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WEIGHTED SUBSEQUENTIAL ERGODIC THEOREMS ON ORLICZ SPACES

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Abstract. For a semifinite von Neumann algebra M, individual convergence of subsequential, $\mathscr{Z}(M)$ (center of M) valued weighted ergodic averages is studied in non commutative Orlicz spaces. In the process, we also derive a maximal ergodic inequality corresponding to such averages in noncommutative L^p $(1 \leqslant p < \infty)$ spaces using the weak $(1,1)$ inequality obtained by Yeadon.

1. Introduction

The connection between ergodic theory and von Neumann algebra dates back to the very inception of theory of operator algebra. The study of pointwise ergodic theorems plays a center role in classical ergodic theory and has a very deep connection with statistical physics as well. However, the study of analogous ergodic theorems in the non commutative settings originated only in the pioneering work of Lance [17] in 1976. After that the theory flourished and many authors extended the results of Lance to various directions. We refer here to [3], [13], [15] and the references therein.

Yeadon [23] first studied the ergodic theorems in the predual of a semifinite von Neumann algebra. He proved a maximal ergodic theorem in noncommutative L^1 space, which still appears frequently in modern proofs of noncommutative ergodic theorems. The corresponding maximal ergodic theorem is extended to noncommutative L^p $(1 < p < \infty)$ space in the celebrated work [14]. Also as a consequence the analogous individual ergodic theorems are proved in the same article.

On the other hand an alternative approach solely based on Yeadon's weak $(1,1)$ inequality was opted by various authors to prove various individual ergodic theorems on non commutative L^p spaces. In [18], the author introduced the notion of noncommutative uniform continuity and bilateral uniform continuity in measure at zero and provided an alternative proof of the individual ergodic theorems from [14]. Several attempts has been made since then to improve the results. One natural generalisation is towards the proof of subsequential ergodic theorems. In [19], first attempt was made to prove an individual ergodic theorem along the so called uniform sequence in the von Neumann algebra setting. Simultaneously weighted ergodic theorems also became an

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interesting area of research. In [7], the authors studied the convergence of standard ergodic averages for actions of free groups and also for the weighted averages. Several other related works are available in the literature. The reader may look into [[1], [2], [11], [12]] and the references therein.

Another extension of these results which has been studied extensively is in the realm of symmetric spaces, in particular, the Orlicz spaces. It is known that the class of Orlicz spaces is significantly wider than the class of L^p spaces. The first account of study of individual ergodic theorems in the case of noncommutative Orlicz spaces is found in [6]. In [5], ergodic theorems for weighted averages are studied in fully symmetric spaces.

In this article we study various ergodic theorems associated with (vector valued) weighted ergodic averages along some special subsequences in noncommutative Orlicz spaces. Before this, ergodic averages with vector valued weights has been studied in [4]. Very recently, in [20], the author studied convergence of (scalar) weighted ergodic averages along subsequences in noncommutative L^p $(1 \leqslant p < \infty)$ spaces.

Our aim in this article is to establish an individual ergodic theorem for positive Dunford- Schwartz operator (see Definition 2.12) with von Neumann algebra valued Besicovitch weighted (see Definition 3.1) ergodic averages along subsequence of density one in Orlicz spaces (see Theorem 3.15). Our proof essentially based upon the notion of bilateral uniform continuity in measure for normed linear spaces.

Now we describe the layout of this article. In §2, we collect all the materials which are essential for this article. In particular, we recall some basic facts about von Neumann algebras M with faithful normal semifinite trace τ and space of τ- measurable operators. We also discuss a few topologies on this space. After that, we recollect the definition of non commutative Orlicz spaces and some of its properties which are essential for this article. We also define Dunford Schwartz operators and bilaterally uniformly equicontinuity in measure (b.u.e.m) at zero of sequences and end this section with the recollection of few important theorems regarding this. §3 begins with the appropriate definition of subsequential weighted ergodic averages. Then we prove a suitable form of maximal ergodic inequality and use it to prove that sequence of averages under study is b.u.e.m at zero, which essentially helps us to obtain a convergence result in $L^1 \cap M$. Finally our main result is achieved.

2. Preliminaries

Throughout this article we assume that M is a semifinite von Neumann algebra with faithful, normal, semifinite (f.n.s.) trace τ represented on a separable Hilbert space $\mathscr{H}$. Let $\mathscr{P}(M)$ (resp. $\mathscr{P}_0(M)$) denotes the collection of all (resp. non-zero) projections in the von Neumann algebra M. For each $e \in \mathscr{P}(M)$ we assign $e^{\perp}$ for the projection $1-e$, where 1 denotes the identity element of M.

Let $\mathscr{B}(\mathscr{H})$ denotes the space of all bounded operators of the Hilbert space $\mathscr{H}$. A closed densely defined operator $x : \mathscr{D}_x \subseteq \mathscr{H} \to \mathscr{H}$ is called affiliated to a M if $y'x \subseteq xy'$ for all $y' \in M'$, where M' denotes the commutant of M which is a von Neumann algebra by its own right. Equivalently, one can define x to be affiliated to M if $u'x = xu'$ holds for all unitary u' in M'. When x is affiliated to M, it is denoted by $x\eta M$. The center of the von Neumann algebra M is defined by $M \cap M'$ and it is denoted by $\mathscr{Z}(M)$.

Now we recall that for two positive, self-adjoint operators x, y defined on $\mathscr{H}$, $x \leqslant y$ is defined as: $\mathscr{D}_y \subseteq \mathscr{D}_x$ and $\|x^{1/2}\xi\|^2 \leqslant \|y^{1/2}\xi\|^2$ for all $\xi \in \mathscr{D}_y$.

PROPOSITION 2.1. *Let x be a positive, self-adjoint operator affiliated to M and $z \in \mathscr{Z}(M)_+$ be such that $z \leqslant C$ for some constant $C > 0$. Then $0 \leqslant zx \leqslant Cx$.*

Proof. First observe that $\mathscr{D}_{zx} = \mathscr{D}_x \subseteq \mathscr{D}_{x^{1/2}}$. Also, $\mathscr{D}_{zx} \subseteq \mathscr{D}_{(zx)^{1/2}}$. Let $\xi \in \mathscr{D}_x$. Then

$$\begin{aligned}
\left\|(zx)^{1/2}\xi\right\|^2 = \langle zx\xi, \xi\rangle &= \langle xz\xi, \xi\rangle \quad (\text{since } zx \subset xz)\\
&= \left\langle x^{1/2}z\xi, x^{1/2}\xi\right\rangle \quad (\text{since } \xi \in \mathscr{D}_{x^{1/2}})\\
&= \left\langle zx^{1/2}\xi, x^{1/2}\xi\right\rangle \quad (\text{since } x\eta M)\\
&\leqslant C\left\|x^{1/2}\xi\right\|^2. \quad \square
\end{aligned}$$

A closed, densely defined operator x affiliated to M is said to be τ-measurable if for every $\varepsilon > 0$ there is a projection e in M such that $e\mathscr{H} \subseteq \mathscr{D}_x$ and $\tau(e^\perp) < \varepsilon$. The set of all τ-measurable operators associated to M is denoted by $L^0(M, \tau)$ or simply L^0. For all $\varepsilon, \delta > 0$, let us define the following neighborhoods of zero.

$$\mathscr{N}(\varepsilon, \delta) := \{x \in L^0 : \exists\, e \in \mathscr{P}(M) \text{ such that } \|xe\| \leqslant \varepsilon \text{ and } \tau(e^\perp) \leqslant \delta\}, \text{ and}$$
$$\mathscr{N}'(\varepsilon, \delta) := \{x \in L^0 : \exists\, e \in \mathscr{P}(M) \text{ such that } \|exe\| \leqslant \varepsilon \text{ and } \tau(e^\perp) \leqslant \delta\}.$$

It is established in [7, Theorem 2.2] that the families $\{\mathscr{N}(\varepsilon, \delta) : \varepsilon > 0, \delta > 0\}$ and $\{\mathscr{N}'(\varepsilon, \delta) : \varepsilon > 0, \delta > 0\}$ generate same topology on L^0, and it is termed as measure topology in the literature. It is also well-known that L^0 becomes a complete, metrizable topological $*$-algebra with respect to the measure topology containing M as a dense subspace [see [10, Theorem 4.12]].

In this article, we also deal with so called almost uniform (a.u) and bilateral almost uniform (b.a.u) convergence of sequences in L^0. We describe it in the following definition.

DEFINITION 2.2. A sequence of operators $\{x_n\}_{n\in\mathbb{N}} \subset L^0$ converges a.u (resp. b.a.u) to $x \in L^0$ if for all $\delta > 0$ there exists a projection $e \in M$ such that

$$\tau(e^\perp) < \delta \text{ and } \lim_{n\to\infty} \|(x_n - x)e\| = 0 \;\; (\text{resp. } \tau(e^\perp) < \delta \text{ and } \lim_{n\to\infty} \|e(x_n - x)e\| = 0).$$

Now we recall the following useful lemma from [19, Lemma 3].

LEMMA 2.3. *If a sequence $\{a_n\}_{n\in\mathbb{N}} \subset M$ is such that for every $\varepsilon > 0$ there is a b.a.u (a.u) convergent sequence $\{b_n\}_{n\in\mathbb{N}} \subset M$ and a positive integer N_0 satisfying $\|a_n - b_n\| < \varepsilon$ for all $n \geqslant N_0$, then $\{a_n\}_{n\in\mathbb{N}}$ converges b.a.u (a.u).*

Next we provide a brief description of noncommutative Orlicz spaces. We follow [21] as our main references.

2.1. Noncommutative Orlicz spaces

Let M be a von Neumann algebra equipped with a f.n.s. trace τ as mentioned above. The trace τ is extended to the positive cone L^0_+ of L^0 as follows. Suppose $x \in L^0_+$ with the spectral decomposition $x = \int_0^\infty \lambda de_\lambda$. Then $\tau(x)$ is defined by

$$\tau(x) := \int_0^\infty \lambda d\tau(e_\lambda).$$

For $0 < p \leqslant \infty$, the noncommutative L^p-space associated to (M,τ) is defined as

$$L^p(M,\tau) := \begin{cases} \{x \in L^0 : \|x\| := \tau(|x|^p)^{1/p} < \infty\} & \text{for } p \neq \infty \\ (M, \|\cdot\|) & \text{for } p = \infty \end{cases}$$

where, $|x| = (x^*x)^{1/2}$. From here onwards we will simply write L^p for noncommutative L^p-spaces.

Let $x \in L^0$. Consider the spectral decomposition $|x| = \int_0^\infty s de_s$. The distribution function of x is defined by

$$(0,\infty) : s \mapsto \lambda_s(x) := \tau(e_s^\perp(|x|)) \in [0,\infty]$$

and the generalised singular number of x is defined by

$$(0,\infty) : t \mapsto \mu_t(x) := \inf\{s > 0 : \lambda_s(x) \leqslant t\} \in [0,\infty].$$

Note that both the functions are decreasing and continuous from right on $(0,\infty)$. Among many other properties of generalised singular number, here we recall the following ones which will be used later.

PROPOSITION 2.4. *Let $a,b,c \in L^0$. Then*

(i) $\mu_t(f(|a|)) = f(\mu_t(a))$, $t > 0$ *and for any continuous increasing function f on $[0,\infty)$ with $f(0) \geqslant 0$.*

(ii) $\mu_t(bac) \leqslant \|b\|\,\|c\|\,\mu_t(a)$ *for all $t > 0$.*

(iii) $\tau(f(|a|)) = \int_0^\infty f(\mu_t(a))dt$ *for any continuous increasing function f on $[0,\infty)$ with $f(0) = 0$.*

Proof. For the proofs we refer to [9, Lemma 2.5 and Corollary 2.8]. □

DEFINITION 2.5. A convex function $\Phi : [0,\infty) \to [0,\infty)$ which is continuous at 0 with $\Phi(0) = 0$ and $\Phi(t) > 0$ when $t \neq 0$ is called an Orlicz function.

It is to be noted that the convexity of the function Φ and continuity at 0 imply that the function is continuous on $[0,\infty)$. Moreover, it is also evident that $\Phi(\lambda t) \leqslant \lambda\Phi(t)$ whenever $0 \leqslant \lambda \leqslant 1$ and $t \in [0,\infty)$, which implies $\Phi(t_1) < \Phi(t_2)$ for all $0 \leqslant t_1 < t_2$. Hence the function Φ is increasing. The following result from [6, Lemma 2.1] is crucial.

LEMMA 2.6. *Let Φ be an Orlicz function. Then for all $\delta > 0$ there exists $u > 0$ satisfying the condition*

$$u \cdot \Phi(t) \geqslant t \quad \textit{whenever } t \geqslant \delta.$$

In particular, $\lim_{t\to\infty} \Phi(t) = \infty$.

Now let Φ be an Orlicz function and consider $x \in L^0_+$ with the spectral decomposition $x = \int_0^\infty \lambda d(e_\lambda)$. Then by means of functional calculus, we have

$$\Phi(x) = \int_0^\infty \Phi(\lambda) de_\lambda.$$

The noncommutative Orlicz space associated to (M,τ) for the Orlicz function Φ is defined as

$$L^\Phi = L^\Phi(M,\tau) := \Big\{x \in L^0 : \tau\Big(\Phi\Big(\frac{|x|}{\lambda}\Big)\Big) < \infty \text{ for some } \lambda > 0\Big\}.$$

The space L^Φ is equipped with the norm (called Luxemburg norm)

$$\|x\| := \inf\Big\{\lambda > 0 : \tau\Big(\Phi\Big(\frac{|x|}{\lambda}\Big)\Big) \leqslant 1\Big\}, \ \ x \in L^\Phi.$$

It follows from [16, Proposition 2.5] that L^Φ equipped with the norm defined above is a Banach space. We now prove the following result.

PROPOSITION 2.7. *Suppose $x \in L^\Phi$, then*

(i) *if $a,b \in M$, then $axb \in L^\Phi$. Moreover, $\|axb\|_\Phi \leqslant \|a\|\,\|b\|\,\|x\|_\Phi$ and*

(ii) *if $\|x\|_\Phi \leqslant 1$, then $\tau(\Phi(|x|)) \leqslant \|x\|_\Phi$.*

Proof. (i) Let $\lambda > 0$ and observe that

$$\begin{aligned}\tau\Big(\Phi\Big(\frac{|axb|}{\|a\|\,\|b\|\,\lambda}\Big)\Big) &= \int_0^\infty \Phi\Big(\mu_t\Big(\frac{axb}{\|a\|\,\|b\|\,\lambda}\Big)\Big)dt \ \ \text{[by } (iii) \text{ of Proposition 2.4]} \\ &\leqslant \int_0^\infty \Phi\Big(\mu_t\Big(\frac{x}{\lambda}\Big)\Big)dt \ \ \text{[by } (ii) \text{ of Proposition 2.4]} \\ &= \tau\Big(\Phi\Big(\frac{|x|}{\lambda}\Big)\Big) \ \ \text{[by } (iii) \text{ of Proposition 2.4]}.\end{aligned} \tag{2.1}$$

Then, note that

$$\inf\Big\{\lambda>0:\tau\Big(\Phi\Big(\frac{|axb|}{\lambda}\Big)\Big)\leqslant 1\Big\}=\inf\Big\{\,\|a\|\,\|b\|\,\lambda>0:\tau\Big(\Phi\Big(\frac{|axb|}{\|a\|\,\|b\|\,\lambda}\Big)\Big)\leqslant 1\Big\}$$
$$=\|a\|\,\|b\|\inf\Big\{\lambda>0:\tau\Big(\Phi\Big(\frac{|axb|}{\|a\|\,\|b\|\,\lambda}\Big)\Big)\leqslant 1\Big\}.$$

Therefore, by Eq. 2.1 we have

$$\begin{aligned}\|axb\|_{\Phi}&=\inf\Big\{\lambda>0:\tau\Big(\Phi\Big(\frac{|axb|}{\lambda}\Big)\Big)\leqslant 1\Big\}\\&\leqslant\|a\|\,\|b\|\inf\Big\{\lambda>0:\tau\Big(\Phi\Big(\frac{|x|}{\lambda}\Big)\Big)\leqslant 1\Big\}\\&=\|a\|\,\|b\|\,\|x\|_{\Phi}\,.\end{aligned}$$

Proof of (ii); it follows immediately from [6, Proposition 2.2]. □

Let us now recall that a Banach space $(E,\|\cdot\|)\subset L^0$ is called fully symmetric if

$$x\in E,\ y\in L^0,\ \int_0^s\mu_t(y)dt\leqslant\int_0^s\mu_t(x)dt\ \ \forall\ s>0\ \Rightarrow\ y\in E\ \text{ and }\|y\|\leqslant\|x\|$$

and a fully symmetric space $(E,\|\cdot\|)\subseteq L^0$ is said to have Fatou Property if

$$\begin{aligned}&x_\alpha\in E_+,\ x_\alpha\leqslant x_\beta\quad\text{for }\alpha\leqslant\beta\text{ and }\sup_\alpha\|x_\alpha\|<\infty\\&\Rightarrow\exists\,x=\sup_\alpha x_\alpha\in E\text{ and }\|x\|=\sup_\alpha\|x_\alpha\|\,.\end{aligned}$$

Now the following proposition holds true.

PROPOSITION 2.8. $(L^{\Phi},\|\cdot\|)$ *is a fully symmetric space with the Fatou property and an exact interpolation space for the Banach couple* (L^1,M).

Proof. Proof follows from [6, Corollary 2.2]. □

As a consequence we remark the following.

REMARK 2.9. It follows from [8, Theorem 4.1] and Proposition 2.8 that unit ball of $(L^{\Phi},\|\cdot\|)$ is closed under measure topology.

DEFINITION 2.10. An Orlicz function Φ is said to satisfy Δ_2 condition if there exists $d>0$ such that

$$\Phi(2t)\leqslant d\Phi(t)\text{ for all }t\geqslant 0.$$

Observe that for every $1\leqslant p<\infty$, $\Phi(u)=\frac{u^p}{p}$, $u\geqslant 0$ is an Orlicz function which satisfy the Δ_2 condition. Also, in this case $L^{\Phi}=L^p$ for all $1\leqslant p<\infty$.

PROPOSITION 2.11. *Let Φ be an Orlicz function satisfying Δ_2 condition. Then the linear subspace $L^1 \cap M$ is dense in $(L^\Phi, \|\cdot\|)$.*

Proof. For the proof we refer to [6, Proposition 2.3]. $\square$

DEFINITION 2.12. A linear map $T : L^1 + M \to L^1 + M$ is called Dunford-Schwartz operator if it contracts both L^1 and M, i.e,

$$\|Tx\|_\infty \leqslant \|x\|_\infty \ \forall\, x \in M \text{ and } \|Tx\|_1 \leqslant \|x\|_1 \ \forall\, x \in L^1.$$

If in addition $T(x) \geqslant 0$ for all $x \geqslant 0$ then we call T is a positive Dunford-Schwartz operator. We write $T \in DS$ (resp. $T \in DS^+$) to denote T is a Dunford-Schwartz operator (resp. positive Dunford-Schwartz operator).

Let $T \in DS$. Then observe that for an Orlicz function Φ the space L^Φ is an exact interpolation space for the Banach couple (L^1, M) (by Proposition 2.8). Therefore we have

$$T(L^\Phi) \subseteq L^\Phi \text{ and } \left\|T : L^\Phi \to L^\Phi\right\| \leqslant 1.$$

DEFINITION 2.13. Let $(X, \|\cdot\|)$ be a normed linear space and $Y \subseteq X$ be such that the identity element of X is a limit point of Y. A family of maps $A_\alpha : X \to L^0$, $\alpha \in I$, is called uniformly equicontinuous in measure (u.e.m) [bilaterally uniformly equicontinuous in measure (b.u.e.m)] at zero on Y if for all $\varepsilon, \delta > 0$, there exists $\gamma > 0$ such that for all $x \in Y$ with $\|x\| < \gamma$ there exists $e \in \mathscr{P}(M)$ such that

$$\tau(e^\perp) < \varepsilon \text{ and } \sup_{\alpha \in I} \|A_\alpha(x)e\|_\infty < \delta \ \ (\text{respectively, } \sup_{\alpha \in I} \|eA_\alpha(x)e\|_\infty < \delta).$$

Now we recall the following significant result from [18, Theorem 2.1] which will play an important role in our studies.

THEOREM 2.14. *Let $(X, \|\cdot\|)$ be a Banach space and $A_n : X \to L^0$ be a sequence of additive maps. If the sequence $\{A_n\}_{n \in \mathbb{N}}$ is b.u.e.m (u.e.m) at zero on X, then the set*

$$\{x \in X : \{A_n(x)\} \text{ converges b.a.u (a.u)}\}$$

is closed in X.

We end this section with a brief introduction to density and lower density of a sequence of natural numbers.

DEFINITION 2.15. A sequence $\mathbf{k} := \{k_j\}_{j \in \mathbb{N}}$ of natural numbers is said to have density (resp, lower density) d if

$$\lim_{n \to \infty} \frac{|\{0, 1, \ldots, n\} \cap \mathbf{k}|}{n+1} = d \ \ (\text{resp, } \liminf_{n \to \infty} \frac{|\{0, 1, \ldots, n\} \cap \mathbf{k}|}{n+1} = d).$$

REMARK 2.16. We remark that if a sequence $\mathbf{k}$ has density d, then $\lim_{n \to \infty} \frac{k_n}{n} = \frac{1}{d}$. Moreover, we recall from [22, Lemma 40] that a sequence $\mathbf{k}$ has positive lower density if and only if $\sup_{n \in \mathbb{N}} \frac{k_n}{n} < \infty$.

3. Convergence along sequence of density one

Throughout this section M is assumed to be a semifinite von Neumann algebra with f.n.s. trace τ and $T \in DS^+$. In this section, we will study the convergence of ergodic averages with M-valued Besicovitch weights (see Definition 3.1 and Definition 3.11) along sequence of density one. In particular, we will prove the b.a.u convergence of sequences of such averages in the spaces L^Φ for some Orlicz function Φ. Convergence of usual vector valued weighted averages in norm and b.a.u topology has already been studied in [4]. In this section, we also extend some of these results. We begin with few definitions of ergodic averages.

DEFINITION 3.1. Let $T \in DS^+$. For $\{b_j\}_{j\in\mathbb{N}} \subset M$ and $\{d_j\}_{j\in\mathbb{N}} \subset M$ and any sequence $\mathbf{k} := \{k_j\}_{j\in\mathbb{N}}$ of natural numbers, define

$$A_n(\{b_j\},\{d_j\},x) := \frac{1}{n}\sum_{j=0}^{n-1} T^j(b_j x d_j), \qquad A_n(\{b_j\},x) := \frac{1}{n}\sum_{j=0}^{n-1} T^j(b_j x);$$

and

$$A_n^{\mathbf{k}}(\{b_j\},\{d_j\},x)) := \frac{1}{n}\sum_{j=0}^{n-1} T^{k_j}(b_{k_j} x d_{k_j}), \qquad A_n^{\mathbf{k}}(\{b_j\},x) := \frac{1}{n}\sum_{j=0}^{n-1} T^{k_j}(b_{k_j} x)$$

for all $n \in \mathbb{N}$ and $x \in L^1 + M$.

Here we observe that when the sequence $\{b_j\}_{j\in\mathbb{N}}$ consists of only scalars $\beta := \{\beta_j\}_{j\in\mathbb{N}}$ and the set $\{d_j\}_{j\in\mathbb{N}}$ consists of only identity of M, then the averages mentioned above will be denoted by $A_n^\beta(x)$ and $A_n^{\beta,\mathbf{k}}(x)$ respectively for $x \in L^1 + M$. Convergence of such averages is studied in [20].

Let us now recall the following maximal ergodic theorem from [23]. This result is crucial in obtaining a maximal ergodic inequality in the form required for our purpose.

THEOREM 3.2. *Let $T \in DS^+$. Then for all $x \in L^1_+$ and $\varepsilon > 0$ there exists $e \in \mathscr{P}(M)$ such that*

$$\tau(e^\perp) \leqslant \frac{\|x\|_1}{\varepsilon} \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|eA_n(\{1\},x)e\| \leqslant \varepsilon.$$

Although the following lemma is a part of the proof of Theorem 2.1 in [5], we include the proof here for the sake of completeness.

LEMMA 3.3. *Let $1 \leqslant p < \infty$, $x \in L^p_+$ and $\varepsilon > 0$. Then there exists $e \in \mathscr{P}(M)$ such that*

$$\tau(e^\perp) \leqslant \Big(\frac{\|x\|_p}{\varepsilon}\Big)^p \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|eA_n(\{1\},x)e\|_\infty \leqslant 2\varepsilon$$

Proof. Consider the spectral decomposition of $x = \int_0^\infty \lambda de_\lambda$. Note that since $\lambda \geqslant \varepsilon \Rightarrow \lambda \leqslant \varepsilon^{1-p}\lambda^p$, we have

$$\int_\varepsilon^\infty \lambda de_\lambda \leqslant \varepsilon^{1-p} \int_\varepsilon^\infty \lambda^p de_\lambda \leqslant \varepsilon^{1-p} x^p.$$

Therefore, we obtain

$$x = \int_0^\varepsilon \lambda de_\lambda + \int_\varepsilon^\infty \lambda de_\lambda \leqslant x_\varepsilon + \varepsilon^{1-p} x^p,$$

where $x_\varepsilon = \int_0^\varepsilon \lambda de_\lambda$. Now since $x^p \in L^1_+$, it follows from Theorem 3.2 that there exists $e \in \mathscr{P}(M)$ such that

$$\tau(e^\perp) \leqslant \frac{\|x^p\|_1}{\varepsilon^p} = \left(\frac{\|x\|_p}{\varepsilon}\right)^p \text{ and } \sup_{n\in\mathbb{N}} \|eA_n(\{1\},x^p)e\| \leqslant \varepsilon^p.$$

Consequently, for all $n \in \mathbb{N}$ we have

$$0 \leqslant eA_n(\{1\},x)e \leqslant eA_n(\{1\},x_\varepsilon)e + \varepsilon^{1-p} eA_n(\{1\},x^p)e.$$

Since $x_\varepsilon \in M$ and $\|T(x_\varepsilon)\|_\infty \leqslant \|x_\varepsilon\|_\infty \leqslant \varepsilon$, we conclude that

$$\sup_{n\in\mathbb{N}} \|eA_n(\{1\},x)e\|_\infty \leqslant 2\varepsilon. \quad \square$$

Now the following result holds.

THEOREM 3.4. *Let $\{b_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $\mathscr{Z}(M)$ and $x \in L^p$ $(1 \leqslant p < \infty)$. Then for all $\varepsilon > 0$ there exists $e \in \mathscr{P}(M)$ such that*

$$\tau(e^\perp) \leqslant 4\left(\frac{\|x\|_p}{\varepsilon}\right)^p \quad \text{and} \quad \sup_{n\in\mathbb{N}} \left\|eA_n(\{b_j\},x)e\right\|_\infty \leqslant 48C\varepsilon,$$

where $C = \sup_{j\in\mathbb{N}} \left\|b_j\right\|_\infty$.

Proof. First consider $x \in L^p_+$ and observe that if $b_j = 1$ for all $j \in \mathbb{N}$, then it follows from Lemma 3.3 that for all $\varepsilon > 0$ there exists $e \in \mathscr{P}(M)$ such that

$$\tau(e^\perp) \leqslant \left(\frac{\|x\|_p}{\varepsilon}\right)^p \quad \text{and} \quad \sup_{n\in\mathbb{N}} \|eA_n(\{1\},x)e\|_\infty \leqslant 2\varepsilon. \tag{3.1}$$

Now consider $\{b_j\}_{j\in\mathbb{N}}$ to be a bounded sequence in $\mathscr{Z}(M)$ with $\left\|b_j\right\|_\infty \leqslant C$ for all $j \in \mathbb{N}$. Then we have $0 \leqslant \mathrm{Re}(b_j) + C \leqslant 2C$ and similarly $0 \leqslant \mathrm{Im}(b_j) + C \leqslant 2C$ for all $j \in \mathbb{N}$. Therefore, we must have for all $j \in \mathbb{N}$

$$0 \leqslant (\mathrm{Re}(b_j) + C)x \leqslant 2Cx \quad \text{and} \quad 0 \leqslant (\mathrm{Im}(b_j) + C)x \leqslant 2Cx.$$

Also, for all $j \in \mathbb{N}$, we have

$$T^j(b_j x) = T^j((\mathrm{Re}(b_j)+C)x) + iT^j((\mathrm{Im}(b_j)+C)x) - (1+i)CT^j(x).$$

Then Eq 3.1 implies that for all $\varepsilon > 0$ there exists $e \in \mathscr{P}(M)$ such that

$$\tau(e^{\perp}) \leqslant \left(\frac{\|x\|_p}{\varepsilon}\right)^p \quad \text{and} \quad \sup_{n\in\mathbb{N}} \left\|eA_n(\{b_j\},x)e\right\|_\infty \leqslant 6C \sup_{n\in\mathbb{N}} \|eA_n(\{1\},x)e\|_\infty \leqslant 12C\varepsilon. \tag{3.2}$$

For $x \in L^p$, write $x = (x_1 - x_2) + i(x_3 - x_4)$, where $x_l \in L^p_+$ and $\|x_l\|_p \leqslant \|x\|_p$ for all $l \in \{1,\ldots,4\}$. Therefore, it follows from Eq 3.2 that there exist projections $e_l \in M$ such that

$$\tau(e_l^{\perp}) \leqslant \left(\frac{\|x\|_p}{\varepsilon}\right)^p \quad \text{and} \quad \sup_{n\in\mathbb{N}} \left\|e_l A_n(\{b_j\},x)e_l\right\|_\infty \leqslant 12C\varepsilon \quad \text{for all } l \in \{1,\ldots,4\}.$$

Now consider $e = \wedge_{l=1}^4 e_l$ to obtain the required result. $\square$

Before we move to our next theorem we need to fix some notations. From here onwards $\mathbf{k} := \{k_j\}_{j\in\mathbb{N}}$ will always denote a strictly increasing sequence of natural numbers. For any sequence $\{b_j\}_{j\in\mathbb{N}} \subset M$ and $n \in \mathbb{N}$, $A_n(\{b_j\},x)$ recall the definition of $A_n^{\mathbf{k}}(\{b_j\},x)$ and $A_n^{\mathbf{k}}(\{b_j\},x)$ from Definition 3.1, where $x \in L^1 + M$.

THEOREM 3.5. *Let $\{b_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $\mathscr{Z}(M)$. If the strictly increasing sequence $\mathbf{k} := \{k_j\}_{j\in\mathbb{N}}$ of natural numbers has lower density $d > 0$, then the sequences $\{A_n(\{b_j\},\cdot)\}_{n\in\mathbb{N}}$ and $\{A_n^{\mathbf{k}}(\{b_j\},\cdot)\}_{n\in\mathbb{N}}$ are b.u.e.m at zero on $(L^\Phi, \|\cdot\|_\Phi)$.*

Proof. It is enough to show that the sequences $\{A_n(\{b_j\},\cdot)\}_{n\in\mathbb{N}}$ and $\{A_n^{\mathbf{k}}(\{b_j\},\cdot)\}_{n\in\mathbb{N}}$ are b.u.e.m at zero on $(L^\Phi_+, \|\cdot\|_\Phi)$. Define, $C := \sup_{j\in\mathbb{N}} \left\|b_j\right\|_\infty$.

Now fix $\varepsilon, \delta > 0$. Then by Lemma 2.6, there exists a $t > 0$ such that

$$t \cdot \Phi(\lambda) \geqslant \lambda \quad \text{for all} \quad \lambda \geqslant \frac{\delta}{2C}.$$

Choose $0 < \gamma < \min\{1, \frac{\delta\varepsilon}{4\times 96Ct}\}$. Let $x \in L^\Phi_+$ with $\|x\|_\Phi < \gamma$ and let $x = \int_0^\infty \lambda de_\lambda$ be its spectral decomposition. Then we can write

$$x = \int_0^{\frac{\delta}{2C}} \lambda de_\lambda + \int_{\frac{\delta}{2C}}^\infty \lambda de_\lambda \leqslant x_\delta + t\int_{\frac{\delta}{2C}}^\infty \Phi(\lambda) de_\lambda \leqslant x_\delta + t\Phi(x),$$

where $x_\delta = \int_0^{\frac{\delta}{2C}} \lambda de_\lambda$ and $\Phi(x) = \int_0^\infty \Phi(\lambda) de_\lambda$.

Observe that, $\|x_\delta\| \leqslant \frac{\delta}{2C}$ and since T is a positive Dunford-Schwarz operator we must have

$$\sup_{n\in\mathbb{N}} \left\|A_n(\{b_j\},x_\delta)\right\| \leqslant \frac{C\delta}{2C} = \frac{\delta}{2}.$$

Also, since $\|x\|_\Phi < \gamma < 1$, by Proposition 2.7 we have $\|\Phi(x)\|_1 \leqslant \|x\|_\Phi$. Furthermore, since $\Phi(x) \in L^1_+$, by Theorem 3.4 we find $e \in \mathscr{P}(M)$ satisfying

$$\tau(e^\perp) < \frac{4 \times 96Ct\,\|\Phi(x)\|_1}{\delta} \leqslant \frac{4 \times 96Ct\,\|x\|_\Phi}{\delta} < \varepsilon$$

and,

$$\sup_{n\in\mathbb{N}} \left\|eA_n(\{b_j\},\Phi(x))e\right\| < \frac{48C\delta}{96Ct} = \frac{\delta}{2t}.$$

Therefore,

$$\begin{aligned}\sup_{n\in\mathbb{N}} \left\|eA_n(\{b_j\},x)e\right\| &\leqslant \sup_{n\in\mathbb{N}} \left\|eA_n(\{b_j\},x_\delta)e\right\| + t\cdot \sup_{n\in\mathbb{N}} \left\|eA_n(\{b_j\},\Phi(x))e\right\| \\ &< \frac{\delta}{2} + t\cdot\frac{\delta}{2t} = \delta.\end{aligned}$$

Hence, the sequence $\{A_n(\{b_j\},\cdot)\}_{n\in\mathbb{N}}$ is b.u.e.m at zero on $(L^\Phi_+, \|\cdot\|_\Phi)$. To show the sequence $\left\{A^{\mathbf{k}}_n(\{b_j\},\cdot)\right\}_{n\in\mathbb{N}}$ is b.u.e.m at zero on $(L^\Phi_+, \|\cdot\|_\Phi)$, we first consider the sequence $\{c_j\}_{j\in\mathbb{N}}$, where for all $j \in \mathbb{N}$, $c_j := \chi_{\mathbf{k}}(j)$.

Observe that for all $n \in \mathbb{N}$,

$$A^{\mathbf{k}}_n(\{b_j\},x) = \frac{k_{n-1}+1}{n} A_{k_{n-1}+1}(\{c_jb_j\},x). \tag{3.3}$$

By the first part of the proof we observe that the sequence $\{A_n(\{c_jb_j,\cdot\})\}_{n\in\mathbb{N}}$ is is b.u.e.m at zero on $(L^\Phi_+, \|\cdot\|_\Phi)$.

Let $K = \sup_{n\in\mathbb{N}} \frac{k_n}{n}$. It follows from Remark 2.16 that $0 < K < \infty$. Let $\varepsilon, \delta > 0$. Let $\gamma > 0$ be such that for all $x \in L^\Phi_+$ there exists $e \in \mathscr{P}(M)$ such that

$$\tau(e^\perp) < \varepsilon \quad \text{and} \quad \sup_{n\in\mathbb{N}} \left\|eA_n(\{c_jb_j\},x)e\right\|_\infty < \frac{\delta}{K}.$$

Consequently,

$$\begin{aligned}\sup_{n\in\mathbb{N}} \left\|eA^{\mathbf{k}}_n(\{b_j\},x)e\right\|_\infty &= \sup_{n\in\mathbb{N}} \frac{k_{n-1}+1}{n} \left\|eA_{k_{n-1}+1}(\{c_jb_j\},x)e\right\|_\infty \\ &\leqslant K \sup_{n\in\mathbb{N}} \left\|eA_n(\{c_jb_j\},x)e\right\|_\infty \\ &< K\frac{\delta}{K} = \delta.\end{aligned}$$

This completes the proof. □

COROLLARY 3.6. *Let $\{\beta_j\}_{j\in\mathbb{N}} \subset l^\infty(\mathbb{C})$. If the strictly increasing sequence $\mathbf{k} := \{k_j\}_{j\in\mathbb{N}}$ of natural numbers has lower density $d > 0$, then the sequences $\{A^\beta_n\}_{n\in\mathbb{N}}$ and $\{A^{\beta,\mathbf{k}}_n\}_{n\in\mathbb{N}}$ are b.u.e.m at zero on $(L^\Phi, \|\cdot\|_\Phi)$.*

REMARK 3.7. Let $\{\beta_j\}_{j\in\mathbb{N}} \subset l^\infty(\mathbb{C})$. Note that it follows from [20, Proposition 3.1] that the sequences $\{A_n^\beta\}_{n\in\mathbb{N}}$ and $\{A_n^{\beta,\mathbf{k}}\}_{n\in\mathbb{N}}$ are b.u.e.m at zero on L^p for $(1 \leqslant p < \infty)$ where the sequence $\mathbf{k} := \{k_j\}_{j\in\mathbb{N}}$ is of lower density $d > 0$. Therefore, Corollary 3.6 substantially improves Proposition 3.1 of [20].

As a consequence we prove the following proposition which is an important ingredient in proving our main result.

PROPOSITION 3.8. *Let $\{b_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $\mathscr{Z}(M)$. If the strictly increasing sequence $\mathbf{k} := \{k_j\}_{j\in\mathbb{N}}$ of natural numbers has lower density $d > 0$, then the sets*

$$\mathscr{S}^{\{b_j\}} := \left\{x \in L^\Phi : \{A_n(\{b_j\},x)\} \text{ converges b.a.u}\right\} \text{ and,}$$
$$\mathscr{S}^{\{b_j\},\mathbf{k}} := \left\{x \in L^\Phi : \{A_n^{\mathbf{k}}(\{b_j\},x)\} \text{ converges b.a.u}\right\}$$

are closed in L^Φ.

Proof. Since $(L^\Phi, \|\cdot\|_\Phi)$ is a Banach space and $\{A_n(\{b_j\},\cdot)\}$ and $\{A_n^{\mathbf{k}}(\{b_j\},\cdot)\}$ are sequences of additive maps, the result follows immediately from Theorem 3.5 and Theorem 2.14. □

REMARK 3.9. Let $\beta := \{\beta_j\}_{j\in\mathbb{N}} \subset l^\infty(\mathbb{C})$ and $\mathbf{k} := \{k_j\}_{j\in\mathbb{N}}$ be as stated in Proposition 3.8. Then we remark that it is evident from Proposition 3.8 that the sets

$$\mathscr{S}^\beta := \left\{x \in L^\Phi : \left\{A_n^\beta(x)\right\} \text{ converges } b.a.u\right\} \text{ and,}$$
$$\mathscr{S}^{\beta,\mathbf{k}} := \left\{x \in L^\Phi : \left\{A_n^{\beta,\mathbf{k}}(x)\right\} \text{ converges } b.a.u\right\}$$

are closed in L^Φ.

In what follows $U(M)$ will always denote the group of unitary operators in M and $\sigma(x)$ will denote the spectrum of an operator in $x \in M$. Let us define,

$$U_f := \{u \in U(M) : \sigma(u) \text{ is finite}\}.$$

DEFINITION 3.10. Let $U_0 \subseteq U(M)$. A function $\psi : \mathbb{N} \to M$ is called a trigonometric polynomial over U_0 if for some $m \in \mathbb{N}$ there exists $\{z_j\}_1^m \subset \mathbb{C}$ and $\{u_j\}_1^m \subset U_0$ such that

$$\psi(k) = \sum_{j=1}^m z_j u_j^k, \quad k \in \mathbb{N}.$$

For a trigonometric polynomial ψ over U_0 as defined above, it is clear that $\|\psi\| \leqslant \sum_{j=1}^m |z_j|$.

DEFINITION 3.11. Let $U_0 \subseteq U(M)$. A sequence $\{b_j\} \subset M$ is called U_0-besicovitch if for all $\varepsilon > 0$ there exists a trigonometric polynomial ψ over U_0 such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left\| b_j - \psi(j) \right\|_\infty \leqslant \varepsilon.$$

A U_0-besicovitch sequence $\{b_j\}$ is called bounded if $\sup_{j\in\mathbb{N}} \left\| b_j \right\|_\infty < \infty$.

Now we recall the following result from [4] regarding the convergence of sequence of ergodic averages and immediately after that we extend it to the case of ergodic averages along a sequence of density 1.

THEOREM 3.12. *Let $\{b_j\}$ and $\{d_j\}$ be U_f-besicovitch sequences such that at least one of which is bounded. Then the averages $A_n(\{b_j\},\{d_j\},x)$ converge a.u for all $x \in L^1 \cap M$.*

Proof. For proof we refer to [4, Theorem 5.1]. □

THEOREM 3.13. *Let $\{b_j\}$ and $\{d_j\}$ be U_f-besicovitch sequences with at least one of them is bounded and $\{k_j\}$ be a strictly increasing sequence of natural numbers of density 1. Then the sequence of averages $A_n^{\mathbf{k}}(\{b_j\},\{d_j\},x))$ converges a.u for all $x \in L^1 \cap M$.*

Proof. Without loss of generality we assume that $\{d_j\}$ is bounded and define $C := \sup_j \left\| d_j \right\| < \infty$. Fix $\varepsilon > 0$ and let $\psi_1(\cdot) = \sum_{i=1}^m z_i u_i^{(\cdot)}$ and $\psi_2(\cdot) = \sum_{i=1}^l w_i v_i^{(\cdot)}$ be such that $\{z_i\},\{w_i\} \subset \mathbb{C}$, $\{u_i\},\{v_i\} \subset U_f$ and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left\| b_j - \psi_1(j) \right\|_\infty \leqslant \varepsilon, \quad \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \left\| d_j - \psi_2(j) \right\|_\infty \leqslant \varepsilon. \tag{3.4}$$

Let $x \in L^1 \cap M$. Note that by Theorem 3.12 the averages $A_n(\{b_j\},\{d_j\},x)$ converges a.u. In particular, the averages $A_n(\{\psi_1(j)\},\{\psi_2(j)\},x)$ converges a.u. Hence the subsequence $A_{k_n}(\{\psi_1(j)\},\{\psi_2(j)\},x)$ converges a.u. Define,

$$M_n(\{\psi_1(j)\},\{d_j\},x) := \frac{1}{k_n} \sum_{j=0}^{n-1} T^{k_j}(\psi_1(k_j) x d_{k_j}), \quad n \in \mathbb{N}.$$

Now, we have

$$\begin{aligned}
&\left\| A_{k_n}(\{\psi_1(j)\},\{\psi_2(j)\},x) - M_n(\{\psi_1(j)\},\{d_j\},x) \right\| \\
&= \left\| \frac{1}{k_n} \sum_{j=0}^{k_n-1} T^j(\psi_1(j) x \psi_2(j)) - \frac{1}{k_n} \sum_{j=0}^{n-1} T^{k_j}(\psi_1(k_j) x d_{k_j}) \right\|
\end{aligned}$$

$$\leqslant \left\| \frac{1}{k_n} \sum_{j=0}^{k_n-1} T^j(\psi_1(j)x\psi_2(j)) - \frac{1}{k_n} \sum_{j=0}^{k_n-1} T^j(\psi_1(j)xd_j) \right\| + \left\| \frac{1}{k_n} \sum_{j=0}^{k_n-1} T^j(\psi_1(j)xd_j) - \frac{1}{k_n} \sum_{j=0}^{n-1} T^{k_j}(\psi_1(k_j)xd_{k_j}) \right\|$$

$$\leqslant \frac{1}{k_n} \sum_{j=0}^{k_n-1} \left\| d_j - \psi_2(j) \right\| \|x\| \|\psi_1\| + \left\| \frac{1}{k_n} \sum_{j=0, j\notin \mathbf{k}}^{k_n-1} T^j(\psi_1(j)xd_j) \right\| \quad (\text{since } \|T\| < 1)$$

$$\leqslant \|\psi_1\| \|x\| \varepsilon + \frac{1}{k_n} \sum_{j=0, j\notin \mathbf{k}}^{k_n-1} \|\psi_1\| C \|x\| \quad (\text{since } \|T\| < 1 \text{ and by Eq. } 3.4)$$

$$\leqslant \|\psi_1\| \|x\| \varepsilon + \frac{k_n - n}{k_n} \|\psi_1\| C \|x\| .$$

Now since $\frac{k_n-n}{k_n} \to 0$ as $n \to \infty$, we can choose $N \in \mathbb{N}$ such that for all $n \geqslant N$ we have

$$\left\| A_{k_n}(\{\psi_1(j)\}, \{\psi_2(j)\}, x) - M_n(\{\psi_1(j)\}, \{d_j\}, x) \right\| < \varepsilon.$$

Hence, it follows from Lemma 2.3 that the sequence $\{M_n(\{\psi_1(j)\}, \{d_j\}, x)\}$ converges a.u. Again, define

$$M_n(\{b_j\}, \{d_j\}, x) := \frac{1}{k_n} \sum_{j=0}^{n-1} T^{k_j}(b_{k_j} x d_{k_j}), \quad n \in \mathbb{N}.$$

Then,

$$\begin{aligned}
&\left\| M_n(\{b_j\}, \{d_j\}, x) - M_n(\{\psi_1(j)\}, \{d_j\}, x) \right\| \\
&= \left\| \frac{1}{k_n} \sum_{j=0}^{n-1} T^{k_j}(b_{k_j} x d_{k_j}) - \frac{1}{k_n} \sum_{j=0}^{n-1} T^{k_j}(\psi_1(k_j) x d_{k_j}) \right\| \\
&\leqslant \frac{1}{k_n} \sum_{j=0}^{n-1} \left\| b_{k_j} - \psi_1(k_j) \right\| \|x\| C \quad (\text{since } \|T\| < 1) \\
&\leqslant \frac{1}{k_n} \sum_{j=0}^{k_n-1} \left\| b_j - \psi_1(j) \right\| \|x\| C \\
&\leqslant \varepsilon \|x\| C (\text{by Eq. } 3.4).
\end{aligned}$$

Hence, an appeal to Lemma 2.3 implies that the sequence $\{M_n(\{b_j\}, \{d_j\}, x)\}$ converges a.u. Now since $\lim_{n\to\infty} \frac{k_n}{n} = 1$ and $A_n^{\mathbf{k}}(\{b_j\}, \{d_j\}, x) = \frac{k_n}{n} M_n(\{b_j\}, \{d_j\}, x)$ for all $n \in \mathbb{N}$, the result follows immediately. $\square$

COROLLARY 3.14. *Let $\{b_j\}_{j\in\mathbb{N}}$ be a U_f-besicovitch sequence and $\{k_j\}_{j\in\mathbb{N}}$ has density* 1. *Let $x \in L^1 \cap M$. Then the averages*

$$\frac{1}{n} \sum_{j=0}^{n-1} T^{k_j}(b_{k_j} x) \quad \text{and} \quad \frac{1}{n} \sum_{j=0}^{n-1} T^{k_j}(x b_{k_j})$$

converges a.u.

As a consequence we obtain the individual ergodic theorem for vector valued Besicovitch weight along a sequence of density one.

THEOREM 3.15. *Assume that the Orlicz function Φ satisfies Δ_2 condition. Let $\mathbf{k} := \{k_j\}$ be a sequence of density 1 and $\{b_j\}_{j\in\mathbb{N}}$ be a bounded U_f-besicovitch sequence in $\mathscr{L}(M)$. Then for every $x \in L^\Phi$ the sequence $\{A_n^{\mathbf{k}}(\{b_j\},x)\}$ converges b.a.u to some $\hat{x} \in L^\Phi$.*

Proof. Define, $\mathscr{S}^{\{b_j\},\mathbf{k}} := \{x \in L^\Phi : \{A_n^{\mathbf{k}}(\{b_j\},x)\}$ converges $b.a.u\}$. Note that, by Proposition 3.8 the set $\mathscr{S}^{\{b_j\},\mathbf{k}}$ is closed in L^Φ. Since $L^1 \cap M$ is dense in L^Φ, we have $\mathscr{S}^{\{b_j\},\mathbf{k}} = L^\Phi$.

Let $x \in L^\Phi$. Then by Proposition 2.7 $\{A_n^{\mathbf{k}}(\{b_j\},x)\}_{n\in\mathbb{N}} \subset L^\Phi$. Also there exists $\hat{x} \in L^0$ such that $A_n^{\mathbf{k}}(\{b_j\},x)$ converges b.a.u to $\hat{x}$, hence in measure. Now since $\|T\| \leqslant 1$, we observe that for all $n \in \mathbb{N}$,

$$\left\|A_n^{\mathbf{k}}(\{b_j\},x)\right\|_\Phi \leqslant \frac{1}{n}\sum_{j=0}^{n-1}\left\|b_{k_j}\right\| \|x\|_\Phi \leqslant C\|x\|_\Phi,$$

where $C = \sup_{j\in\mathbb{N}}\left\|b_j\right\| < \infty$. Therefore, for all $n \in \mathbb{N}$, $A_n^{\mathbf{k}}(\{b_j\},x)$ belongs to the closed ball of $(L^\Phi, \|\cdot\|_\Phi)$ of radius $C\|x\|_\Phi$. Consequently by Remark 2.9, $\hat{x} \in L^\Phi$. □

REMARK 3.16.

1. Following Definition 3.10 and 3.11 one can always define a scalar valued Besicovitch sequence. In particular, a scalar valued trigonometric polynomial is a function $P : \mathbb{N} \to \mathbb{C}$ satisfying

$$P(k) = \sum_{j=1}^{s} r_j \lambda_j^k, \quad k \in \mathbb{Z}$$

   for some $\{r_j\}_{j=1}^s \subset \mathbb{C}$ and $\{\lambda_j\}_{j=1}^s \subset \mathbb{C}^1$, where $\mathbb{C}^1 := \{z \in \mathbb{C} : |z| = 1\}$. A sequence $\{\beta_j\}_{j=1}^\infty$ of complex numbers is called a Besicovitch sequence if for all $\varepsilon > 0$ there exists a trigonometric polynomial P such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1}\left|\beta_j - P(j)\right| < \varepsilon.$$

   The sequence $\{\beta_j\}_{j=1}^\infty$ is bounded if $\sup_{j\in\mathbb{N}}\left|\beta_j\right| < \infty$.

2. Very recently in [20, Corollary 3.2], the author proved the conclusion of Theorem 3.15 when $x \in L^p$ $(1 \leqslant p < \infty)$ and also under the hypothesis that the Besicovitch weights are scalar valued. Hence our theorem generalises Corollary 3.2 of [20].

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REFERENCES

[1] CLAIRE ANANTHARAMAN, JEAN-PHILIPPE ANKER, MARTINE BABILLOT, ALINE BONAMI, BRUNO DEMANGE, SANDRINE GRELLIER, FRANÇOIS HAVARD, PHILIPPE JAMING, EMMANUEL LESIGNE, PATRICK MAHEUX, JEAN-PIERRE OTAL, BARBARA SCHAPIRA, AND JEAN-PIERRE SCHREIBER, *Théorèmes ergodiques pour les actions de groupes*, Monographies de L'Enseignement Mathématique [Monographs of L'Enseignement Mathématique], vol. 41, L'Enseignement Mathématique, Geneva, 2010, With a foreword in English by Amos Nevo, MR 2643350.

[2] PANCHUGOPAL BIKRAM AND DIPTESH SAHA, *On the non-commutative neveu decomposition and ergodic theorems for amenable group action*, Journal of Functional Analysis **284** (2023), no. 1, 109706.

[3] J.-P. CONZE AND N. DANG-NGOC, *Ergodic theorems for noncommutative dynamical systems*, Inventiones Mathematicae **46** (1978), no. 1, 1–15, MR 500185.

[4] DOĞAN ÇÖMEZ AND SEMYON LITVINOV, *Ergodic averages with vector-valued Besicovitch weights*, Positivity **17** (2013), no. 1, 27–46, MR 3027644.

[5] VLADIMIR CHILIN AND SEMYON LITVINOV, *Ergodic theorems in fully symmetric spaces of τ-measurable operators*, Studia Math. **228** (2015), no. 2, 177–195, MR 3422091.

[6] VLADIMIR CHILIN AND SEMYON LITVINOV, *Individual ergodic theorems in noncommutative Orlicz spaces*, Positivity **21** (2017), no. 1, 49–59, MR 3612983.

[7] VLADIMIR CHILIN, SEMYON LITVINOV, AND ADAM SKALSKI, *A few remarks in non-commutative ergodic theory*, Journal of Operator Theory **53** (2005), no. 2, 331–350, MR 2153152.

[8] P. G. DODDS, T. K. DODDS, F. A. SUKOCHEV, AND O. YE. TIKHONOV, *A non-commutative Yosida-Hewitt theorem and convex sets of measurable operators closed locally in measure*, Positivity **9** (2005), no. 3, 457–484, MR 2188531.

[9] THIERRY FACK AND HIDEKI KOSAKI, *Generalized s-numbers of τ-measurable operators*, Pacific J. Math. **123** (1986), no. 2, 269–300, MR 840845.

[10] FUMIO HIAI, *Lectures on selected topics in von Neumann algebras*, EMS Series of Lectures in Mathematics, EMS Press, Berlin, [2021] ©2021, MR 4331436.

[11] GUIXIANG HONG, BEN LIAO, AND SIMENG WANG, *Noncommutative maximal ergodic inequalities associated with doubling conditions*, Duke Math. J. **170** (2021), no. 2, 205–246, MR 4202493.

[12] GUIXIANG HONG, *Non-commutative ergodic averages of balls and spheres over Euclidean spaces*, Ergodic Theory Dynam. Systems **40** (2020), no. 2, 418–436, MR 4048299.

[13] RYSZARD JAJTE, *Strong limit theorems in noncommutative probability*, Lecture Notes in Mathematics, vol. 1110, Springer-Verlag, Berlin, 1985, MR 778724.

[14] MARIUS JUNGE AND QUANHUA XU, *Noncommutative maximal ergodic theorems*, Journal of the American Mathematical Society **20** (2007), no. 2, 385–439, MR 2276775.

[15] BURKHARD KÜMMERER, *A non-commutative individual ergodic theorem*, Inventiones Mathematicae **46** (1978), no. 2, 139–145, MR 482260.

[16] WOLFGANG KUNZE, *Noncommutative Orlicz spaces and generalized Arens algebras*, Math. Nachr. **147** (1990), 123–138, MR 1127316.

[17] E. CHRISTOPHER LANCE, *Ergodic theorems for convex sets and operator algebras*, Inventiones Mathematicae **37** (1976), no. 3, 201–214, MR 428060.

[18] SEMYON LITVINOV, *Uniform equicontinuity of sequences of measurable operators and non-commutative ergodic theorems*, Proc. Amer. Math. Soc. **140** (2012), no. 7, 2401–2409, MR 2898702.

[19] SEMYON LITVINOV AND FARRUKH MUKHAMEDOV, *On individual subsequential ergodic theorem in von Neumann algebras*, Studia Math. **145** (2001), no. 1, 55–62, MR 1828992.

[20] MORGAN O'BRIEN, *Some noncommutative subsequential weighted individual ergodic theorems*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **24** (2021), no. 3, Paper No. 2150018, 17, MR 4348153.

[21] GILLES PISIER AND QUANHUA XU, *Non-commutative L^p-spaces*, Handbook of the geometry of Banach spaces, Vol. 2, North-Holland, Amsterdam, 2003, pp. 1459–1517, MR 1999201.

[22] JOSEPH ROSENBLATT, *Norm convergence in ergodic theory and the behavior of Fourier transforms*, Canad. J. Math. **46** (1994), no. 1, 184–199, MR 1260343.

[23] F. J. YEADON, *Ergodic theorems for semifinite von Neumann algebras, I*, Journal of the London Mathematical Society, second series **16** (1977), no. 2, 326–332, MR 487482.

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