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HYPERCYCLICITY CRITERIA ON NON–ARCHIMEDEAN BANACH SPACES OF COUNTABLE TYPE

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Abstract. In this paper we initiate the investigation of non-archimedean linear dynamics. We study the hypercyclicity in Non-archimedean setting. Then, we give some Hypercyclicity Criteria of operators on Non-archimedean Banach spaces of Coutable Type.

1. Introduction

Hypercyclicity is the study of operators that possess a dense orbit. There is many approaches for proving the hypercyclicity of an operator. The first one is the Birkhoff's theorem [12] which establishes the equivalence between the hypercyclicity of an operator on separable F-space and his topological transitivity. This theorem is a direct application of the Baire category theorem. The best known and earliest examples of hypercyclic operators are due to Birkhoff [12], MacLane [19] and Rolewicz [25]. Motivated by these examples, researchers began to study the dynamical properties of general operators. One of the first important result is the Hypercyclicity Criterion which was developed independently by Kitai [17] and Gethner and Shapiro [15]. It states that an operator $\mathbf{T}$ is hypercyclic if it has a right inverse S and a dense subset D such that the orbits by $\mathbf{T}$ and S of the elements of D tend to zero. Some generalizations of this criterion have been later introduced and investigated. Hence, during the last years Hypercyclicity Criterion on Banach or Fréchet spaces has attracted many Mathematicians working in linear Functional Analysis. However, all these investigations were considered over the fields of real or complex numbers. But now the Analysis over non-archimedean valued fields is a well-established discipline, which was developed intensively, since the forties of the last century, thanks to the efforts of many Mathematicians ([20], [21], [24], [26], [29]). We deal here with the concept of hypercyclicity on infinite-dimensinal non-archimedean Banach spaces. Since the definition of the hypercyclicity requires a countable dense set in the space, this concept only occurs in separable spaces. But in the non-archimedean context, this assumption entails that the non-archimedean field must be separable as well, which is not always the case, as for the real or complex cases. In non-archimedean setting, we introduce an other concept

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which fits better the non-archimedean structure. By linearizing the notion of separability, we obtain a useful generalization for every scalar field, namely a space of countable type. We say that a space is of countable type if it contains a countable set whose linear hull is dense. So, every separable space is of countable type, and if $\mathbb{K}$ is itself separable, the two notions are equivalent. The dynamics of linear operators and, as a special case, a study of hypercyclic operators was actively developed for the last thirty years. This study is strongly related to the unsolved problem in the Hilbert space operator theory, which is the Invariant Subspace Problem. This problem deals with the question wether every operator on a separable infinite-dimensional Hilbert space has a non-trivial closed invariant subspace. It is easy to check that if every nonzero vector of a space $\mathbf{E}$ is hypercyclic for an operator $\mathbf{T}$, then $\mathbf{T}$ has no closed invariant subsets, and so no closed invariant subspaces as well.

Recently, F. Mukhamedov and O. Khakimov [17] studied the hypercyclicity and supecyclicity of operators on a separable non-archimedean Fréchet space. But this automatically implies that the field $\mathbb{K}$ must be separable, which is not always true as in the classical case.

The purpose of this paper is to initiate the study of linear dynamics in the non-archimedean setting. We begin by the concept of hypercyclicity. Hence, we give some hypercyclicity criteria for an operator on infinite-dimensional non-archimedean Banach space of countable type. In section one, we introduce the definitions and notations we will use in this paper. In section two, we describe some facts about the linear dynamics. In section three, we give some basic results about the infinite-dimensional non-archimedean Banach spaces of countable type. We establish that each one of these spaces is linearly homeomorphic to a non-archimedean Banach space $c_0(\mathbb{N})$ of null scalar sequences. Then, up to linear homeomorphisms, there is only one infinite-dimensional non-archimedean Banach space of countable type, namely $c_0(\mathbb{N})$. Hence, for the study of the concept of hypercyclicity, or any other topological concept, in any infinite-dimensional non-archimedean Banach space of countable type, it is completely sufficient to undertake this study in $c_0(\mathbb{N})$ endowed with the topology induced by the non-archimedean norm $\|.\|_\infty$. In the forth section, we give the main results of this paper, namely Theorems 14, 15 and 17, which are hypercyclicity criteria of an operator defined on any infinite-dimensional non-archimedean Banach space of countable type.

2. Preliminaries

Throughtout the present paper, $\mathbb{K}$ will denote a complete valued field with a non-archimedean non-trivial absolute value $|.|$. $\mathbb{K}$ is said to be spherically complete if every shrinking sequence of closed balls in $\mathbb{K}$ has a non-empty intersection. Clearly the spherical completion implies completion, but the converse is not true in general [29]. Normed spaces over $\mathbb{K}$ are defined in a natural way. We say that a norm $\|.\|$ on a $\mathbb{K}$-vector space $\mathbf{E}$ is non-archimedean if it satisfies the strong triangle inequality: $\|x+y\| \leqslant \max\{\|x\|,\|y\|\}$ for all $x,y\in\mathbf{E}$. We say that a normed space is non-archimedean if its topology is defined by a non-archimedean norm. Let $\mathbf{E}$ be a non-archimedean normed space. For any subset A of $\mathbf{E}$, $[A]$ will denote the linear hull of A in $\mathbf{E}$. Let $t\in]0,1]$, nonzero elements x and y of $\mathbf{E}$ are called $\mathbf{T}$-orthogonal if $d(x,[y])\geqslant t.\|x\|$, where $d(x,[y]) = \inf\{\|x-z\|/z \in [y]\}$ is the distance of x to $[y]$. We write $x \perp_t y$. If $t = 1$, we say that x and y are orthogonal, and we write $x \perp y$. We check easily that $x \perp_t y$ if, and only if, $\|\alpha x + \beta y\| \geqslant t.\max\{\|\alpha x\|, \|\beta y\|\}$ for all $\alpha, \beta \in \mathbb{K}$. We say that a family of nonzero elements $(x_i)_{i\in I}$ of $\mathbf{E}$ is $\mathbf{T}$-orthogonal if for each $i \in I, x_i \perp_t x_j$ for all $j \in I\backslash\{i\}$. Clearly, $(x_i)_{i\in I}$ is $\mathbf{T}$-orthogonal if, and only if, for each finite $J \subset I$, $(x_i)_{i\in J}$ is $\mathbf{T}$-orthogonal if, and only if, for each distinct $i_1,\ldots,i_n \in I$, and for each $\lambda_1,\ldots,\lambda_n \in \mathbb{K}$, $\|\sum_{k=1}^n \lambda_k x_{i_k}\| \geqslant t.\max_{1\leqslant k\leqslant n}\|\lambda_k x_{i_k}\|$.

If, in addition, $\mathbf{E} = \overline{[x_i/i \in I]}$, then we say that $(x_i)_{i\in I}$ is a $\mathbf{T}$-orthogonal basis of $\mathbf{E}$. More if $\|x_i\| = 1$ for all $i \in I$, we say that $(x_i)_{i\in I}$ is a $\mathbf{T}$-orthonormal basis of $\mathbf{E}$.

If $(x_i)_{i\in I}$ is a $\mathbf{T}$-orthogonal basis of $\mathbf{E}$, for every $x \in \mathbf{E}$, there is a unique family $(\lambda_i)_{i\in I} \in \mathbb{K}^I$ such that: $x = \sum_{i\in I}\lambda_i x_i$ and $\|x\| \geqslant t.\sup_{i\in I}\|\lambda_i x_i\|$.

Recall that $x = \sum_{i\in I}\lambda_i x_i$ if, and only if, for all $\varepsilon > 0$, there exists J_0 finite $\subset I$ such that $\|x - \sum_{i\in J}\lambda_i x_i\| < \varepsilon$ for all J finite with $J_0 \subset J \subset I$. In that case, there exists a countable set $I_0 \subset I$ such that $x_i = 0$ for all $i \in I\backslash I_0$. Hence, $x = \sum_{i\in I}\lambda_i x_i = \sum_{i\in I_0}\lambda_i x_i$.

We note that if $(x_i)_{i\in I}$ is a $\mathbf{T}$-orthogonal family in $\mathbf{E}$ and $(\lambda_i)_{i\in I}$ is a family of nonzero elements of $\mathbb{K}$, then $(\lambda_i x_i)_{i\in I}$ is also a $\mathbf{T}$-orthogonal family in $\mathbf{E}$. In particular, if we take $\pi \in \mathbb{K}$ with $0 < |\pi| < 1$, then we can choose $(\lambda_i)_{i\in I} \in \mathbb{K}^I$ such that $|\pi| \leqslant \|\lambda_i x_i\| \leqslant 1$ for all $i \in I$.

As a consequence, if $(x_i)_{i\in I}$ is a $\mathbf{T}$-orthogonal basis of $\mathbf{E}$, without loss of generality, we can suppose that $(x_i)_{i\in I}$ satisfies $|\pi| \leqslant \|x_i\| \leqslant 1$ for all $i \in I$.

Let $c_0(I) = \{(\lambda_i)_{i\in I} \in \mathbb{K}^I/\lim_{i\in I}\lambda_i = 0\}$. Endowed with the non-archimedean norm $\|(\lambda_i)_i\|_\infty = \sup_{i\in I}|\lambda_i|$, $c_0(I)$ is a non-archimedean Banach space.

For each $i \in I$, let $e_i = (\delta_{ij})_{j\in I}$, with δ_{ij} is the Kronecker symbol. We denote $\varphi(I)$ the linear hull of $\{e_i/i \in I\}$. Clearly $(e_i)_{i\in I}$ forms an orthonormal basis of $c_0(I)$.

We note that if $(x_i)_{i\in I}$ is $\mathbf{T}$-orthogonal family of nonzero elements of non-archimedean normed space $\mathbf{E}$ such that $\mathbf{E} = \overline{[x_i/i \in I]}$, then $(x_i)_{i\in I}$ is a $\mathbf{T}$-orthogonal basis of $\mathbf{E}$.

Let $l^\infty(I) = \{(\lambda_i)_{i\in I} \in K^I/\sup_i|\lambda_i| < +\infty\}$. $(l^\infty(I), \|.\|_\infty)$ is a non-archimedean Banach space containing $c_0(I)$ as a closed subspace.

3. Linear dynamics

Linear dynamics is a rapidly evolving area of operator theory. The study of orbits, and in particular cyclic vectors, has a long history and several classical problems and results in Analysis can be viewed as problems and results on orbits of operators. In recent years there has been growing interest to study orbits of operators in more detailed way than was done before. Hypercyclicity is the study of operators thas possess a dense orbits. It is a wide-spread phenomenon in Analysis. The field of hypercyclicity was born in 1982 with the Ph.D dissertation of Kitai [18]. The name of hypercyclicity was suggested in 1986 by Beauzamy [5] because of its connection to the much older concept in operator theory of a cyclic operator. A more systematic study of hypercyclic operators started in 1987 with Gethner and Shapiro [15]. In all what follows, an operator on a topological vector space $\mathbf{X}$ will be a continuous linear mapping $\mathbf{X} \to \mathbf{X}$. $L(\mathbf{X})$ is the space of all operators on $\mathbf{X}$. A linear dynamical system is a pair $(\mathbf{X}, \mathbf{T})$ consisting of a separable Fréchet space $\mathbf{X}$ and an operator $\mathbf{T}: \mathbf{X} \to \mathbf{X}$. Let $\mathbf{T} \in L(\mathbf{X})$ and $x \in \mathbf{X} \backslash \{0\}$, the $\mathbf{T}$-orbit of $\mathbf{X}$ is the set $O(x, \mathbf{T}) = \{\mathbf{T}^n.x/n \in N\}$. $\mathbf{T}$ is called hypercyclic if there exists $x \in \mathbf{X}$ such that its $\mathbf{T}$-orbit $O(x, T)$ is dense in $\mathbf{X}$. Then, $\mathbf{X}$ is called a $\mathbf{T}$-hypercyclic vector. The set of all $\mathbf{T}$-hypercyclic vectors is denoted by $HC(\mathbf{T})$. A first example of a hypercyclic operator was provided in 1929 by Birkhoff [12], who showed the existence of an entire function f whose successive translates by a nonzero constant λ are arbitrarily close to any function in the space of entire functions $H(\mathbb{C})$. A second example of a hypercyclic operator was provided in 1952 by MacLane [19], who showed that there exists a universal entire function f whose successive derivatives are dense in $H(\mathbb{C})$.

We note that the importance of the field of hypercyclicity is made evident by its connection to the famous Invariant Subspace Problem in operator theory, which asks wether every operator on a Hilbert space has a non-trivial closed $\mathbf{T}$-invariant subspace, and which still open until now. We stress that the definition of hypercyclicity can be difficult to work with, since it is not always easy to find a hypercyclic vector for a given operator. Fortunately there exist a number of approaches for establishing hypercyclicity. The first characterization of hypercyclicity is due to Birkhoff [12], it relates the concept of hypercyclicity to the well known notion of topological transitivity. $\mathbf{T} \in L(\mathbf{X})$ is called topologically transitive if for each pair of nonempty open sets $U, V \subset \mathbf{X}$, there exists $n \in \mathbb{N}$ such that $\mathbf{T}^n(U) \cap V \neq \emptyset$.

THEOREM 1. (Birkhoff Transitivity Theorem) *Let* $\mathbf{T}$ *be a continous map on separable complete metric space* $\mathbf{X}$ *without isolated points. Then the following assertions are equivalent:*

(i) $\mathbf{T}$ *is topologically transitive;*

(ii) There exists some $x \in \mathbf{X}$ *such that* $O(x, \mathbf{T})$ *is dense in* $\mathbf{X}$.

If one of these conditions holds, then the set of points of $\mathbf{X}$ *with dense orbit is a dense* G_δ*-set.*

Proof. [16], p. 10, Theorem 1.16. □

Another approach for establishing the hypercyclicity of an operator is by linear conjugacy.

DEFINITION 2. Let $(\mathbf{X}, \mathbf{T})$ and $(\mathbf{Y}, \mathbf{S})$ be linear dynamical systems.

(a) $\mathbf{T} \in L(\mathbf{X})$ is called linearly quasi-conjugate to $\mathbf{S} \in L(\mathbf{Y})$ if there exists an operator $\varphi : \mathbf{Y} \to \mathbf{X}$ with a dense range such that $\mathbf{T}\varphi = \varphi\mathbf{S}$, that is the following diagram commutes:

$$\begin{array}{ccc} & S & \\ Y & \rightarrow & Y \\ \varphi \downarrow & & \downarrow \varphi \\ X & \rightarrow & X \\ & T & \end{array}$$

(b) If φ is a linear homeomorphism, then $\mathbf{S}$ and $\mathbf{T}$ are called linearly conjugate.

The linear conjugacy introduces an equivalence relation among dynamical systems. And the conjugate dynamical systems have the same dynamical behaviour.

THEOREM 3. (1) *If* $\mathbf{T}$ *is linearly quasi-conjugate to* $\mathbf{S}$ *then:* $\mathbf{S}$ *is hypercyclic implies* $\mathbf{T}$ *is hypercyclic.*

(2) *If* $\mathbf{T}$ *and* $\mathbf{S}$ *are linearly conjugate then:* $\mathbf{S}$ *is hypercyclic if, and only if,* $\mathbf{T}$ *is hypercyclic.*

Proof. [16], p. 11, Proposition 1.19. □

We note also the very useful result of Shapiro [28], which shows that in search of hypercyclic operators on a space $\mathbf{X}$, it can be useful to find hypercyclic operators on smaller spaces.

THEOREM 4. (Hypercyclicity Comparison Principle) *Suppose* $\mathbf{X}$ *and* $\mathbf{Y}$ *are two normed spaces,* $\mathbf{Y}$ *is continuously and densely embedded in* $\mathbf{X}$*, and* $\mathbf{T}$ *is a linear transformation on* $\mathbf{X}$ *that maps the smaller space* $\mathbf{Y}$ *to itself and is continuous in the topology of each space. Then,* $\mathbf{T}$ *is hypercyclic on the larger space* $\mathbf{X}$ *whenever* $\mathbf{T}$ *is hypercyclic on* $\mathbf{Y}$.

In particular, if x *is an hypercyclic vector for* $T_{|\mathbf{Y}}$*, then* x *is an hypercyclic vector for* $\mathbf{T}$.

Proof. See [28]. □

While the theorem of Birkhoff characterizes hypercyclicity by topological transitivity, the most useful result in this theory is the so called Hypercyclicity Criterion, which is a set of sufficient conditions for realizing the hypercyclicity, discovered by Kitai in her Thesis [18], but she never published it. This criterion was rediscovered later by Gethner and Shapiro [15]. Its proof is easy, and it is quite easy to use nevertheless his seemingly complicated statement. They proved the following theorem:

THEOREM 5. (Kitai-Gethner-Shapiro Hypercyclicity Criterion) *Let* $\mathbf{T}$ *be an operator on a Fréchet space* $\mathbf{X}$*. If there exist a dense set* D *and a map* $\mathbf{S}:\mathbf{X}\to\mathbf{X}$ *such that:*

(1) $\mathbf{TS}=\mathbf{I}$;
(2) $\mathbf{T}^n\to 0$ *on* D;
(3) $\mathbf{S}^n\to 0$ *on* D.

Then, $\mathbf{T}$ *is hypercyclic.*

Using this result, Gethner and Shapiro presented much simpler proofs for the results of Birkhoff, MacLane and Rolewicz, as well as other examples of hypercyclic operators. Later, Bés and Peris [11] weakened the sufficient condition of the criterion and this is the form which is widely utilized in this theory.

THEOREM 6. (Kitai-Bés-Peris Hypercyclicity Criterion) *Let* $\mathbf{X}$ *be a separable Fréchet space and* $\mathbf{T}:\mathbf{X}\to\mathbf{X}$ *an operator.*

If there exist an increasing sequence (n_k) *of positive integers, two dense sets* D_1 *and* D_2*, and a sequence of maps* $\mathbf{S}_{n_k} : D_2 \to \mathbf{X}$ *such that:*

(1) $\mathbf{T}^{n_k}.x \to 0$ *for all* $x \in D_1$*;*

(2) $\mathbf{S}_{n_k}.y \to 0$ *for all* $y \in D_2$*;*

(3) $\mathbf{T}^{n_k}\mathbf{S}_{n_k}.y \to y$ *for all* $y \in D_2$*.*

Then, $\mathbf{T}$ *is hypercyclic.*

Proof. See [10], [11] and [4]. □

4. Non-archimedean Banach spaces of countable type

The separable Banach spaces played a special role in Archimedean analysis. We note that the useful property of this class of spaces is, in general, the existence of a countable subset whose linear hull is dense rather the existence of a dense countable subset. However, these properties are equivalent thanks to the separability of $\mathbb{R}$ and $\mathbb{C}$. But in non-archimedean analysis this is not the case, and the situation is completely different if $\mathbb{K}$ is not separable. In that case the only separable non-archimedean Banach space is $\{0\}$, since if $\mathbf{E}$ is a non-trivial separable non-archimedean Banach space over K, any one-dimensinal subspace of $\mathbf{E}$ is homeomorphic to $\mathbb{K}$, and $\mathbb{K}$ must be separable. However, there exist non-archimedean Banach spaces over a non separable non-archimedean field $\mathbb{K}$ that possess a countable subset which its linear hull is dense, such as all finite-dimensional Banach spaces and the non-archimedean Banach space of null sequences $c_0(\mathbb{N})$. Thus, the concept of separability is of no use in the theory of non-archimedean Banach spaces if the field is not separable. However, if we linearize this concept we obtain a new one which generalizes the separability and fits well the non-archimedean setting. In all what follows $\mathbf{E}$ will denote a non-archimedean Banach space over a non-archimedean valued field $\mathbb{K}$.

DEFINITION 7. $\mathbf{E}$ is of countable type if it contains a countable set whose linear hull is dense in $\mathbf{E}$.

It is evident that all separable non-archimedean Banach spaces are of countable type. And the converse is true if $\mathbb{K}$ is separable. Thus, the two notions are equivalent if the field $\mathbb{K}$ is separable.

We can check easily that:

(1) Every finite product of spaces of countable type is of countable type;

(2) Every countable direct sum of spaces of countable type is of countable type;

(3) Every subspace of a space of countable type is of countable type;

(4) Every continuous linear image of a space of countable type is of countable type.

PROPOSITION 8. *Let* F *be a dense subspace of* $\mathbf{E}$*. Then,* $\mathbf{E}$ *is of countable type if, and only if,* F *is of countable type.*

Proof. Suppose that F is of countable type, and let A be a countable subset of F such that $F = \overline{[A]}$. Then, $\mathbf{E} = \overline{[A]}$ and $\mathbf{E}$ is of countable type. Conversely, if $\mathbf{E}$ is of countable type, let D be a countable subset of $\mathbf{E}$ such that $\mathbf{E} = \overline{[D]}$. For each $x \in D$, there exists a countable subset A_x of F such that $x \in \overline{A_x}$. Let $A = \cup_{x\in D} A_x$. Then, A is countable and $F = \overline{[A]}$. Therefore, F is of countable type. □

THEOREM 9. *If* $\mathbf{E}$ *is of countable type, then for each* $t \in]0,1[$, $\mathbf{E}$ *has a countable* $\mathbf{T}$*-orthogonal basis.*

Proof. [21], p. 30, Theorem 2.3.7. □

THEOREM 10. *If* $\mathbf{E}$ *is of countable type and* $\mathbb{K}$ *is spherically complete, then* $\mathbf{E}$ *has a countable orthogonal basis.*

Proof. [21], p. 36, Theorem 2.3.25. □

COROLLARY 11. *Each non-archimedean Banach space of countable type is linearly homeomorphic to* $c_0(\mathbb{N})$.

Proof. Let $\mathbf{E}$ be a non-archimedean Banach space of countable type. Then, for $t \in]0,1[$, there exists (e_n) a $\mathbf{T}$-orthogonal basis of $\mathbf{E}$. So, for each $x \in \mathbf{E}$, there exists a unique $(\lambda_n) \in \mathbb{K}^{\mathbb{N}}$ such that $x = \sum_n \lambda_n e_n$ and $\|x\| \geqslant t.\sup_n \|\lambda_n e_n\|$.

Let $\pi \in \mathbb{K}$ such that $0 < |\pi| < 1$. Without loss of generality we can assume that $|\pi| \leqslant \|e_n\| \leqslant 1$ for all $n \in \mathbb{N}$. Define the linear map

$$f: \begin{array}{ccc} \mathbf{E} & \to & c_0(\mathbb{N}) \\ \sum_n \lambda_n e_n & \mapsto & (\lambda_n) \end{array}$$

f is well defined since for each $x = \sum_n \lambda_n e_n$ we have:

$$|\lambda_n| = \frac{\|\lambda_n e_n\|}{\|e_n\|} \leqslant \frac{\|\lambda_n e_n\|}{|\pi|} \to 0 \quad (n \to +\infty).$$

On the other hand, $\|x\| = \|\sum_n \lambda_n e_n\| \leqslant \sup_n \|\lambda_n e_n\| \leqslant \sup_n |\lambda_n| = \|f(x)\|_\infty$.

And by $\mathbf{T}$-orthogonality, $\|x\| \geqslant t.\sup_n \|\lambda_n e_n\| \geqslant t.|\pi| \sup_n |\lambda_n| = t.|\pi| \, \|f(x)\|_\infty$.

Hence, $t.|\pi| \, \|f(x)\|_\infty \leqslant \|x\| \leqslant \|f(x)\|_\infty$. Therefore, f is a homeomorphism. □

REMARK 12. As an important consequence of a Corollary 11, we can state that there exists only one non-archimedean Banach space of countable type, namely $c_0(\mathbb{N})$, or equivalently $c_0(I)$ with I is any infinte countable set.

By the same arguments, if $\mathbf{E}$ is a non-archimedean Banach space of countable type, for each $t \in]0,1[$, we can choose the $\mathbf{T}$-orthogonal basis for $\mathbf{E}$ of the form $(e_n)_{n\in\mathbb{Z}}$. Thus, for each $x \in \mathbf{E}$, there exists a unique $(\lambda_n) \in \mathbb{K}^{\mathbb{Z}}$ such that: $x = \sum_n \lambda_n e_n$ and $\|x\| \geqslant t.\sup_{n\in\mathbb{Z}} \|\lambda_n e_n\|$.

In a such case, $\mathbf{E}$ is linearly homeomorphic to $c_0(\mathbb{Z}) = \{(\lambda_i)_{i\in\mathbb{Z}} \in K^{\mathbb{Z}} / \lim_i \lambda_i = 0\}$.

5. Hypercyclicity criteria on $c_0(\mathbb{Z})$ and $c_0(\mathbb{N})$

Let $(e_i)_{i\in\mathbb{Z}}$ be the canonical basis of $c_0(\mathbb{Z})$. For each $j\in\mathbb{N}$, let $\mathbf{E}_j=[e_i/|i|\leqslant j]$. $(\mathbf{E}_j)_{j\in\mathbb{N}}$ is an increasing sequence of subspaces of $c_0(\mathbb{Z})$ such that $c_0(\mathbb{Z})=\cup_{j\in\mathbb{N}}\mathbf{E}_j$.

Before stating the first Hypercyclicity Criterion, we give the following lemma.

LEMMA 13. *Let $\mathbf{E}$ be a non-archimedean Banach space with a dense sequence (e_n), and let $\mathbf{T}$ be an operator on $\mathbf{E}$. If x is nonzero vector of $\mathbf{E}$ such that there exists an increasing sequence of positive integers (n_k) satisfying $\lim_k\|\mathbf{T}^{n_k}.x-e_k\|=0$.*

Then, x is a $\mathbf{T}$-hypercyclic vector.

Proof. Let y be an element of $\mathbf{E}$, and let $\varepsilon>0$.

There exists $k\in\mathbb{N}$ such that $\max\{\|y-e_k\|,\|\mathbf{T}^{n_k}.x-e_k\|\}<\varepsilon$.

Hence, $\|\mathbf{T}^{n_k}.x-y\|\leqslant\max\{\|\mathbf{T}^{n_k}.x-e_k\|,\|y-e_k\|\}<\varepsilon$. Therefore, $O(\mathbf{T},x)$ is dense in $\mathbf{E}$. □

Now, we give the Hyperciclicity Criterion of an operator on $c_0(\mathbb{Z})$.

THEOREM 14. *Let $\mathbf{T}$ be an operator on $c_0(\mathbb{Z})$ satisfying the following property: For each $\varepsilon>0$, $j\in\mathbb{N}$ and $x,y\in\mathbf{E}_j$, there exist $n\in\mathbb{N}$ and $z\in\mathbf{E}_{j+n}$ such that:*

$$\max\{\|z\|,\|\mathbf{T}^n.z-y\|,\|\mathbf{T}^n.x\|\}<\varepsilon.$$

Then, $\mathbf{T}$ is hypercyclic.

Proof. Let $l_k=\sum_{|i|\leqslant k}l_{ki}e_i$ be a basis of $c_0(\mathbb{Z})$. Let $n_0=1$ and $a=\|\mathbf{T}\|$. We apply the hypothesis of the theorem to $\varepsilon_1=\dfrac{1}{2^1a^1}$ and $x_1=y_1=l_1\in E_1$, then there exist $n_1>n_0$ and $z_1\in E_{n_0+n_1}$ such that:

$$\max\{\|z_1\|,\|\mathbf{T}^{n_1}.z_1-y_1\|,\|\mathbf{T}^{n_1}.x_1\|\}<\varepsilon_1.$$

Now; let $\varepsilon_2=\dfrac{1}{2^2a^{a_{n_1}}}$, $y_2=l_2\in\mathbf{E}_2$, and $x_2=z_1\in\mathbf{E}_{n_0+n_1}$. Then there exist $n_2>n_1$ and $z_2\in\mathbf{E}_{n_0+n_1+n_2}$ such that:

$$\max\{\|z_2\|,\|\mathbf{T}^{n_2}.z_2-y_2\|,\|\mathbf{T}^{n_2}.x_2\|\}<\varepsilon_2.$$

Hence, by induction, we construct an increasing sequence (n_k) of positive integers and sequences (x_k), (y_k), (z_k) in $c_0(\mathbb{Z})$ such that for all $k\geqslant2$ we have:

$$\begin{aligned}
&y_k=l_k;\\
&z_k\in E_{n_0+n_1+\ldots+n_k};\\
&x_k=z_1+z_2+\ldots+z_{k-1};\\
&\max\{\|z_k\|,\|\mathbf{T}^{n_k}.z_k-y_k\|,\|\mathbf{T}^{n_k}.x_k\|\}<\varepsilon_k=\frac{1}{2^ka^{n_{k-1}}}.
\end{aligned}$$

Let $z = \sum_{k\geqslant 1} z_k$. Clearly $z \in c_0(\mathbb{Z})$.

Let $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\max\{\varepsilon_k, \frac{1}{2^k}\} < \varepsilon$ for all $k \geqslant k_0$. For all $k \geqslant k_0$ we have:

$$\begin{aligned}\|\mathbf{T}^{n_k}.z - l_k\| &\leqslant \max\{\|\mathbf{T}^{n_k}.\sum_{1\leqslant i\leqslant k-1} z_i\|, \|\mathbf{T}^{n_k}.z_k - l_k\|, \|\sum_{i>k}\mathbf{T}^{n_k}.z_i\|\} \\ &< \max\{\varepsilon_k, \sup_{i>k}\|\mathbf{T}^{n_k}\|.\|z_i\|\} \\ &< \max\{\varepsilon_k, \frac{1}{2^{k+1}}\} < \varepsilon.\end{aligned}$$

Hence, $\lim_k \|\mathbf{T}^{n_k}.z - l_k\| = 0$. So, by lemma 13, $\mathbf{T}$ is hypercyclic. □

Now, let Let $(e_i)_{i\in\mathbb{N}}$ be the canonical basis of $c_0(\mathbb{N})$. For each $j \in \mathbb{N}$, let $F_j = [e_i/i \leqslant j]$. $(F_j)_{j\in\mathbb{N}}$ is an increasing sequence of subspaces of $c_0(\mathbb{N})$ such that $c_0(\mathbb{N}) = \cup_{j\in\mathbb{N}} F_j$. In the next theorem, we give the Hyperciclicity Criterion of an operator on $c_0(\mathbb{N})$.

THEOREM 15. *Let* $\mathbf{T}$ *be an operator on* $c_0(\mathbb{N})$ *satisfying the following property: There exists an increasing sequence* (n_k) *of positive integers such that for each* $\varepsilon > 0$, *and for each* $y \in F_{n_k}$, *there exists* $z \in F_{n_{k+1}}$ *such that:*

$$\max\{\|z\|, \|\mathbf{T}^{n_k}.z - y\|\} < \varepsilon.$$

Then, $\mathbf{T}$ *is hypercyclic.*

Proof. Let $a = \|\mathbf{T}\|$ and $\varepsilon_k = \frac{1}{2^{n_k} a^{2n_k}}$ for all $k \in \mathbb{N}$. And let $l_k = \sum_{i\leqslant k} l_{ki} e_i$ $(k \in \mathbb{N})$ be a basis of $c_0(\mathbb{N})$ such that $\|l_k\| < \frac{1}{2^{n_k} a^{n_k}}$ for all $k \in \mathbb{N}$.

Let $z_0 = 0$. For $y_1 = l_1 \in F_1$, there exists $z_1 \in F_{n_1}$ such that:

$$\max\{\|z_1\|, \|\mathbf{T}.z_1 - y_1\|\} < \varepsilon_1.$$

For $y_2 = l_2 + z_1 \in F_{n_1}$, there exists $z_2 \in F_{n_2}$ such that:

$$\max\{\|z_2\|, \|\mathbf{T}^{n_1}.z_2 - (l_2 + z_1)\|\} < \varepsilon_2.$$

For all $k \geqslant 3$, let $y_k = l_k + z_{k-1} + \ldots + z_1 \in F_{n_{k-1}}$, then there exists $z_k \in F_{n_k}$ such that:

$$\max\{\|z_k\|, \|\mathbf{T}^{n_{k-1}}.z_k - (l_k + z_{k-1} + \ldots + z_1)\|\} < \varepsilon_k.$$

For all $k \geqslant 2$, we have:

$$\|\mathbf{T}^{n_{k-1}}.z_k\| \leqslant a^{n_{k-1}}\|z_k\| < \frac{1}{2^{n_k} a^{n_k}}.$$

$\|\mathbf{T}^{n_{k-1}} z_k - l_k\| \leqslant \max\{\|\mathbf{T}^{n_{k-1}} z_k\|, \|l_k\|\} < \frac{1}{2^{n_k} a^{n_k}}$.

Hence, $\|\sum_{1\leqslant i\leqslant k-1} z_i\| < \varepsilon_k$. Then, $\|\mathbf{T}^{n_{k-1}}.\sum_{1\leqslant i\leqslant k-1} z_i\| < a^{n_{k-1}}\varepsilon_k < \frac{1}{2^{n_k}}$.

Let $z = \sum_{i\geqslant 1} z_i$. Clearly $z \in c_0(\mathbb{N})$.

Let $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\frac{1}{2^{n_k}} < \varepsilon$ for all $k \geqslant k_0$.
For all $k \geqslant k_0$ we have:

$$\begin{aligned}\|\mathbf{T}^{n_{k-1}}.z - l_k\| &\leqslant \max\{\|\mathbf{T}^{n_{k-1}}.\sum_{1\leqslant i\leqslant k-1} z_i\|, \|\mathbf{T}^{n_{k-1}}.z_k - l_k\|, \|\sum_{i>k}\mathbf{T}^{n_{k-1}}.z_i\|\}\\ &\leqslant \max\{\frac{1}{2^{n_k}}, \frac{1}{2^{n_k}a^{n_k}}, \sup_{i>k}\|\mathbf{T}^{n_{k-1}}\|.\|z_i\|\}\\ &\leqslant \frac{1}{2^{n_k}} < \varepsilon.\end{aligned}$$

Hence, $\lim_k \|\mathbf{T}^{n_{k-1}}.z - l_k\| = 0$. Therefore, by lemma 13, $\mathbf{T}$ is hypercyclic. □

We have the following Corollary which is a spacial case for Theorem 15.

COROLLARY 16. *Let* $\mathbf{T}$ *be an operator on* $c_0(\mathbb{N})$ *satisfying the following property:*
For each $\varepsilon > 0$*, and for each* $y \in F_m$ $(m \in \mathbb{N})$*, there exists* $z \in F_{2m}$ *such that:*

$$\max\{\|z\|, \|\mathbf{T}^m.z - y\|\} < \varepsilon.$$

Then, $\mathbf{T}$ *is hypercyclic.*

Now, we give an other Hyperciclicity Criterion of an operator on $c_0(\mathbb{N})$.

THEOREM 17. *Let* $\mathbf{T}$ *be an operator on* $c_0(\mathbb{N})$ *satisfying the following property:*
For each $\varepsilon > 0$*, and for each* $x, y \in F_m$ $(m \in \mathbb{N})$*, there exists* $z \in F_{m+1}$ *such that:*

$$\max\{\|z\|, \|\mathbf{T}^m.z - y\|, \|\mathbf{T}^m.x\|\} < \varepsilon.$$

Then, $\mathbf{T}$ *is hypercyclic.*

Proof. Let $l_k = \sum_{i\leqslant k} l_{ki}e_i$ $(k \in \mathbb{N})$ be a basis of $c_0(\mathbb{N})$. And let $a = \|\mathbf{T}\|$ and $\varepsilon_k = \frac{1}{2^{n_k}a^{2n_k}}$ for all $k \in \mathbb{N}$.
For $x_1 = y_1 = l_1 \in F_1$, there exists $z_1 \in F_2$ such that:

$$\max\{\|z_1\|, \|\mathbf{T}.z_1 - y_1\|, \|\mathbf{T}.x_1\|\} < \varepsilon_1.$$

For $y_2 = l_2 \in F_2$ and $x_2 = z_0 + z_1 \in F_2$, there exists $z_2 \in F_3$ such that:

$$\max\{\|z_2\|, \|\mathbf{T}^2.z_2 - y_2\|, \|\mathbf{T}^2.x_2\|\} < \varepsilon_2.$$

By induction, for all $n \geqslant 3$, let $y_n = l_n \in F_n$ and $x_n = z_0 + z_1 + \ldots + z_{n-1} \in F_n$, then there exists $z_n \in F_{n+1}$ such that:

$$\max\{\|z_n\|, \|\mathbf{T}^n.z_n - y_n\|, \|\mathbf{T}^n.x_n\|\} < \varepsilon_n.$$

Let $z = \sum_n z_n$. Evidently $z \in c_0(\mathbb{N})$.
Let $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\max\{\varepsilon_n, \frac{1}{2^n}\} < \varepsilon$ for all $n \geqslant n_0$.

For all $n \geqslant n_0$ we have:

$$\begin{aligned} \|\mathbf{T}^n.z - l_n\| &\leqslant \max\{\|\mathbf{T}^n. \sum_{0\leqslant k\leqslant n-1} z_k\|, \|\mathbf{T}^n.z_n - l_n\|, \|\sum_{k>n} \mathbf{T}^n.z_k\|\} \\ &< \max\{\varepsilon_n, \sup_{k>n} \|\mathbf{T}^n\|.\|z_k\|\} \\ &< \max\{\varepsilon_n, \frac{1}{2^n}\} < \varepsilon. \end{aligned}$$

Hence, $\lim_n \|\mathbf{T}^n.z - l_n\| = 0$. So, by lemma 13, $\mathbf{T}$ is hypercyclic. □

REMARK 18. We note that by the assumptions given in Theorems 14, 15 and 17 we must have $\|\mathbf{T}\| > 1$.

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