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BOUNDS FOR THE EIGENVALUES OF NON–MONIC OPERATOR POLYNOMIALS

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Abstract. In this paper, we give some new bounds for the eigenvalues of non-monic operator polynomials by applying several numerical radius inequalities to the Frobenius companion matrices of these polynomials. Our bounds refine the existing bounds for monic matrix polynomials.

1. Introduction

Let $B(H)$ be the space of all bounded linear operators on a complex Hilbert space H with inner product $\langle\cdot,\cdot\rangle$. For $A \in B(H)$, let $\sigma(A)$, $r(A)$, $\omega(A)$, and $\|A\|$, be the spectrum, the spectral radius, the numerical radius, and the operator norm of A, respectively. Let

$$P(z) = A_m z^m + A_{m-1} z^{m-1} + \cdots + A_1 z + A_0$$

be a non-monic operator polynomial of degree $m \geqslant 2$, with operator coefficients $A_i \in B(H)$ for $i = 0, 1, \cdots, m$, and A_m is a uniformly positive operator. The Frobenius companion matrix of $P(z)$ is the $m \times m$ operator matrix defined by

$$C(P) = \begin{bmatrix} -A_{m-1}A_m^{\frac{m-1}{1-m}} & -A_{m-2}A_m^{\frac{m-2}{1-m}} & \cdots & -A_1 A_m^{\frac{1}{1-m}} & -A_0 \\ A_m^{\frac{1}{1-m}} & 0 & \cdots & 0 & 0 \\ 0 & A_m^{\frac{1}{1-m}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m^{\frac{1}{1-m}} & 0 \end{bmatrix},$$

where $A_m^{\frac{1}{1-m}}$ denotes the $(1-m)$-th root of A_m. In particular, if $A_m = I$, then $P(z)$ is the monic operator polynomial. Here, I is the identity operator in $B(H)$.

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The Frobenius companion matrix establishes an important connection between matrix analysis and polynomial geometry. A scalar λ is called an *eigenvalue* of $P(z)$ if there exists a non-zero vector $x \in H$ such that $P(\lambda)x = 0$. It is clear that λ is an eigenvalue of the non-monic operator polynomial $P(z)$ if and only if $\lambda \in \sigma_p(C(P))$. Here, $\sigma_p(A)$ is the point spectrum of $A \in B(H)$. Hence, if $\lambda \in \sigma_p(C(P))$, then

$$|\lambda| \leqslant r(C(P)) \leqslant \omega(C(P)).$$

Computing and locating the eigenvalues of matrix polynomials, known as the polynomial eigenvalue problem, is a very important topic in scientific computation. In recent years, polynomial eigenvalue problems have received extensive attention and achieved fruitful results, see [2, 3, 8, 10, 12, 11, 13, 14, 15]. When the coefficients of the polynomial are generalized from matrices to operator we obtain the operator polynomial eigenvalue problem, which has also received much attention. However to our knowledge little or nothing has been published on bounds for eigenvalues of non-monic operator polynomials. In this paper, we give some new bounds for the eigenvalues of non-monic operator polynomials by applying several numerical radius inequalities to the Frobenius companion matrices of these polynomials. Our bounds refine the existing bounds for monic matrix polynomials.

2. Preliminaries

To prove our main results, we need the following well-known lemmas. The first lemma is the extension of Buzano inequality and it is introduced in [4].

LEMMA 1. ([4]) *Let $x, y, e \in H$ with $\|e\| = 1$ and $\alpha \in \mathbb{C} \setminus \{0\}$. Then*

$$|\langle x, e\rangle\langle e, y\rangle| \leqslant \frac{1}{|\alpha|}(\max\{1, |\alpha - 1|\}\|x\|\|y\| + |\langle x, y\rangle|).$$

In particular, if $\alpha = 2$, then the above inequality becomes the Buzano inequality ([5])

$$|\langle x, e\rangle\langle e, y\rangle| \leqslant \frac{1}{2}(\|x\|\|y\| + |\langle x, y\rangle|),$$

for any $x, y \in H$ and $e \in H$ with $\|e\| = 1$.

The next lemma is the upper bound for the numerical radius of the operator matrix $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$, which is based on Lemma 1.

LEMMA 2. *Let $X, Y \in B(H)$, $T = \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}$, and $\alpha \in \mathbb{C} \setminus \{0\}$. Then*

$$\omega^2(T) \leqslant \frac{\max\{1, |\alpha - 1|\}}{2|\alpha|} \max\{\|X^*X + YY^*\|, \|XX^* + Y^*Y\|\} + \frac{1}{|\alpha|} \max\{\omega(XY), \omega(YX)\}.$$

Proof. Define $N := \begin{pmatrix} XX^* & 0 \\ 0 & YY^* \end{pmatrix}$, $P := \begin{pmatrix} Y^*Y & 0 \\ 0 & X^*X \end{pmatrix}$, and $Q := \begin{pmatrix} XY & 0 \\ 0 & YX \end{pmatrix}$, to simplify notations. Then

$$T^2 = Q,\ P+N = \begin{pmatrix} XX^*+Y^*Y & 0 \\ 0 & X^*X+YY^* \end{pmatrix}. \tag{1}$$

$$T^*T = P,\ TT^* = N. \tag{2}$$

Now, for every unit vector $x \in H \oplus H$, and for $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned}
&|\alpha||\langle Tx,x\rangle|^2 \\
&= |\alpha||\langle Tx,x\rangle\langle x,T^*x\rangle| \\
&\leqslant \max\{1,|\alpha-1|\}\|Tx\|\|T^*x\| + |\langle Tx,T^*x\rangle| \quad \text{(by Lemma 1)} \\
&= \max\{1,|\alpha-1|\}\sqrt{\langle Tx,Tx\rangle\langle T^*x,T^*x\rangle} + |\langle T^2x,x\rangle| \\
&= \max\{1,|\alpha-1|\}\sqrt{\langle T^*Tx,x\rangle\langle TT^*x,x\rangle} + |\langle Qx,x\rangle| \quad \text{(by (1))} \\
&= \max\{1,|\alpha-1|\}\sqrt{\langle Px,x\rangle\langle Nx,x\rangle} + |\langle Qx,x\rangle| \quad \text{(by (2))} \\
&\leqslant \frac{\max\{1,|\alpha-1|\}}{2}(\langle Px,x\rangle + \langle Nx,x\rangle) + \omega(Q) \\
&\quad \text{(by arithematic-Geometric mean inequality and by definition of } \omega(Q)) \\
&= \frac{\max\{1,|\alpha-1|\}}{2}\langle (P+N)x,x\rangle + \max\{\omega(XY),\omega(YX)\} \\
&\leqslant \frac{\max\{1,|\alpha-1|\}}{2}\|P+N\| + \max\{\omega(XY),\omega(YX)\} \\
&= \frac{\max\{1,|\alpha-1|\}}{2}\max\{\|X^*X+YY^*\|,\|XX^*+Y^*Y\|\} + \max\{\omega(XY),\omega(YX)\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\omega^2(T) &= \sup\{|\langle Tx,x\rangle|^2 : x \in H \oplus H, \|x\| = 1\} \\
&\leqslant \frac{\max\{1,|\alpha-1|\}}{2|\alpha|}\max\{\|X^*X+YY^*\|,\|XX^*+Y^*Y\|\} \\
&\quad + \frac{1}{|\alpha|}\max\{\omega(XY),\omega(YX)\}. \quad \square
\end{aligned}$$

LEMMA 3. ([7]) *Let $H_1, H_2, \cdots, H_n$ be Hilbert spaces, and let $A_1 \in B(H_1)$, $A_2 \in B(H_2, H_1)$, $\cdots, A_n \in B(H_n, H_1)$. Then*

$$\omega\left(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\omega(A_1) + \sqrt{\omega^2(A_1) + \sum_{j=2}^{n}\|A_j\|^2}\right).$$

LEMMA 4. ([7]) *Let* $H_1, H_2, \cdots, H_n$ *be Hilbert spaces, and let* $A_1 \in B(H_1)$, $A_2 \in B(H_2, H_1), \cdots, A_n \in B(H_n, H_1)$. *Then*

$$\omega\left(\begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|A_1\| + \sqrt{\|\sum_{j=1}^{n} A_j A_j^*\|}\right).$$

LEMMA 5. ([6]) *Let* $A, B \in B(H)$, $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $r = 1, 2, 3, \cdots$. *Then*

$$\omega^2(A+B) \leqslant \omega^2(A) + \omega^2(B) + \frac{\max\{1, |\alpha - 1|\}}{|\alpha|}\beta + \frac{2}{|\alpha|}\omega(BA),$$

where

$$\beta = (\||A|^2 + |B^*|^2\|^r - \inf_{\|x\|=1} \varphi(x))^{1/r},$$
$$\varphi(x) = (\langle A^*Ax, x\rangle^{r/2} - \langle BB^*x, x\rangle^{r/2})^2.$$

Using Lemma 1, we give a generalized numerical radius inequality for the sum of two operators.

LEMMA 6. *Let* $A, B \in B(H)$ *and* $\alpha \in \mathbb{C} \setminus \{0\}$. *Then*

$$\omega(A+B) \leqslant \sqrt{\omega^2(A) + \omega^2(B) + \frac{2\max\{1, |\alpha - 1|\}}{|\alpha|}\|A\|\|B\| + \frac{2}{|\alpha|}\omega(B^*A)}.$$

Proof. For every unit vector $x \in H$, and for $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned}
&|\langle (A+B)x, x\rangle|^2 \\
&\leqslant (|\langle Ax, x\rangle| + |\langle Bx, x\rangle|)^2 \\
&= |\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 + 2|\langle Ax, x\rangle||\langle Bx, x\rangle| \\
&= |\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 + 2|\langle Ax, x\rangle||\langle x, Bx\rangle| \\
&\leqslant |\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 + \frac{2\max\{1, |\alpha - 1|\}}{|\alpha|}\|Ax\|\|Bx\| + \frac{2}{|\alpha|}|\langle Ax, Bx\rangle| \\
&\qquad \text{(by Lemma 1)} \\
&= |\langle Ax, x\rangle|^2 + |\langle Bx, x\rangle|^2 + \frac{2\max\{1, |\alpha - 1|\}}{|\alpha|}\|Ax\|\|Bx\| + \frac{2}{|\alpha|}|\langle B^*Ax, x\rangle| \\
&\leqslant \omega^2(A) + \omega^2(B) + \frac{2\max\{1, |\alpha - 1|\}}{|\alpha|}\|A\|\|B\| + \frac{2}{|\alpha|}\omega(B^*A).
\end{aligned}$$

Thus,

$$\begin{aligned}
\omega^2(A+B) &= \sup\{|\langle (A+B)x, x\rangle|^2 : x \in H, \|x\| = 1\} \\
&\leqslant \omega^2(A) + \omega^2(B) + \frac{2\max\{1, |\alpha - 1|\}}{|\alpha|}\|A\|\|B\| + \frac{2}{|\alpha|}\omega(B^*A),
\end{aligned}$$

and so

$$\omega(A+B) \leqslant \sqrt{\omega^2(A)+\omega^2(B)+\frac{2\max\{1,|\alpha-1|\}}{|\alpha|}\|A\|\|B\|+\frac{2}{|\alpha|}\omega(B^*A)}. \quad \square$$

REMARK 1. When $\alpha = 2$, the inequality in Lemma 6 becomes

$$\omega(A+B) \leqslant \sqrt{\omega^2(A)+\omega^2(B)+\|A\|\|B\|+\omega(B^*A)},$$

which is given in [1].

LEMMA 7. ([9]) *Let $M = [m_{ij}] \in M_n(\mathbb{C})$ such that $m_{ij} \geqslant 0$ for $i, j = 1,2,\cdots,m$. Then*

$$\omega(M) = \frac{1}{2}r([m_{ij}+m_{ji}]).$$

LEMMA 8. ([9]) *Let $T_n \in M_n(\mathbb{C})$ be the tridiagonal matrix given by*

$$T_n = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 \\ \vdots & \vdots\ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{1}{2} \\ 0 & 0 & \cdots & \frac{1}{2} & 0 \end{bmatrix}.$$

Then the eigenvalues of T_n are given by

$$\lambda_j = \cos\frac{\pi j}{n+1}, \quad j = 1,2,\cdots,n.$$

LEMMA 9. *Let L be the $m \times m$ block matrix given by*

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ A_m^{\frac{1}{1-m}} & 0 & \cdots & 0 & 0 \\ 0 & A_m^{\frac{1}{1-m}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m^{\frac{1}{1-m}} & 0 \end{bmatrix},$$

where $A_m^{\frac{1}{1-m}}$ is a uniformly positive operator. Then

$$\omega(L) = \|A_m^{\frac{1}{1-m}}\| \cos\left(\frac{\pi}{m+1}\right).$$

Proof. By applying Lemma 7, we have

$$\omega(L)=r\left(\begin{bmatrix} 0 & \frac{1}{2}A_m^{\frac{1}{1-m}} & 0 & \cdots & 0\\ \frac{1}{2}A_m^{\frac{1}{1-m}} & 0 & \frac{1}{2}A_m^{\frac{1}{1-m}} & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & \frac{1}{2}A_m^{\frac{1}{1-m}}\\ 0 & 0 & \cdots & \frac{1}{2}A_m^{\frac{1}{1-m}} & 0 \end{bmatrix}\right).$$

Note that

$$T=\begin{bmatrix} 0 & \frac{1}{2}A_m^{\frac{1}{1-m}} & 0 & \cdots & 0\\ \frac{1}{2}A_m^{\frac{1}{1-m}} & 0 & \frac{1}{2}A_m^{\frac{1}{1-m}} & \cdots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \cdots & 0 & \frac{1}{2}A_m^{\frac{1}{1-m}}\\ 0 & 0 & \cdots & \frac{1}{2}A_m^{\frac{1}{1-m}} & 0 \end{bmatrix}=T_m\otimes A_m^{\frac{1}{1-m}},$$

where T_m is the $m\times m$ tridiagonal matrix given in Lemma 8, and $\otimes$ is the Kronecker product of T_m and $A_m^{\frac{1}{1-m}}$. So by Lemma 8, we get

$$\begin{aligned}\sigma(T)&=\sigma(T_m)\bullet\sigma(A_m^{\frac{1}{1-m}})\\ &=\left\{\cos\frac{\pi}{m+1},\cdots,\cos\frac{m\pi}{m+1}\right\}\bullet\sigma(A_m^{\frac{1}{1-m}})\end{aligned}$$

Since $A_m^{\frac{1}{1-m}}$ is a uniformly positive operator, we have $\omega(L)=\|A_m^{\frac{1}{1-m}}\|\cos\left(\frac{\pi}{m+1}\right)$. □

3. Some new bounds for the eigenvalues of non-monic operator polynomials

Our first result refines and generalizes the inequality given by Jaradat and Kittaneh [9].

THEOREM 1. *Let $P(z)=A_mz^m+A_{m-1}z^{m-1}+\cdots+A_1z+A_0$ be a non-monic operator polynomial, where $A_i\in B(H)$, $i=0,1,\cdots,m$, A_m is a uniformly positive operator, and $C(P)$ is the Frobenius companion matrix of $P(z)$. Then for $\alpha\in\mathbb{C}\setminus\{0\}$, we have*

$$\omega(C(P))\leqslant\frac{1}{2}\left(\omega(A_{m-1}A_m^{-1})+\sqrt{\omega^2(A_{m-1}A_m^{-1})+\sum_{i=1}^{m-2}\|A_iA_m^{\frac{i}{1-m}}\|^2}\right)+\beta,$$

where

$$\beta=\sqrt{\frac{\max\{1,|\alpha-1|\}}{2|\alpha|}\zeta+\frac{1}{|\alpha|}\max\left\{\omega(A_0(A_m^{\frac{1}{1-m}})_{m-1}),\omega((A_m^{\frac{1}{1-m}})_{m-1}A_0)\right\}},$$

$$\zeta=\max\left\{\|A_0^*A_0+(A_m^{\frac{1}{1-m}})_{m-1}(A_m^{\frac{1}{1-m}})^*_{m-1}\|,\|A_0A_0^*+(A_m^{\frac{1}{1-m}})^*_{m-1}(A_m^{\frac{1}{1-m}})_{m-1}\|\right\}.$$

Proof. Using the triangle inequality of numerical radius, we have

$$\begin{aligned}
\omega(C(P)) &\leqslant \omega\left(\begin{bmatrix} -A_{m-1}A_m^{-1} & \cdots & -A_1A_m^{\frac{1}{1-m}} & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}\right) \\
&\quad + \omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & -A_0 \\ A_m^{\frac{1}{1-m}} & 0 & \cdots & 0 & 0 \\ 0 & A_m^{\frac{1}{1-m}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m^{\frac{1}{1-m}} & 0 \end{bmatrix}\right) \\
&\leqslant \frac{1}{2}\left(\omega(A_{m-1}A_m^{-1}) + \sqrt{\omega^2(A_{m-1}A_m^{-1}) + \sum_{i=1}^{m-2}\|A_iA_m^{\frac{i}{1-m}}\|^2}\right) \\
&\quad + \omega\left(\begin{bmatrix} 0_r & -A_0 \\ (A_m^{\frac{1}{1-m}})_{m-1} & 0_q \end{bmatrix}\right) \quad \text{(by Lemma 3)},
\end{aligned}$$

where $0_r = 0_q^t = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}_{1\times(m-1)}$ and

$$(A_m^{\frac{1}{1-m}})_{m-1} = \begin{pmatrix} A_m^{\frac{1}{1-m}} & 0 & \cdots & 0 \\ 0 & A_m^{\frac{1}{1-m}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m^{\frac{1}{1-m}} \end{pmatrix}_{(m-1)\times(m-1)}.$$

By using Lemma 2, we have

$$\omega(C(P)) \leqslant \frac{1}{2}\left(\omega(A_{m-1}A_m^{-1}) + \sqrt{\omega^2(A_{m-1}A_m^{-1}) + \sum_{i=1}^{m-2}\|A_iA_m^{\frac{i}{1-m}}\|^2}\right) + \beta,$$

where

$$\beta = \sqrt{\frac{\max\{1,|\alpha-1|\}}{2|\alpha|}\zeta + \frac{1}{|\alpha|}\max\left\{\omega(A_0(A_m^{\frac{1}{1-m}})_{m-1}), \omega((A_m^{\frac{1}{1-m}})_{m-1}A_0)\right\}},$$

$$\zeta = \max\left\{\|A_0^*A_0 + (A_m^{\frac{1}{1-m}})_{m-1}(A_m^{\frac{1}{1-m}})_{m-1}^*\|, \|A_0A_0^* + (A_m^{\frac{1}{1-m}})_{m-1}^*(A_m^{\frac{1}{1-m}})_{m-1}\|\right\}. \quad \square$$

The following corollary can be immediately obtained by Theorem 1.

COROLLARY 1. *Let $P(z)=z^m+A_{m-1}z^{m-1}+\cdots+A_1z+A_0$ be a monic matrix polynomial, where A_i, $i=0,1,\cdots,m-1$ are $n\times n$ complex matrices, and $C(P)$ is the Frobenius companion matrix of $P(z)$. Then for $\alpha\in\mathbb{C}\setminus\{0\}$, we have*

$$\omega(C(P))\leqslant\frac{1}{2}\left(\omega(A_{m-1})+\sqrt{\omega^2(A_{m-1})+\sum_{i=1}^{m-2}\|A_i\|^2}\right)+\beta,$$

where

$$\beta=\sqrt{\frac{\max\{1,|\alpha-1|\}}{2|\alpha|}\max\{\||A_0|^2+I\|,\||A_0^*|^2+I\|\}+\frac{1}{|\alpha|}\omega(A_0)}.$$

Proof. The proof follows by substituting $A_m=I$. □

REMARK 2. Let $\alpha=2$ in Corollary 1. Then we observe that

$$\omega(C(P))\leqslant\frac{1}{2}\left(\omega(A_{m-1})+\sqrt{\omega^2(A_{m-1})+\sum_{i=1}^{m-2}\|A_i\|^2}\right)+\beta,$$

where

$$\beta=\frac{1}{2}\sqrt{\max\{\||A_0|^2+I\|,\||A_0^*|^2+I\|\}+2\omega(A_0)}.$$

It should be mentioned that the above inequality refines the inequality [9],

$$\omega(C(P))\leqslant\frac{1}{2}\left(\omega(A_{m-1})+\sqrt{\omega^2(A_{m-1})+\sum_{i=1}^{m-2}\|A_i\|^2}\right)+\max\left\{1,\frac{\|A_0\|+1}{2}\right\}.$$

In fact, when $\max\{\||A_0|^2+I\|,\||A_0^*|^2+I\|\}=\||A_0|^2+I\|$, then we can obtain

$$\begin{aligned}\beta=\frac{1}{2}\sqrt{\||A_0|^2+1\|+2\omega(A_0)}&\leqslant\frac{1}{2}\sqrt{\||A_0|^2\|+1+2\omega(A_0)}\\&\leqslant\frac{1}{2}\sqrt{\|A_0\|^2+1+2\|A_0\|}\\&=\frac{1}{2}\sqrt{(\|A_0\|+1)^2}\\&=\frac{\|A_0\|+1}{2}.\end{aligned}$$

And when $\max\{\||A_0|^2+I\|,\||A_0^*|^2+I\|\}=\||A_0^*|^2+I\|$, the proof is similar to the above. Thus,

$$\begin{aligned}\omega(C(P))\leqslant&\frac{1}{2}\left(\omega(A_{m-1})+\sqrt{\omega^2(A_{m-1})+\sum_{i=1}^{m-2}\|A_i\|^2}\right)\\&+\frac{1}{2}\sqrt{\max\{\||A_0|^2+I\|,\||A_0^*|^2+I\|\}+2\omega(A_0)}\\\leqslant&\frac{1}{2}\left(\omega(A_{m-1})+\sqrt{\omega^2(A_{m-1})+\sum_{i=1}^{m-2}\|A_i\|^2}\right)+\max\left\{1,\frac{\|A_0\|+1}{2}\right\}.\end{aligned}$$

REMARK 3. (1) If λ is an eigenvalue of the non-monic operator polynomial $P(z)$, then by Theorem 1, we have

$$|\lambda| \leqslant \frac{1}{2}\left(\omega(A_{m-1}A_m^{-1}) + \sqrt{\omega^2(A_{m-1}A_m^{-1}) + \sum_{i=1}^{m-2}\|A_iA_m^{\frac{i}{1-m}}\|^2}\right) + \beta. \tag{3}$$

(2) If λ is an eigenvalue of the non-monic operator polynomial $P(z)$ and A_0 is an invertible operator, then $\frac{1}{\lambda}$ is an eigenvalue of the polynomial $Q(z)$ defined by

$$Q(z) = Iz^m + A_0^{-1}A_1z^{m-1} + \cdots + A_0^{-1}A_{m-1}z + A_0^{-1}A_m.$$

Hence, by Theorem 1, we have

$$\frac{1}{|\lambda|} \leqslant \frac{1}{2}\left(\omega(A_0^{-1}A_1) + \sqrt{\omega^2(A_0^{-1}A_1) + \sum_{i=2}^{m-1}\|A_0^{-1}A_i\|^2}\right) + \eta,$$

where

$$\eta = \sqrt{\frac{\max\{1,|\alpha-1|\}}{2|\alpha|}\max\{\||A_0^{-1}A_m|^2 + I\|, \||(A_0^{-1}A_m)^*|^2 + I\|\} + \frac{1}{|\alpha|}\omega(A_0^{-1}A_m)}.$$

Equivalently,

$$|\lambda| \geqslant \frac{2}{\omega(A_0^{-1}A_1) + \sqrt{\omega^2(A_0^{-1}A_1) + \sum_{i=2}^{m-1}\|A_0^{-1}A_i\|^2} + 2\eta}. \tag{4}$$

The upper bound (3) and the lower bound (4) determine an annulus in the complex plane containing all eigenvalues of the non-monic operator polynomial $P(z)$.

Following, we will give some new upper bounds of numerical radius inequalities of the Frobenius companion matrix $C(P)$ of the non-monic operator polynomial $P(z)$.

THEOREM 2. *Let $P(z) = A_mz^m + A_{m-1}z^{m-1} + \cdots + A_1z + A_0$ be a non-monic operator polynomial, where $A_i \in B(H)$, $i = 0,1,\cdots,m$, A_m is a uniformly positive operator, and $C(P)$ is the Frobenius companion matrix of $P(z)$. Then for $\alpha \in \mathbb{C}\setminus\{0\}$, we have*

$$\omega(C(P)) \leqslant \frac{1}{2}\left(\|A_{m-1}A_m^{-1}\| + \sqrt{\|\sum_{i=1}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^*\|}\right) + \beta,$$

where

$$\beta = \sqrt{\frac{\max\{1,|\alpha-1|\}}{2|\alpha|}\zeta + \frac{1}{|\alpha|}\max\left\{\omega(A_0(A_m^{\frac{1}{1-m}})_{m-1}), \omega((A_m^{\frac{1}{1-m}})_{m-1}A_0)\right\}},$$

$$\zeta = \max\left\{\|A_0^*A_0 + (A_m^{\frac{1}{1-m}})_{m-1}(A_m^{\frac{1}{1-m}})_{m-1}^*\|, \|A_0A_0^* + (A_m^{\frac{1}{1-m}})_{m-1}^*(A_m^{\frac{1}{1-m}})_{m-1}\|\right\}.$$

Proof. Using the triangle inequality, we have

$$\omega(C(P)) \leqslant \omega\left(\begin{bmatrix} -A_{m-1}A_m^{-1} & \cdots & -A_1 A_m^{\frac{1}{1-m}} & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}\right)$$

$$+\,\omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & -A_0 \\ A_m^{\frac{1}{1-m}} & 0 & \cdots & 0 & 0 \\ 0 & A_m^{\frac{1}{1-m}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m^{\frac{1}{1-m}} & 0 \end{bmatrix}\right)$$

$$\leqslant \frac{1}{2}\left(\|A_{m-1}A_m^{-1}\| + \sqrt{\|\sum_{i=1}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^*\|}\right)$$

$$+\,\omega\left(\begin{bmatrix} 0_r & -A_0 \\ (A_m^{\frac{1}{1-m}})_{m-1} & 0_q \end{bmatrix}\right) \quad \text{(by Lemma 4)},$$

where

$$0_r = 0_q^t = \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix}_{1\times(m-1)},\ (A_m^{\frac{1}{1-m}})_{m-1} = \begin{pmatrix} A_m^{\frac{1}{1-m}} & 0 & \cdots & 0 \\ 0 & A_m^{\frac{1}{1-m}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_m^{\frac{1}{1-m}} \end{pmatrix}_{(m-1)\times(m-1)}.$$

And using Lemma 2, we have

$$\omega(C(P)) \leqslant \frac{1}{2}\left(\|A_{m-1}A_m^{-1}\| + \sqrt{\|\sum_{i=1}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^*\|}\right) + \beta,$$

where

$$\beta = \sqrt{\frac{\max\{1,|\alpha-1|\}}{2|\alpha|}\zeta + \frac{1}{|\alpha|}\max\left\{\omega(A_0(A_m^{\frac{1}{1-m}})_{m-1}), \omega((A_m^{\frac{1}{1-m}})_{m-1}A_0)\right\}},$$

$$\zeta = \max\left\{\|A_0^*A_0 + (A_m^{\frac{1}{1-m}})_{m-1}(A_m^{\frac{1}{1-m}})_{m-1}^*\|, \|A_0A_0^* + (A_m^{\frac{1}{1-m}})_{m-1}^*(A_m^{\frac{1}{1-m}})_{m-1}\|\right\}. \quad \square$$

Now let $C(P) = A + L$, where

$$A = \begin{bmatrix} -A_{m-1}A_m^{\frac{m-1}{1-m}} & \cdots & -A_1A_m^{\frac{1}{1-m}} & -A_0 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$L = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ A_m^{\frac{1}{1-m}} & 0 & \cdots & 0 & 0 \\ 0 & A_m^{\frac{1}{1-m}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_m^{\frac{1}{1-m}} & 0 \end{bmatrix}.$$

Then we get the following results by applying the generalized numerical radius inequality for the sum of two operators.

THEOREM 3. *Let $P(z) = A_m z^m + A_{m-1} z^{m-1} + \cdots + A_1 z + A_0$ be a non-monic operator polynomial, where $A_i \in B(H)$, $i = 0, 1, \cdots, m$, A_m is a uniformly positive operator, and $C(P)$ is the Frobenius companion matrix of $P(z)$. Then for $\alpha \in \mathbb{C} \setminus \{0\}$ and $r = 1, 2, 3, \cdots$, we have*

$$\omega(C(P)) \leqslant \sqrt{\|A_m^{\frac{1}{1-m}}\|^2 \cos^2\left(\frac{\pi}{m+1}\right) + \frac{1}{4}\gamma_1^2 + \frac{\max\{1, |\alpha - 1|\}}{|\alpha|}\beta + \frac{1}{|\alpha|}\gamma_2},$$

where

$$\gamma_1 = \left(\omega(A_{m-1}A_m^{\frac{m-1}{1-m}}) + \sqrt{\omega^2(A_{m-1}A_m^{\frac{m-1}{1-m}}) + \sum_{i=0}^{m-2} \|A_i A_m^{\frac{i}{1-m}}\|^2}\right),$$

$$\gamma_2 = \left(\omega(A_{m-2}A_m^{-1}) + \sqrt{\omega^2(A_{m-2}A_m^{-1}) + \sum_{i=1}^{m-3} \|A_i A_m^{\frac{i+1}{1-m}}\|^2}\right),$$

$$\beta = (\| |A_m^{\frac{1}{1-m}}|^2 + \sum_{i=0}^{m-1} (A_i A_m^{\frac{i}{1-m}})(A_i A_m^{\frac{i}{1-m}})^* \|^r - \inf_{\|x\|=1} \varphi(x))^{1/r},$$

$$\varphi(x) = ((\sum_{i=1}^{m-1} \langle |A_m^{\frac{1}{1-m}}|^2 x_i, x_i \rangle)^{r/2} - \langle \sum_{i=0}^{m-1} (A_i A_m^{\frac{i}{1-m}})(A_i A_m^{\frac{i}{1-m}})^* x_1, x_1 \rangle^{r/2})^2.$$

Proof. Using the decomposition of $C(P) = A + L$, we have

$$L^*L = \begin{bmatrix} |A_m^{\frac{1}{1-m}}|^2 & 0 & \cdots & 0 & 0 \\ 0 & |A_m^{\frac{1}{1-m}}|^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & |A_m^{\frac{1}{1-m}}|^2 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$AA^* = \begin{bmatrix} \sum_{i=0}^{m-1} (A_i A_m^{\frac{i}{1-m}})(A_i A_m^{\frac{i}{1-m}})^* & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

$$L^*L+AA^* = \begin{bmatrix} |A_m^{\frac{1}{1-m}}|^2 + \sum_{i=0}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^* & 0 & \cdots & 0 & 0 \\ 0 & |A_m^{\frac{1}{1-m}}|^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & |A_m^{\frac{1}{1-m}}|^2 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{aligned} \|L^*L+AA^*\|^r &= \max\left\{ \left\| |A_m^{\frac{1}{1-m}}|^2 + \sum_{i=0}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^* \right\|, \left\| |A_m^{\frac{1}{1-m}}|^2 \right\| \right\}^r \\ &= \left\| |A_m^{\frac{1}{1-m}}|^2 + \sum_{i=0}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^* \right\|^r, \end{aligned}$$

and $\omega(L) = \|A_m^{\frac{1}{1-m}}\| \cos\left(\frac{\pi}{m+1}\right)$, (by Lemma 9)

$$AL = \begin{bmatrix} -A_{m-2}A_m^{-1} & -A_{m-3}A_m^{\frac{m-2}{1-m}} & \cdots & -A_1A_m^{\frac{2}{1-m}} & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned} &\omega(C(P)) \\ &= \omega(L+A) \\ &\leqslant \sqrt{\omega^2(L) + \omega^2(A) + \frac{\max\{1,|\alpha-1|\}}{|\alpha|}\beta + \frac{2}{|\alpha|}\omega(AL)} \quad \text{(by Lemma 5)} \\ &= \sqrt{\|A_m^{\frac{1}{1-m}}\|^2\cos^2\left(\frac{\pi}{m+1}\right) + \omega^2(A) + \frac{\max\{1,|\alpha-1|\}}{|\alpha|}\beta + \frac{2}{|\alpha|}\omega(AL)} \\ &\leqslant \sqrt{\|A_m^{\frac{1}{1-m}}\|^2\cos^2\left(\frac{\pi}{m+1}\right) + \frac{1}{4}\gamma_1^2 + \frac{\max\{1,|\alpha-1|\}}{|\alpha|}\beta + \frac{1}{|\alpha|}\gamma_2} \quad \text{(by Lemma 3)}, \end{aligned}$$

where

$$\begin{aligned} \gamma_1 &= \left(\omega(A_{m-1}A_m^{\frac{m-1}{1-m}}) + \sqrt{\omega^2(A_{m-1}A_m^{\frac{m-1}{1-m}}) + \sum_{i=0}^{m-2}\|A_iA_m^{\frac{i}{1-m}}\|^2} \right), \\ \gamma_2 &= \left(\omega(A_{m-2}A_m^{-1}) + \sqrt{\omega^2(A_{m-2}A_m^{-1}) + \sum_{i=1}^{m-3}\|A_iA_m^{\frac{i+1}{1-m}}\|^2} \right), \end{aligned}$$

$$\begin{aligned}\beta &= (\|L^*L+AA^*\|^r - \inf_{\|x\|=1}\varphi(x))^{1/r}\\ &= (\||A_m^{\frac{1}{1-m}}|^2 + \sum_{i=0}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^*\|^r - \inf_{\|x\|=1}\varphi(x))^{1/r},\\ \varphi(x) &= (\langle L^*Lx,x\rangle^{r/2} - \langle AA^*x,x\rangle^{r/2})^2\\ &= ((\sum_{i=1}^{m-1}\langle |A_m^{\frac{1}{1-m}}|^2x_i,x_i\rangle)^{r/2} - \langle \sum_{i=0}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^*x_1,x_1\rangle^{r/2})^2. \quad \square\end{aligned}$$

THEOREM 4. *Let $P(z)=A_mz^m+A_{m-1}z^{m-1}+\cdots+A_1z+A_0$ be a non-monic operator polynomial, where $A_i\in B(H)$, $i=0,1,\cdots,m$, A_m is a uniformly positive operator, and $C(P)$ is the Frobenius companion matrix of $P(z)$. Then for $\alpha\in\mathbb{C}\setminus\{0\}$, we have*

$$\omega(C(P)) \leqslant \sqrt{\frac{1}{4}\gamma^2 + \|A_m^{\frac{1}{1-m}}\|^2\cos^2\left(\frac{\pi}{m+1}\right) + \frac{2\max\{1,|\alpha-1|\}}{|\alpha|}\varepsilon},$$

where

$$\gamma = \left(\omega(A_{m-1}A_m^{-1}) + \sqrt{\omega^2(A_{m-1}A_m^{-1}) + \sum_{i=1}^{m-2}\|A_iA_m^{\frac{i}{1-m}}\|^2}\right),$$

and

$$\varepsilon = \sqrt{m-1}\|A_m^{\frac{1}{1-m}}\|\left(\sum_{i=0}^{m-1}\|A_iA_m^{\frac{i}{1-m}}\|^2\right)^{1/2}.$$

Proof. Since $C(P)=A+L$ and $L^*A=0$, we have

$$\begin{aligned}&\omega(C(P))\\ &= \omega(A+L)\\ &\leqslant \sqrt{\omega^2(A)+\omega^2(L)+\frac{2\max\{1,|\alpha-1|\}}{|\alpha|}\|A\|\|L\|} \quad \text{(by Lemma 6)}\\ &\leqslant \sqrt{\left(\frac{1}{2}\gamma\right)^2+\omega^2(L)+\frac{2\max\{1,|\alpha-1|\}}{|\alpha|}\|A\|\|L\|} \quad \text{(by Lemma 3)}\\ &\leqslant \sqrt{\frac{1}{4}\gamma^2+\|A_m^{\frac{1}{1-m}}\|^2\cos^2\left(\frac{\pi}{m+1}\right)+\frac{2\max\{1,|\alpha-1|\}}{|\alpha|}\varepsilon}. \quad \square\end{aligned}$$

Applying Lemma 4 easily yields the following theorem.

THEOREM 5. *Let $P(z)=A_mz^m+A_{m-1}z^{m-1}+\cdots+A_1z+A_0$ be a non-monic operator polynomial, where $A_i\in B(H)$, $i=0,1,\cdots,m$, A_m is a uniformly positive operator, and $C(P)$ is the Frobenius companion matrix of $P(z)$. Then*

$$\omega(C(P)) \leqslant \frac{1}{2}\left(\|A_{m-1}A_m^{-1}\| + \sqrt{\|\sum_{i=0}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^*\|}\right) + \|A_m^{\frac{1}{1-m}}\|\cos\frac{\pi}{m+1}.$$

Proof. Since $C(P)=A+L$, and by applying the triangle inequality and Lemma 4 to $C(P)$, we have

$$\begin{aligned}\omega(C(P)) &\leqslant \omega(A)+\omega(L)\\ &\leqslant \frac{1}{2}\left(\|A_{m-1}A_m^{-1}\|+\sqrt{\|\sum_{i=0}^{m-1}(A_iA_m^{\frac{i}{1-m}})(A_iA_m^{\frac{i}{1-m}})^*\|}\right)+\|A_m^{\frac{1}{1-m}}\|\cos\frac{\pi}{m+1}. \quad \square\end{aligned}$$

Next, we will give an example to show that in some special cases the upper bound of our conclusion is smaller than that obtained in [8, 9].

EXAMPLE 1. Consider the 2×2 monic matrix polynomial $P(z)=Iz^2-A_1z-A_0$ whose coefficient matrices are given by

$$A_1=\begin{bmatrix}2&0\\0&2\end{bmatrix},\quad A_0=\begin{bmatrix}2&3\\5&0\end{bmatrix}.$$

The upper bounds for the eigenvalues of this polynomial $P(z)$ estimated by different mathematicians are given in Table 1.

Result	Upper bound
Jaradat and Kittaneh [9]	5.2687
Higham and Tisseur [8]	5.9717

Table 1: *Upper bounds for other mathematicians*

If λ is a eigenvalue of the polynomial $P(z)=Iz^2-A_1z-A_0$, then Theorem 1 and 2 give $|\lambda|\leqslant 5.2368$, Theorem 3 gives $|\lambda|\leqslant 5.2158$, Theorem 4 gives $|\lambda|\leqslant 2.8526$, and Theorem 5 gives $|\lambda|\leqslant 4.4437$ which are better than the estimates mentioned in Table 1.

Finally, we remark that the lower bound counterpart of the upper bound obtained in this section can be derived by considering the polynomial $A_0^{-1}z^mP(\frac{1}{z})$, whose eigenvalues are the reciprocal of the non-monic operator polynomial $P(z)$. This enables us to describe a new annulus containing the eigenvalues of the non-monic operator polynomial $P(z)$.

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