdot,\eta))-\eta$ has to vanish at all points z_1 near to λ^1. This gives that for each $\eta\in\mathbb{D}\bigcup E_1$,

$$g_1(\theta(z_1,\eta))\equiv\eta,\ \forall z_1\in\mathbb{D}.$$

Combining with (2.4), we obtain that

$$g_1(\theta(z_1,z_2))=z_2,\quad \theta(z_1,z_2)=z_2 g_2(\theta(z_1,z_2))$$

for all $(z_1,z_2)\in\mathbb{D}^2$, and this gives that

$$\theta(z_1,0)=0, g_1(0)=0.$$

Let $E_2\subseteq\mathbb{T}$ be a measurable subset of positive measure such that for each $\zeta\in E_2$, $\theta(\zeta,z_2)$ is a one-variable inner function. Thus

$$g_1(\theta(\zeta,z_2))=z_2$$

for each $\zeta\in E_2$, this implies that $\theta(\zeta,\cdot)$ is injective on $\mathbb{D}$, and hence $\theta(\zeta,z_2)$ is Möbius map. Thus

$$\theta(\zeta,z_2)=\beta(\zeta)\frac{z_2-\alpha(\zeta)}{1-\overline{\alpha(\zeta)}z_2},$$

where $|\beta(\zeta)|=1$. Note that $\theta(\zeta,0)=0$, we obtain that

$$\theta(\zeta,z_2)=\beta(\zeta)z_2$$

for all $\zeta\in E_2$. By $\theta(\zeta,z_2)=z_2 g_2(\theta(\zeta,z_2))$ for all $z_2\in\mathbb{D}$, we have

$$g_2(\beta(\zeta)z_2)=\beta(\zeta),$$

this implies that $g_2=\beta(\zeta)\equiv\beta$ is a constant, and thus

$$\theta(z_1,z_2)=\beta z_2$$

is a one-variable inner function, which is a contradiction, and this proves that for each $\eta \in \mathbb{D}\bigcup E_1$, $\theta(\cdot,\eta)$ is injective on $\mathbb{D}$.

On the other hand, for each $\zeta \in \mathbb{D}\bigcup E_2$, $\theta(\zeta,\cdot)$ is also injective on $\mathbb{D}$. □

For $\eta \in E_2$, $\theta(\cdot,\eta)$ is injective, and is a one-variable inner function, thus $\theta(z_1,\eta)$ is a Möbius map. Also we have for $\zeta \in E_2$, $\theta(\zeta,z_2)$ is a Möbius map. Therefore, by Corollary B in [10], $\theta(z_1,z_2)$ is a rational inner function with degree at most $(1,1)$.

Therefore we proved the following theorem.

THEOREM 2.6. *Given an inner function $\theta \in H^2(\mathbb{D}^2)$, A necessary condition for $\mathscr{K}_\theta$ to have complete Nevanlinna-Pick property is that $\theta(z_1,z_2)$ is a rational function with degree at most $(1,1)$.*

In [10], Guo and Wang determined the following characterization of rational inner function with degree at most $(1,1)$.

LEMMA 2.7. ([10], Theorem 1.2) *Given an inner function $\theta \in H^2(\mathbb{D}^2)$, If θ is a rational inner function having degree at most (1,1). This turns out to be equivalent to that θ has one of the following forms:*

1. $\theta(z_1,z_2) = \beta h_a(z_1)$ *or* $\theta(z_1,z_2) = \beta h_a(z_2)$, *for some* $|a| < 1$, $|\beta| = 1$;
2. $\theta(z_1,z_2) = \beta h_a(z_1)h_b(z_2)$, *for some* $|a| < 1$, $|b| < 1$, $|\beta| = 1$;
3. $\theta(z_1,z_2) = \beta \frac{z_1z_2+az_1+bz_2+c}{1+\bar{a}z_2+\bar{b}z_1+\bar{c}z_1z_2}$ *for some* $|\beta| = 1$ *and* $c \neq ab$.

LEMMA 2.8. *Suppose $\theta(z_1,z_2) = \beta h_a(z_1)$ for some $|a| < 1$, $|\beta| = 1$. Then $\mathscr{K}_\theta$ has complete Nevanlinna-Pick property.*

Proof. If $\theta(z_1,z_2) = \beta h_a(z_1) = \beta \frac{a-z_1}{1-\bar{a}z_1}$, then it is easy to check that

$$k^\theta(z,w) = \frac{1-|a|^2}{(1-a\bar{w}_1)(1-\bar{a}z_1)(1-\bar{w}_2z_2)}$$

is a complete Nevanlinna-Pick kernel. □

LEMMA 2.9. *Suppose $\theta(z_1,z_2) = \beta h_a(z_1)h_b(z_2)$, for some $|a| < 1$, $|b| < 1$, $|\beta| = 1$. Then $\mathscr{K}_\theta$ does not have the complete Nevanlinna-Pick property.*

Proof. We can assume that $\theta(0,0) = 0$, so let $\theta(z_1,z_2) = \beta z_1 h_b(z_2)$. Suppose $k^\theta(z,w)$ is a complete Nevanlinna-Pick kernel, we have $1 - \frac{1}{k^\theta(z,w)} \succeq 0$, then

$$1 - \frac{(1-\bar{w}_1z_1)(1-\bar{w}_2z_2)}{1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2)} \succeq 0.$$

Rewrite it as

$$\frac{\bar{w}_1 z_1 + \bar{w}_2 z_2 - \bar{w}_1 z_1 \bar{w}_2 z_2 - \bar{w}_1 z_1 \frac{\bar{b}-\bar{w}_2}{1-b\bar{w}_2}\frac{b-z_2}{1-\bar{b}z_2}}{1-\bar{w}_1 z_1 \frac{\bar{b}-\bar{w}_2}{1-b\bar{w}_2}\frac{b-z_2}{1-\bar{b}z_2}} \succeq 0. \tag{2.5}$$

Thus a necessary condition for $k^{\theta}(z,w)$ to be a complete Nevanlinna-Pick kernel is that (2.5) holds.

Let $\mathscr{H} = \mathscr{H}\left(\frac{1}{1-\bar{w}_1 z_1 \frac{\bar{b}-\bar{w}_2}{1-b\bar{w}_2}\frac{b-z_2}{1-\bar{b}z_2}}\right)$. Indeed, it follows from the proof of Lemma 2.5 that

$$\mathscr{H} = \left\{ f(z_1,z_2) = \sum_{n=0}^{\infty} a_n \left(z_1 \frac{b-z_2}{1-\bar{b}z_2}\right)^n, \sum_{n=0}^{\infty} |a_n|^2 < +\infty \right\}.$$

We claim that

$$z_1, \quad \frac{b-z_2}{1-\bar{b}z_2} \notin \mathscr{H}.$$

If $z_1 \in \mathscr{H}$, then $z_1 = \sum_{n=0}^{\infty} a_n \left(z_1 \frac{b-z_2}{1-\bar{b}z_2}\right)^n$. It follows that $a_0 = 0$, and

$$1 = a_1 \frac{b-z_2}{1-\bar{b}z_2} + a_2 z_1 \left(\frac{b-z_2}{1-\bar{b}z_2}\right)^2 + \cdots,$$

which is a contradiction. If $\frac{b-z_2}{1-\bar{b}z_2} \in \mathscr{H}$, then

$$\frac{b-z_2}{1-\bar{b}z_2} = \sum_{n=0}^{\infty} b_n \left(z_1 \frac{b-z_2}{1-\bar{b}z_2}\right)^n.$$

By taking $z_2 = h_b(z_2)$, we get that $b_0 = 0$, and

$$1 = b_1 z_1 + b_2 z_1^2 z_2 + \cdots,$$

which is also a contradiction. Since $1 \in \mathscr{H}$, we get that z_1, $\frac{b-z_2}{1-\bar{b}z_2}$ are not multipliers of $\mathscr{H}$.

Let $w_2 = w_1$ and $z_2 = z_1$ in (2.5), then (2.5) becomes

$$\frac{2\bar{w}_1 z_1 - \bar{w}_1^2 z_1^2 - \bar{w}_1 z_1 \frac{\bar{b}-\bar{w}_1}{1-b\bar{w}_1}\frac{b-z_1}{1-\bar{b}z_1}}{1-\bar{w}_1 z_1 \frac{\bar{b}-\bar{w}_1}{1-b\bar{w}_1}\frac{b-z_1}{1-\bar{b}z_1}} = \bar{w}_1 z_1 \frac{2-\bar{w}_1 z_1 - \frac{\bar{b}-\bar{w}_1}{1-b\bar{w}_1}\frac{b-z_1}{1-\bar{b}z_1}}{1-\bar{w}_1 z_1 \frac{\bar{b}-\bar{w}_1}{1-b\bar{w}_1}\frac{b-z_1}{1-\bar{b}z_1}}.$$

By Lemma 2.4, we know that

$$\frac{1-\frac{1}{\sqrt{2}}\frac{\bar{b}-\bar{w}_1}{1-b\bar{w}_1}\frac{1}{\sqrt{2}}\frac{b-z_1}{1-\bar{b}z_1}}{1-\bar{w}_1 z_1 \frac{\bar{b}-\bar{w}_1}{1-b\bar{w}_1}\frac{b-z_1}{1-\bar{b}z_1}}, \quad \frac{1-\frac{\bar{w}_1}{\sqrt{2}}\frac{z_1}{\sqrt{2}}}{1-\bar{w}_1 z_1 \frac{\bar{b}-\bar{w}_1}{1-b\bar{w}_1}\frac{b-z_1}{1-\bar{b}z_1}}$$

is not positive, and therefore $\Phi(z_1) = (\frac{z_1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\frac{b-z_1}{1-\bar{b}z_1}) \notin \mathrm{Mult}_1(\mathscr{H}\otimes\mathbb{C}^2, \mathscr{H})$. It follows that

$$\frac{1-\frac{\bar{w}_1}{\sqrt{2}}\frac{z_1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{\bar{b}-\bar{w}_1}{1-b\bar{w}_1}\frac{1}{\sqrt{2}}\frac{b-z_1}{1-\bar{b}z_1}}{1-\bar{w}_1 z_1 \frac{\bar{b}-\bar{w}_1}{1-b\bar{w}_1}\frac{b-z_1}{1-\bar{b}z_1}}$$

is not positive, and hence (2.5) is not positive. Therefore $k^\theta(z,w)$ does not have complete Nevanlinna-Pick property. $\square$

LEMMA 2.10. *Suppose $\theta(z_1,z_2)\in H^2(\mathbb{D}^2)$ is an inner function. If $\mathscr{K}_\theta$ has complete Nevanlinna-Pick property, then for all $|\beta|=1$, $\mathscr{K}_{\beta\theta}$ also has complete Nevanlinna-Pick property.*

Proof. Indeed,

$$\begin{aligned} 1-\overline{\beta\theta(w_1,w_2)}\beta\theta(z_1,z_2) &= 1-|\beta|^2\overline{\theta(w_1,w_2)}\theta(z_1,z_2) \\ &= 1-\overline{\theta(w_1,w_2)}\theta(z_1,z_2). \end{aligned}$$

So $k^\theta(z,w)$ is a complete Nevanlinna-Pick kernel if and only if $k^{\beta\theta}(z,w)$ is a complete Nevanlinna-Pick kernel. $\square$

LEMMA 2.11. *Suppose $\theta(z_1,z_2)=\beta\frac{z_1z_2+az_1+bz_2+c}{1+\bar{a}z_2+\bar{b}z_1+\bar{c}z_1z_2}$ for some $|\beta|=1$ and $c\neq ab$. Then $\mathscr{K}_\theta$ does not have complete Nevanlinna-Pick property.*

Proof. We assume that $\theta(0,0)=0$, then we can let

$$c=0,\quad \theta(z_1,z_2)=\beta\frac{z_1z_2+az_1+bz_2}{1+\bar{a}z_2+\bar{b}z_1}.$$

Suppose $k^\theta(z,w)$ is a complete Nevanlinna-Pick kernel, then by proof of Lemma 2.5, we have that (2.4) holds, i.e. there exists $g_1,g_2\in H^2(\mathbb{D})$ such that

$$z_1z_2+\theta(z_1,z_2)=z_1g_1(\theta(z_1,z_2))+z_2g_2(\theta(z_1,z_2)).$$

If $z_1=0$, then we have $g_2(\frac{\beta bz_2}{1+\bar{a}z_2})=\frac{\beta b}{1+\bar{a}z_2}$, and thus $g_2(z_2)=\beta b-\bar{a}z_2$. On the other hand, if $z_2=0$, then we also have $g_1(\frac{\beta az_1}{1+\bar{b}z_1})=\frac{\beta a}{1+\bar{b}z_1}$, thus $g_1(z_1)=\beta a-\bar{b}z_1$. Therefore

$$z_1z_2+\theta(z_1,z_2)=z_1(\beta a-\bar{b}\theta(z_1,z_2))+z_2(\beta b-\bar{a}\theta(z_1,z_2)),$$

this implies that

$$\theta(z_1,z_2)=\beta\frac{-\bar{\beta}z_1z_2+az_1+bz_2}{1+\bar{a}z_2+\bar{b}z_1},$$

and thus $\beta=-1$.

A necessary condition for $k^{\theta}(z,w)$ to be a complete Nevanlinna-Pick kernel is that (2.4) holds, and then β must be -1. If $\theta(z_1,z_2) = -\frac{z_1z_2+az_1+bz_2}{1+\bar{a}z_2+\bar{b}z_1}$ such that $\mathscr{K}_\theta$ has complete Nevanlinna-Pick property, then by Lemma 2.10, we know that for all $|\alpha|=1$, $\mathscr{K}_{\alpha\theta}$ also has complete Nevanlinna-Pick property, which is a contradiction. Therefore, $\mathscr{K}_\theta$ does not have complete Nevanlinna-Pick property. □

The same example appears in the following Remarks.

REMARK 2.12. Suppose $\theta(z_1,z_2) = z_1\frac{a-z_2}{1-\bar{a}z_2}$ for $|a|<1$, then for every $f(z_1) \in H^2(\mathbb{D})$, there exists $g_1(z_1) = af(z_1), g_2(z_1) = \bar{a}z_1f(z_1) \in H^2(\mathbb{D})$ such that (2.3) holds. However, by Lemma 2.9, $\mathscr{K}_\theta$ does not have complete Nevanlinna-Pick property.

Thus, combining Lemma 2.8 with Lemma 2.9, Lemma 2.11, we obtain Theorem 1.1, which we restate as follows.

THEOREM 2.13. *Suppose $\theta(z_1,z_2) \in H^2(\mathbb{D}^2)$ is an inner function. Then the Beurling type quotient module $\mathscr{K}_\theta$ has complete Nevanlinna-Pick property if and only if $\theta(z_1,z_2) = \beta h_a(z_1)$ or $\theta(z_1,z_2) = \beta h_a(z_2)$, for some $|a|<1$, $|\beta|=1$.*

In [8], Chu characterized the de Branges-Rovnyak spaces $\mathscr{H}_\theta$ with complete Nevanlinna-Pick property. If θ is an inner function on $\mathbb{D}$, there are a large number of θ such that $H^2(\mathbb{D}) \ominus \theta H^2(\mathbb{D})$ has the complete Nevanlinna-Pick property. In [12], Luo and Zhu proved sub-Bergman Hilbert space

$$\mathscr{H}\left(\frac{1-\overline{\theta(w_1)}\theta(z_1)}{(1-\bar{w}_1z_1)^2}\right)$$

on $\mathbb{D}$ has the complete Nevanlinna-Pick property if and only if θ is a Möbius map. Theorem 1.1 implies that there be few $\mathscr{K}_\theta$ with complete Nevanlinna-Pick property, and we will see that $H^2(\mathbb{D}^2)$ does not have complete Nevanlinna-Pick property. It seems that the complete Nevanlinna-Pick property of the quotient spaces is closed relate to the Nevanlinna-Pick property of space which is sited.

REMARK 2.14. $H^2(\mathbb{D}^2)$ does not have the complete Nevanlinna-Pick property.

Proof. By definition, we need to check $1-(1-\bar{w}_1z_1)(1-\bar{w}_2z_2)$ is not positive semi-definite. Rewrite it as

$$\bar{w}_1z_1\left(1-\frac{1}{2}\bar{w}_2z_2\right)+\bar{w}_2z_2\left(1-\frac{1}{2}\bar{w}_1z_1\right). \tag{2.6}$$

Let $\alpha \in \mathbb{D}, |\alpha|^2 = \frac{1}{2}$. Since αz_i for $i=1,2$ is not a contractive multiplier in a constant space, by Lemma 2.4, we know that $1-\frac{1}{2}\bar{w}_iz_i$ is not positive. Let $z_2=z_1$ and $w_2=w_1$ in (2.6), then

$$\bar{w}_1z_1\left(1-\frac{1}{2}\bar{w}_2z_2\right)+\bar{w}_2z_2\left(1-\frac{1}{2}\bar{w}_1z_1\right) = 2\bar{w}_1z_1\left(1-\frac{1}{2}\bar{w}_1z_1\right)$$

is not positive, and hence (2.6) is not positive. Therefore $H^2(\mathbb{D}^2)$ does not have the complete Nevanlinna-Pick property. □

We will give two examples which can be calculated explicitly.

EXAMPLE 2.15. If $\theta(z_1,z_2) = z_1z_2$ or $\theta(z_1,z_2) = z_1^2$, then $\mathscr{K}_\theta$ does not have complete Nevanlinna-Pick property.

(1) By definition, we need to check that

$$1 - \frac{1}{k^\theta(z,w)} = \frac{\bar{w}_1z_1 + \bar{w}_2z_2 - 2\bar{w}_1z_1\bar{w}_2z_2}{1 - \bar{w}_1z_1\bar{w}_2z_2}$$

is not positive semi-definite. Rewrite this as

$$[\bar{w}_1z_1(1-\bar{w}_2z_2) + \bar{w}_2z_2(1-\bar{w}_1z_1)]\sum_{n=0}^{\infty}(\bar{w}_1z_1\bar{w}_2z_2)^n. \tag{2.7}$$

Note that for $i = 1,2,\ 1-\bar{w}_iz_i$ is not positive, and by the same argument as Remark 2.14, we know that

$$\bar{w}_1z_1(1-\bar{w}_2z_2) + \bar{w}_2z_2(1-\bar{w}_1z_1)$$

is not positive. Therefore (2.7) is not positive.

(2) By definition, we need to check that

$$1 - \frac{1}{k^\theta(z,w)} = \frac{\bar{w}_1z_1 + \bar{w}_2z_2 - \bar{w}_1z_1\bar{w}_2z_2 - \bar{w}_1^2z_1^2}{1 - \bar{w}_1^2z_1^2}$$

is not positive semi-definite. Rewrite this as

$$(\bar{w}_1z_1 + \bar{w}_2z_2)\sum_{n=0}^{\infty}(-1)^n(\bar{w}_1z_1)^n, \tag{2.8}$$

and since $\sum_{n=0}^{\infty}(-1)^n(\bar{w}_1z_1)^n$ is not positive, we know that (2.8) is not positive.

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