Operators and Matrices
Volume 18, Number 3 (2024), 693–696
doi:10.7153/oam-2024-18-41

CONTINUITY BETWEEN SOME SPECIAL KÖTHE SPACES

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(*Communicated by R. Curto*)

Abstract. In this work, we gave some necessary and sufficient conditions for a special lower triangular map from a nuclear Köthe space to a nuclear G_1-space to be linear and continuous. Also, we considered its upper triangular version.

1. Preliminaries

Suppose that $A = (a_n^k)_{n,k\in\mathbb{N}}$ is a Köthe matrix, that is, A satisfies $0 \leqslant a_n^k \leqslant a_n^{k+1}$ for all $n,k \in \mathbb{N}$ and $\sup\limits_k a_n^k > 0$ for all $n \in \mathbb{N}$. The ℓ^1-Köthe space $\lambda(A)$ is the space defined by $\lambda(A) = \{x = (x_n) : ||x||_k = \sum\limits_{n\in\mathbb{N}} |x_n| a_n^k < \infty$ for all $k \in \mathbb{N}\}$ and the topological dual of $\lambda(A)$ is isomorphic to the space of all sequences $u = (u_n)$ for which $|u_n| \leqslant C a_n^k$ for some k and $C > 0$.

The ℓ^1-Köthe space $\lambda(A)$ is a Fréchet space with the topology generated by the system of seminorms $\{||.||_k : k \in \mathbb{N}\}$.

We know that $\lambda(A)$ is a nuclear space if and only if for each $k \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $\sum\limits_{n=1}^{\infty} \dfrac{a_n^k}{a_n^m} < \infty$.

If $\lambda(A)$ is nuclear, then the fundamental system of seminorms $||x||_k = \sum\limits_{n\in\mathbb{N}} |x_n| a_n^k$ can be replaced by the equivalent system of seminorms $||x||_k = \sup\limits_{n\in\mathbb{N}} |x_n| a_n^k$.

The space ℓ^1 is a Köthe space $\lambda(A)$ where $A = (a_n^k)$ with $a_n^k = 1$ for all $n,k \in \mathbb{N}$.

The space of all sequences $w = \mathbb{K}^{\mathbb{N}}$ is a Köthe space $\lambda(A)$ where $A = (a_n^k)$ with $a_n^k = e_1 + e_2 + \ldots + e_k = (1,1,\ldots,1,0,0,0,\ldots)$ (here 1 ends in the k^{th} place).

Let $(\alpha_n)_n$ be a monotonically increasing sequence of positive numbers with limit ∞ as n goes to ∞. Then the Köthe spaces given by the matrices $(e^{k\alpha_n})$ and $(e^{-\frac{\alpha_n}{k}})$ are called the infinite and finite type power series spaces, respectively. Power series spaces play an important role in the structure theory of spaces of entire functions on $\mathbb{C}$ and spaces of holomorphic functions on the unit disc.

Mathematics subject classification (2020): 47B37, 46A45.
Keywords and phrases: ℓ^1-Köthe spaces, nuclear spaces.

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Paper OaM-18-41

Smooth sequence spaces G_∞ and G_1 appear in [4] and they are generalizations of power series spaces.

The ℓ^1-Köthe space $\lambda(A)$ is called a G_∞ space if its Köthe matrix $A=(a_n^k)_{n,k\in\mathbb{N}}$ satisfies:

$$a_n^1 = 1, \quad a_n^k \leqslant a_{n+1}^k \qquad \text{for all} \ \ k,n\in\mathbb{N};$$

for all k there exists j such that $(a_n^k)^2 = O(a_n^j)$.

The ℓ^1-Köthe space $\lambda(B)$ is called a G_1 space if its Köthe matrix $B=(b_n^k)_{n,k\in\mathbb{N}}$ satisfies:

$$0 < b_{n+1}^k \leqslant b_n^k < 1 \qquad \text{for all} \ \ k,n\in\mathbb{N};$$

for all k there exists j such that $b_n^k = O((b_n^j)^2)$.

Finally, by [1], we know that a linear map T between the ℓ^1-Köthe spaces $\lambda(A)$ and $\lambda(B)$ is continuous if and only if for all k there is m such that $\sup_n \dfrac{||Te_n||_k}{||e_n||_m} < \infty$.

For the terminology used, but not defined here we refer to [2], [3] and [4].

The aim in this work is to generalize the results given in [5].

2. Continuity between special Köthe spaces

THEOREM 1. *Let $\lambda(A)$ be a nuclear Köthe space and $\lambda(B)$ be a nuclear G_1-Köthe space. Let $T:\lambda(A)\to\lambda(B)$ be a linear map given by the lower triangular matrix $C=(c_j^n)_{j,n\in\mathbb{N}}$ where $((c_n^n)^{-1})_{n\in\mathbb{N}}$ is a bounded sequence.*

Then T is continuous if and only if for all n $(c_j^n)_j\in\lambda(B)$ for $j\geqslant n$ and $\lambda(A)\subset\lambda(B)$.

Proof. Notice

$$Te_n = \begin{cases} c_j^n, & j\geqslant n \\ 0, & j<n \end{cases};$$

$$||Te_n||_k = \sup_{j\geqslant n}|c_j^n|b_j^k \quad \text{and} \quad ||e_n||_m = a_n^m.$$

Suppose that T is continuous. Then for all k there exists m such that $\sup_n \dfrac{||Te_n||_k}{||e_n||_m} < \infty$ i.e. for all k there exist m and $\rho>0$ such that $\sup_{j\geqslant n}|c_j^n|b_j^k \leqslant \rho a_n^m$ for all $n\in\mathbb{N}$.

By taking $j=n$, we get that $|c_n^n|b_n^k\leqslant\rho a_n^m$ for all $n\in\mathbb{N}$.

Since $((c_n^n)^{-1})$, $n\in\mathbb{N}$ is a bounded sequence, this means $b_n^k\leqslant Ca_n^m$ for all $n\in\mathbb{N}$ for some $C>0$ and hence $\lambda(A)\subset\lambda(B)$.

Notice that $Te_n=(0,0,\ldots,0,c_n^n,c_{n+1}^n,c_{n+2}^n,\ldots)\in\lambda(B)$ for all n. Since $(c_j^n)_j$ is a subsequence of Te_n for $j\geqslant n$, we get $(c_j^n)_j\in\lambda(B)$ for $j\geqslant n$.

Conversely, suppose that for all n $(c_j^n)_j\in\lambda(B)$ for $j\geqslant n$ and $\lambda(A)\subset\lambda(B)$.

So, since B is a G_1-set and $\lambda(A) \subset \lambda(B)$ for all k there exist $m_1 = m_1(k)$, $C_1 > 0$, $m_2 = m_2(m_1)$, $C_2 > 0$ such that

$$\begin{aligned} ||Te_n||_k &= \sup_{j \geqslant n} |c_j^n| b_j^k \leqslant C_1 \sup_{j \geqslant n} |c_j^n| (b_j^{m_1})^2 \\ &\leqslant C_1 (\sup_{j \geqslant n} |c_j^n| b_j^{m_1}) b_n^{m_1} \\ &\leqslant C_2 C_1 (\sup_{j \geqslant n} |c_j^n| b_j^{m_1}) a_n^{m_2} \leqslant C a_n^{m_2} \end{aligned}$$

for some $C > 0$. So, for all k there exists m_2 such that $\sup_n \frac{||Te_n||_k}{||e_n||_{m_2}} < \infty$, i.e. $T : \lambda(A) \to \lambda(B)$ is continuous. $\square$

THEOREM 2. *Let $\lambda(A)$ be a nuclear G_∞-Köthe space and $\lambda(B)$ be a nuclear Köthe space. Let $T : \lambda(A) \to \lambda(B)$ be a linear map given by the upper triangular matrix $C = (c_j^n)_{j,n \in \mathbb{N}}$ where $((c_n^n)^{-1})_{n \in \mathbb{N}}$ is a bounded sequence.*

Then T is continuous if and only if for all n $(c_j^n)_n \in \lambda(A)'$ for $j \leqslant n$ and $\lambda(A) \subset \lambda(B)$.

Proof. Notice

$$Te_n = \begin{cases} c_j^n, & j \leqslant n \\ 0, & j > n \end{cases};$$

$$||Te_n||_k = \sup_{1 \leqslant j \leqslant n} |c_j^n| b_j^k \quad \text{and} \quad ||e_n||_m = a_n^m.$$

Suppose that T is continuous. Then for all k there exist m and $\rho > 0$ such that $\sup_{1 \leqslant j \leqslant n} |c_j^n| b_j^k \leqslant \rho a_n^m$ for all $n \in \mathbb{N}$, in particular $|c_j^n| b_j^k \leqslant \rho a_n^m$ for all $n \in \mathbb{N}$ and for all $j = 1, 2, \ldots, n$.

So, we have $|c_j^n| \leqslant C a_n^m$, i.e. $(c_j^n)_n \in \lambda(A)'$ for $j \leqslant n$.

Now choose $j = n$. Then for all k there exist m and $\rho > 0$ such that $|c_n^n| b_n^k \leqslant \rho a_n^m$ for all n. Since $((c_n^n)^{-1})$, $n \in \mathbb{N}$ is a bounded sequence, we have that for all k there exist m and $\tilde{C} > 0$ such that $b_n^k \leqslant \tilde{C} a_n^m$ for all $n \in \mathbb{N}$ and hence $\lambda(A) \subset \lambda(B)$.

Conversely, suppose $(c_j^n)_n \in \lambda(A)'$ and $\lambda(A) \subset \lambda(B)$ for all $j \leqslant n$.

So there exist m_1, $C_1 > 0$ for all k there exist $m_2 = m_2(k)$, $C_2 > 0$ such that

$$\begin{aligned} ||Te_n||_k &= \sup_{1 \leqslant j \leqslant n} |c_j^n| b_j^k \leqslant C_1 \sup_{1 \leqslant j \leqslant n} a_n^{m_1} b_j^k \\ &\leqslant C_2 C_1 \sup_{1 \leqslant j \leqslant n} a_n^{m_1} a_j^{m_2} \\ &\leqslant C_2 C_1 a_n^{m_1} a_n^{m_2} \leqslant C_2 C_1 (a_n^m)^2 \end{aligned}$$

where $m = \max\{m_1, m_2\}$ since A is a G_∞-set.

Also, since $\lambda(A)$ is a G_∞-space, there exists $j = j(m)$ such that $\sup_n \frac{(a_n^m)^2}{a_n^j} < \infty$, i.e. there exist $j = j(m)$ and $\tilde{C} > 0$ such that $(a_n^m)^2 \leqslant \tilde{C} a_n^j$. Thus, for all k there exist j and $\tilde{C} > 0$ such that $||Te_n||_k \leqslant \tilde{C}C_2C_1a_n^j$, i.e. $\sup_n \frac{||Te_n||_k}{||e_n||_j} < \infty$. Hence, T is continuous. $\square$

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(Received December 7, 2021)

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