Operators and Matrices
Volume 18, Number 3 (2024), 753–761 doi:10.7153/oam-2024-18-45

MAXIMAL DIMENSION OF AFFINE SUBSPACES OF SPECIFIC MATRICES

ELENA RUBEI

(*Communicated by M. Omladič*)

Abstract. For every $n \in \mathbb{N}$ and every field K, let $M(n \times n, K)$ be the set of $n \times n$ matrices over K, let $N(n,K)$ be the set of nilpotent $n \times n$ matrices over K and let $D(n,K)$ be the set of $n \times n$ matrices over K which are diagonalizable over K, that is, which are diagonalizable in $M(n \times n, K)$. Moreover, if K is a field with an involutory automorphism, let $R(n,K)$ be the set of normal $n \times n$ matrices over K.

In this short note we prove that the maximal dimension of an affine subspace in $N(n,K)$ is $\frac{n(n-1)}{2}$ and, if the characteristic of the field is zero, an affine not linear subspace in $N(n,K)$ has dimension less than or equal to $\frac{n(n-1)}{2} - 1$. Moreover we prove that the maximal dimension of an affine subspace in $R(n,\mathbb{C})$ is n, the maximal dimension of a linear subspace in $D(n,\mathbb{R})$ is $\frac{n(n+1)}{2}$, while the maximal dimension of an affine not linear subspace in $D(n,\mathbb{R})$ is $\frac{n(n+1)}{2} - 1$.

1. Introduction

There is a wide literature on the maximal dimension of linear or affine subspaces of matrices with specific characteristics. In particular we quote the following results. For every $m, n \in \mathbb{N}$ and every field K, let $M(m \times n, K)$ be the vector space of $m \times n$ matrices over K. Let $N(n,K)$ be the set of nilpotent $n \times n$ matrices over K and let $D(n,K)$ be the set of $n \times n$ matrices over K which are diagonalizable over K, that is, which are diagonalizable in $M(n \times n, K)$. Moreover, if K is a field with an involutory automorphism, let $R(n,K)$ be the set of normal $n \times n$ matrices over K, that is, the set of matrices commuting with the conjugate transpose, where conjugation is given by the involutory automorphism.

THEOREM 1. Gerstenhaber, Serezhkin *Let K be a field. The maximal dimension of a linear subspace in $N(n,K)$ is $\frac{n(n-1)}{2}$.*

The theorem above was proved by Gerstenhaber under the assumption that K has at least n elements (see [10]) and the result was generalized by Serezhkin for any field in [16]. We mention also that in [11], the authors gave a new simple proof of the result. In [12] and [7] the authors generalized it as follows:

Mathematics subject classification (2020): 15A30.
Keywords and phrases: Affine subspaces, nilpotent matrices, normal matrices, diagonalizable matrices.

THEOREM 2. (Quinlan, De Seguins Pazzis) *Let K be a field. The maximal dimension of a linear subspace of $n \times n$ matrices over K with no nonzero eigenvalue in K is $\frac{n(n-1)}{2}$.*

As observed in [6], the statement above is equivalent to the statement that the maximal dimension of an affine subspace of invertible $n \times n$ matrices over K is $\frac{n(n-1)}{2}$. Obviously, invertibility is equivalent to being of rank n, so we can say that this is a theorem on maximal dimension of an affine subspace of matrices with constant rank. We say that an affine subspace S of $M(m \times n, K)$ has constant rank r if every matrix of S has rank r and we say that a linear subspace S of $M(m \times n, K)$ has constant rank r if every nonzero matrix of S has rank r. There are many other theorems on the maximal dimension of a linear or an affine subspace of matrices with constant rank. For instance, the papers [2], [4], [5], [17] deal with the maximal dimension of a linear subspace with constant rank r in $M(m \times n, \mathbb{C})$ or in the space of the complex symmetric $n \times n$ matrices; we quote also the more recent results on the maximal dimension of an affine subspace with constant rank r in the space of the $m \times n$ matrices, in the space of symmetric $n \times n$ matrices and in the space of antisymmetric (i.e. skew-symmetric) $n \times n$ matrices, see [8], [9], [13], [14], [15]. There are also theorems on the maximal dimension of a linear or an affine subspace of matrices whose rank is bounded below or above or both below and above, see for instance [1], [7], [8], [15] and the introduction of the last paper for a more detailed description of the results.

In this short note we focus on the maximal dimension of affine subspaces in $N(n, K)$ for any field K and on the maximal dimension of affine subspaces in $R(n, \mathbb{C})$ and in $D(n, \mathbb{R})$; precisely we prove the following theorems.

THEOREM 3. *Let K be a field. The maximal dimension of an affine subspace in $N(n, K)$ is $\frac{n(n-1)}{2}$.*

Moreover, if the characteristic of K is zero, an affine not linear subspace in $N(n, K)$ has dimension less than or equal to $\frac{n(n-1)}{2} - 1$.

THEOREM 4. *The maximal dimension of an affine subspace in $R(n, \mathbb{C})$ is n.*

THEOREM 5. *The maximal dimension of a linear subspace in $D(n, \mathbb{R})$ is $\frac{n(n+1)}{2}$, while the maximal dimension of an affine not linear subspace in $D(n, \mathbb{R})$ is $\frac{n(n+1)}{2} - 1$.*

2. Proof of the theorems

NOTATION 6. Let $n \in \mathbb{N} - \{0\}$. For any field K, we denote the $n \times n$ identity matrix over K by I_n^K. We omit the superscript and the subscript when it is clear from the context.

We denote by $E_{i,j}^K$ the $n \times n$ matrix, whose (i, j)-entry is 1 and all the other entries are zero. We omit the superscript when it is clear from the context.

We denote by $A(n, K)$ the subspace of the antisymmetric (also called skew-symmetric) matrices of $M(n \times n, K)$.

For any square matrix A, let $S_2(A)$ be the sum of the 2×2 principal minors of A. For any complex matrix A, let A^* be the transpose of the conjugate matrix of A. For any $m\times n$ matrix A over a field K, let $f_A : K^n \to K^m$ be the linear map

$$x \longmapsto Ax.$$

To prove Theorem 3 we follow the guidelines of the proof of Theorem 1 in [11] and firstly we need to generalize some lemmas in [11].

LEMMA 7. *Let $n\in\mathbb{N}-\{0\}$ and K be a field. Let $R\in M(n\times n,K)$ and $U\in N(n,K)$. Then*

$$S_2(R)-S_2(R+U)=\mathrm{tr}(RU).$$

Proof. • First let us suppose that U is in Jordan form. Then

$$\begin{aligned}
&S_2(R)-S_2(R+U)\\
&=\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}(R_{i,i}R_{j,j}-R_{i,j}R_{j,i})-\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}((R+U)_{i,i}(R+U)_{j,j}-(R+U)_{i,j}(R+U)_{j,i})\\
&=\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}(R_{i,i}R_{j,j}-R_{i,j}R_{j,i})-\sum_{\substack{i,j\in\{1,\dots,n\}\\ i<j}}(R_{i,i}R_{j,j}-(R+U)_{i,j}R_{j,i})\\
&=\sum_{i\in\{1,\dots,n-1\}}U_{i,i+1}R_{i+1,i}=\sum_{j\in\{1,\dots,n\}}\sum_{i\in\{1,\dots,n\}}R_{j,i}U_{i,j}\\
&=\sum_{j\in\{1,\dots,n\}}(RU)_{j,j}=tr(RU).
\end{aligned}$$

• Now let U be generic. Let $C\in GL(n,K)$ be such that $C^{-1}UC$ is in Jordan form. Then

$$\begin{aligned}
S_2(R)-S_2(R+U)&=S_2(C^{-1}RC)-S_2(C^{-1}(R+U)C)\\
&=S_2(C^{-1}RC)-S_2(C^{-1}RC+C^{-1}UC)\\
&=tr(C^{-1}RCC^{-1}UC)=tr(RU),
\end{aligned}$$

where in the last but one equality we have used the previous item. □

LEMMA 8. *Let $n\in\mathbb{N}-\{0\}$ and K be a field. Let $P,A,B\in M(n\times n,K)$ such that P, $P+A$, $P+B$, $P+A+B\in N(n,K)$. Then* $\mathrm{tr}(AB)=0$.

Proof. By the previous lemma, we have:

$$S_2(A)-S_2(A+P)=\mathrm{tr}(AP)$$

(since P is nilpotent) and

$$S_2(A)-S_2(A+B+P)=\mathrm{tr}(A(B+P))$$

(since $B+P$ is nilpotent). Hence we get:

$$\begin{aligned}\operatorname{tr}(AB) &= \operatorname{tr}(A(B+P)) - \operatorname{tr}(AP)\\ &= S_2(A) - S_2(A+B+P) - S_2(A) + S_2(A+P)\\ &= -S_2(A+B+P) + S_2(A+P) = -0+0 = 0,\end{aligned}$$

where the last but one equality holds because $A+P$ and $A+B+P$ are nilpotent. $\square$

The following theorem will be useful to prove Theorem 4 and probably it is well-known; for the convenience of the reader we include the proof here.

THEOREM 9. *Let K be a field. If Z is a linear subspace of $M(n\times n,K)$ such that A,B are simultaneously diagonalizable for every $A,B\in Z$, then there exists a basis $\mathscr{B}$ of K^n such that every element of $\mathscr{B}$ is an eigenvector of every element of Z.*

To prove it, we need the following proposition.

PROPOSITION 10. *Let K be a field. If $\{A_1,\dots,A_r\}$ is a subset of $M(n\times n,K)$ such that A_i and A_j are simultaneously diagonalizable for every $i,j\in\{1,\dots,r\}$, then $A_1,\dots,A_r$ are collectively simultaneously diagonalizable, that is, there exists a basis $\mathscr{B}$ of K^n such that every element of $\mathscr{B}$ is an eigenvector of A_i for every $i\in\{1,\dots,r\}$.*

Proof. We prove the statement by induction on n.

In the case $n=1$ there is nothing to prove.

So we suppose that the statement is true for $k\times k$ matrices with $k<n$ and we prove it for $n\times n$ matrices.

Let $A_1,\dots,A_r$ be pairwise simultaneously diagonalizable $n\times n$ matrices.

If, for every $i=1,\dots,r$, the matrix A_i has only one eigenvalue, then, for every $i=1,\dots,r$, the matrix A_i is a multiple of the identity matrix and the statement is obvious.

So we can suppose that for some i the matrix A_i has more than one eigenvalue, for instance we can suppose that A_1 has more than one eigenvalue. Let E be an eigenspace of A_1 and let E' be the sum of the other eigenspaces of A_1; call m the dimension of E. Let $\mathscr{A}$ be an ordered basis of E and $\mathscr{A}'$ be an ordered basis of E'; let $\mathscr{B}$ be the ordered basis of K^n defined as the union of $\mathscr{A}$ and $\mathscr{A}'$.

For every $i=2,\dots,r$, it follows from the simultaneous diagonalizabilty of A_i and A_1 that the spaces E and E' are preserved by f_{A_i}. Hence, for every $i=1,\dots,r$, the matrix C_i expressing f_{A_i} (see Notation 6) in the basis $\mathscr{B}$ is a block diagonal matrix with the first block $m\times m$, call it H_i, and the second block $(n-m)\times(n-m)$, call it G_i:

$$C_i = \begin{pmatrix} H_i & 0\\ 0 & G_i\end{pmatrix}.$$

For every $i,j\in\{1,\dots,r\}$ the matrix C_i and the matrix C_j commute and they are diagonalizable since they are similar respectively to A_i and A_j; therefore the matrix H_i and the matrix H_j commute and they are diagonalizable and the matrix G_i and the matrix

G_j commute and they are diagonalizable. So the matrices $H_1,\dots,H_r$ are pairwise simultaneously diagonalizable and the matrices $G_1,\dots,G_r$ are pairwise simultaneously diagonalizable.

Hence, by induction assumption, the matrices $H_1,\dots,H_r$ are collectively simultaneously diagonalizable and the matrices $G_1,\dots,G_r$ are collectively simultaneously diagonalizable. Therefore the matrices $C_1,\dots,C_r$ are collectively simultaneously diagonalizable (they are diagonalizable by a block diagonal matrix whose first block is a matrix diagonalizing $H_1,\dots,H_r$ and whose second block is a matrix diagonalizing $G_1,\dots,G_r$). Thus the matrices $A_1,\dots,A_r$ are collectively simultaneously diagonalizable. $\square$

Proof of Theorem 9. Let $\{A_1,\dots,A_r\}$ be a basis of Z; by Proposition 10 there exists a basis $\mathscr{B}$ of K^n such that every element of $\mathscr{B}$ is an eigenvector of A_i for every $i\in\{1,\dots,r\}$. But then every element of $\mathscr{B}$ is an eigenvector of any linear combination of the matrices A_i, that is of every element of Z. $\square$

As we have already said the proof of Theorem 3 is very similar to the proof of Theorem 1 in [11] but we need to use Lemma 8.

Proof of Theorem 3. Let $P\in M(n\times n,K)$ and let Z be a linear subspace of $M(n\times n,K)$. Let $S=P+Z$ and suppose $S\subset N(n,K)$. We want to show that $\dim(S)$, that is $\dim(Z)$, is less than or equal to $\frac{n(n-1)}{2}$. We can suppose that P is in Jordan form.

Let T be the set of the strictly upper triangular matrices.

Consider the bilinear form on $M(n\times n,K)$ defined by $(A,B)\mapsto tr(AB)$. It is nondegenerate and, with respect to this bilinear form, we have that

$$T^{\perp}=\{A\in M(n\times n,K)\,|\,A\text{ upper triangular}\}.$$

Let $Z_1=Z\cap T$ and let Z_2 be a subspace such that $Z=Z_1\oplus Z_2$. Observe that

$$Z_2\cap T^{\perp}\subset T, \tag{1}$$

in fact, if A is an upper triangular matrix such that $P+A$ is nilpotent, then A is strictly upper triangular (remember that P is in Jordan form and nilpotent).

Moreover, for the definition of Z_1 and Z_2, we have that $Z_2\cap T=\{0\}$. From this and from (1), we get:

$$Z_2\cap T^{\perp}=\{0\}. \tag{2}$$

Obviously, since $Z_1\subset T$, we have that

$$T^{\perp}\subset Z_1^{\perp}. \tag{3}$$

Finally, from Lemma 8, we have that

$$Z_2\subset Z_1^{\perp} \tag{4}$$

From (2), (3) and (4), we get that $Z_2\oplus T^{\perp}\subset Z_1^{\perp}$, hence

$$\dim(Z_2)+\dim(T^{\perp})\leqslant\dim(Z_1^{\perp})=n^2-\dim(Z_1),$$

thus

$$\dim(Z) \leqslant n^2 - \dim(T^{\perp}) = \frac{n(n-1)}{2}.$$

Suppose now that the characteristic of K is 0 and that $P \notin Z$. Let $\{z_1, \ldots, z_h\}$ be a basis of Z. Then the span of P and Z is generated by $P, P+z_1, \ldots, P+z_h$. The additive semigroup generated by $P, P+z_1, \ldots, P+z_h$ consists only of nilpotents. Theorem 3 in the paper [11] states that the linear space generated by a set $\mathscr{E}$ of $n \times n$ matrices over a field of characteristic 0 consists only of nilpotents if and only if the additive semigroup generated by $\mathscr{E}$ consists only of nilpotents. Hence the linear space generated by $P, P+z_1, \ldots, P+z_h$, that is the span of P and Z, consists only of nilpotents. Hence, by Theorem 1 the dimension of the span of P and Z is less than or equal to $\frac{n(n-1)}{2}$, thus the dimension of Z is less than or equal to $\frac{n(n-1)}{2} - 1$. □

REMARK 11. The second statement of Theorem 3 need not be true if the characteristic of the field K is not zero: take $K = \mathbb{Z}/2$, $n = 2$ and $S = E_{1,2} + \langle E_{1,2} + E_{2,1} \rangle$; it consists only of $E_{1,2}$ and of $E_{2,1}$, which are both nilpotent, and its dimension is 1, that is $\frac{n(n-1)}{2}$.

Proof of Theorem 4. Let $P \in M(n \times n, \mathbb{C})$ and let Z be a linear subspace of $M(n \times n, \mathbb{C})$. Let $S = P + Z$ and suppose $S \subset R(n, \mathbb{C})$. We want to show that $\dim(S)$, that is $\dim(Z)$, is less than or equal to n.

Let $A \in Z$; then

$$(P+sA)(P^*+sA^*) = (P^*+sA^*)(P+sA) \quad \forall s \in \mathbb{R},$$

and this implies

$$PP^* + s(AP^* + PA^*) + s^2 AA^* = P^*P + s(P^*A + A^*P) + s^2 A^*A \quad \forall s \in \mathbb{R},$$

hence (since P is normal)

$$s(AP^* + PA^*) + s^2 AA^* = s(P^*A + A^*P) + s^2 A^*A \quad \forall s \in \mathbb{R},$$

in particular, if we take $s = 1$ and $s = -1$, we get that A is normal. Hence $Z \subset R(n, \mathbb{C})$.

So, to prove our statement, it is sufficient to prove that, if Z is a linear subspace in $R(n, \mathbb{C})$, then $\dim(Z) \leqslant n$.

Let $A, B \in Z$. Hence $A + zB \in Z \subset R(n, \mathbb{C})$ for any $z \in \mathbb{C}$, thus

$$(A+zB)(A^* + \overline{z}B^*) = (A^* + \overline{z}B^*)(A+zB),$$

that is

$$AA^* - A^*A + z(BA^* - A^*B) + \overline{z}(AB^* - B^*A) + |z|^2(BB^* - B^*B) = 0.$$

Since A and B are normal, we get that, for any $r, s \in \mathbb{R}$,

$$(r+is)(BA^* - A^*B) + (r-is)(AB^* - B^*A) = 0.$$

If we take first $r=1$ and $s=0$ and then $r=0$ and $s=1$, we get respectively

$$(BA^*-A^*B)+(AB^*-B^*A)=0$$

and

$$(BA^*-A^*B)-(AB^*-B^*A)=0.$$

Therefore

$$BA^*-A^*B=AB^*-B^*A=0.$$

In particular A and B^* are simultaneously diagonalizable and so, being normal, they are unitarily simultaneously diagonalizable. Since a unitary matrix diagonalizing B^* diagonalizes also B, we get that A and B are simultaneously diagonalizable. So we have proved that every couple of matrices in Z is simultaneously diagonalizable. By Theorem 9 there is a basis $\mathscr{B}$ of $\mathbb{C}^n$ such that every element of $\mathscr{B}$ is an eigenvector of every element of Z. Let C be an $(n\times n)$-matrix whose columns are the elements of $\mathscr{B}$. Hence $C^{-1}ZC$ is a linear subspace of diagonal matrices. So

$$\dim(Z)=\dim(C^{-1}ZC)\leqslant n.$$

So we have proved that the dimension of an affine subspace in $R(n,\mathbb{C})$ is less than or equal to n. Obviously the diagonal $n\times n$ matrices form a linear subspace in $R(n,\mathbb{C})$ of dimension n, so the maximal dimension of an affine subspace in $R(n,\mathbb{C})$ is equal to n. □

REMARK 12. Oberve that the proof of Theorem 4 depends in essential way on the properties of the complex field and for any field with an involutory automorphism the result need not be true, as we can see if we take the involutory automorphism equal to the identity (so that the normal matrices coincide with the symmetric ones).

Proof of Theorem 5. The first statement is obvious, in fact a linear subspace contained in $D(n,\mathbb{R})$ can intersect $A(n,\mathbb{R})$ only in 0, so its dimension must be less than or equal to $\frac{n(n+1)}{2}$ and the linear subspace of the symmetric matrices achieves this dimension.

Now let S be an affine not linear subspace contained in $D(n,\mathbb{R})$; let

$$S=P+Z,$$

where Z is a linear subspace (and obviously $P\notin Z$). We want to show that

$$\dim(Z)\leqslant\frac{n(n+1)}{2}-1.$$

Let W be the span of P and Z. If the dimension of Z were greater than or equal to $\frac{n(n+1)}{2}$, then the dimension of W would be greater than or equal to $\frac{n(n+1)}{2}+1$, so the intersection of W and $A(n,\mathbb{R})$ would contain a nonzero matrix Y.

Since $A(n,\mathbb{R})\cap D(n,\mathbb{R})=\{0\}$ and $W\setminus Z\subset D(n,\mathbb{R})$, we would have that $Y\in Z$. Hence $P+tY$ would be in S for every $t\in\mathbb{R}$, therefore it would be diagonalizable over

$\mathbb{R}$ for every $t \in \mathbb{R}$. But Y is nonzero and antisymmetric, hence it has at least a nonzero pure imaginary eigenvalue; so, since the roots of a complex polynomial vary continously as a function of the coefficients (see [3]), for a sufficiently large t at least one of the complex eigenvalues of $P+tY$ is not real, hence $P+tY$ cannot be diagonalizable over $\mathbb{R}$. So we get a contradiction, hence we must have $\dim(S) = \dim(Z) \leqslant \frac{n(n+1)}{2} - 1$.

Now let U be the span of the matrices $E_{i,i}$ for $i = 2,\ldots,n$ and of $E_{i,j} + E_{j,i}$ for $i,j \in \{1,\ldots,n\}$ with $i < j$. Obviously $E_{1,1} + U$ is an affine not linear subspace of dimension $\frac{n(n+1)}{2} - 1$ whose elements are symmetric and hence diagonalizable over $\mathbb{R}$. Thus we have proved that the maximal dimension of an affine not linear subspace in $D(n,\mathbb{R})$ is $\frac{n(n+1)}{2} - 1$. □

REMARK 13. One can wonder if the only linear subspace of maximal dimension in $D(n,\mathbb{R})$ is the subspace of the symmetric matrices, but the answer is obviously no: for instance the linear subspace $\left\{ \begin{pmatrix} a & b \\ 2b & c \end{pmatrix} \mid a,b,c \in \mathbb{R} \right\}$ is contained in $D(2,\mathbb{R})$ and has dimension 3.

Acknowledgements. This work was supported by the National Group for Algebraic and Geometric Structures, and their Applications (GNSAGA-INdAM).

The author wishes to thank the anonymous referee for his/her comments, which helped to improve the paper.

REFERENCES

[1] H. FLANDERS, *On spaces of linear transformations with bounded rank*, J. London Math. Soc., **37** (1962), pp. 10–16.

[2] F. R. GANTMACHER, *The Theory of Matrices*, vol. 2 Chelsea, Publishing Company New York 1959.

[3] G. HARRIS, C. MARTIN, *The roots of a complex polynomial vary continously as a function of the coefficients*, Proc. A.M.S., **100** (1987), pp. 390–392.

[4] W. V. D. HODGE, D. PEDOE, *Methods of Algebraic Geometry*, vol. 2, Cambridge University Press-Cambidge 1994.

[5] B. ILIC, J. M. LANDSBERG, *On symmetric degeneracy loci, spaces of symmetric matrices of constant rank and dual varieties*, Mathematischen Annlen, **314** (1999), pp. 159–174.

[6] C. DE SEGUINS PAZZIS, *Large affine spaces of non-singular matrices*, Trans. A.M.S., **365** (2013), pp. 2569–2596.

[7] C. DE SEGUINS PAZZIS, *On the matrices of given rank in a large subspace*, Linear Algebra Appl., **435** (2011), pp. 147–151.

[8] C. DE SEGUINS PAZZIS, *On affine spaces of alternating matrices with constant rank*, Linear and Multilinear Algebra, published on line, (2024), https://doi.org/10.1080/03081087.2024.2356827.

[9] C. DE SEGUINS PAZZIS, *On affine spaces of rectangular matrices with constant rank*, arXiv:2405.02689.

[10] M. GERSTENHABER, *On Nilalgebras and Linear Varieties of Nilpotent Matrices, I*, Amer. J. Math., **80** (1958), pp. 614–622.

[11] B. MATHES, M. OMLADIČ, H. RADJAVI, *Linear subspaces of nilpotent matrices*, Linear Algebra Appl., **149** (1991), pp. 215–225.

[12] R. QUINLAN, *Spaces of matrices without non-zero eigenvalues in their field of definition, and a question of Szechtman*, Linear Algebra Appl., **434** (2011), pp. 1580–1587.

[13] E. RUBEI, *Affine subspaces of matrices with constant rank*, Linear Algebra Appl., **644** (2022), pp. 259–269.
[14] E. RUBEI, *Affine subspaces of antisymmetric matrices with constant rank*, Linear and Multilinear Algebra, **72** (2024), pp. 1741–1750.
[15] E. RUBEI, *Affine subspaces of matrices with rank in a range*, arXiv:2405.04694, to appear in The Electronic Journal of Linear Algebra.
[16] V. N. SEREZHKIN, *Linear transformations preserving nilpotency* (Russian), Vestsi Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk, **6** (1985), pp. 46–50, 125.
[17] R. WESTWICK, *Spaces of matrices of fixed rank*, Linear and Multilinear Algebra, **20** (1987), pp. 171–174.

(Received September 16, 2023)

Elena Rubei
Dipartimento di Matematica e Informatica
"U. Dini"
viale Morgagni 67/A, 50134 Firenze, Italia
e-mail: elena.rubei@unifi.it