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SEVERAL PROPERTIES OF THE SPECTRUM AND LOCAL SPECTRUM OF CLASS A_n OPERATORS

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Abstract. In this article, we establish some conditions which imply the normality of class A_n. Also, we prove that if T is a class A_n and $\mathcal{M}$ is an invariant subspace of T such that $T|_{\mathcal{M}}$ is a normal operator with $0 \notin \sigma_p(T|_{\mathcal{M}})$, then $\mathcal{M}$ reduces T. Moreover, we show that Weyl's theorem holds for every class A_n operator and some results related to the Riesz idempotent of class A_n operators. By using the spectral properties of class A_n operators, we prove that a class A_n contraction is the direct sum of a unitary and a $C_{\cdot 0}$ completely non-unitary contraction. In addition, the existence of a nontrivial hyperinvariant subspace of a class A_n operator will be shown.

1. Introduction

Throughout this paper let $\mathcal{H}$ be a separable complex Hilbert space with inner product $\langle .,. \rangle$. Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. If $T \in \mathcal{B}(\mathcal{H})$ we shall write $\ker(T)$ and $\operatorname{ran}(T)$ for the null space and range of T. Also, let $\sigma(T)$, $\sigma_a(T)$, $\sigma_r(T)$ and $\sigma_p(T)$ denote the spectrum, approximate point spectrum, the residual spectrum and the point spectrum of T. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is T-invariant (or $\mathcal{M}$ is invariant under T) if $T(\mathcal{M}) \subseteq \mathcal{M}$. We say that an operator T has a non-trivial invariant closed subspace $\mathcal{M}$ of $\mathcal{H}$ if $\mathcal{M}$ is closed, T-invariant, and $\{0\} \neq \mathcal{M} \neq \mathcal{H}$. A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is called T-hyperinvariant if $S(\mathcal{M}) \subseteq \mathcal{M}$ for every $S \in \mathcal{B}(\mathcal{H})$ such that $TS = ST$ (see [26]). An operator T is said to be positive (denoted by $T \geqslant 0$) if $\langle Tx, x \rangle \geqslant 0$ for all $x \in \mathcal{H}$, and also T is said to be strictly positive (denoted by $T > 0$) if it is positive and invertible. The polar decomposition states that the operator T can be uniquely decomposed as $T = U|T|$, where U is a partial isometry, $|T| = (T^*T)^{\frac{1}{2}}$ and $\ker(T) = \ker(|T|) = \ker(U)$, which is one of the most important results in operator theory ([10], [15], [18] and [32]).

An easy extension of normal operators, hyponormal operators have been studied by many researchers. Though there are many unsolved interesting problems for this class (for example, the invariant subspace problem), one of recent trends in operator theory is to study natural extensions of hyponormal operators. Here we introduce some of these non-hyponormal operators. Following [13], an operator $T \in \mathcal{B}(\mathcal{H})$ is said to be hyponormal, if $T^*T \geqslant TT^*$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be paranormal

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[11], if $\|Tx\|^2 \leqslant \|T^2x\|$ for every unit vector $x \in \mathcal{H}$. Further, $T \in \mathcal{B}(\mathcal{H})$ is said to be n-paranormal operator [7], if $\|Tx\|^{n+1} \leqslant \|T^{n+1}x\| \, \|x\|^n$ for all $x \in \mathcal{H}$.

T. Furuta el at. [13] introduced a very interesting class of bounded linear Hilbert space operator: class A defined by $|T^2| \geqslant |T|^2$, and they showed that class A is a subclass of paranormal operators. Hence, we have

$$\{\text{Hyponormal}\} \subset \{\text{class } A\} \subset \{\text{paranormal}\} \subset \{n\text{-paranormal}\}.$$

The Riesz idempotent E_λ of an operator T with respect to an isolated point λ of $\sigma(T)$ is defined as follows.

$$E_\lambda = \frac{1}{2\pi i} \int_{\partial D_\lambda} (z-T)^{-1} \, dz \tag{1.1}$$

It satisfies $\sigma(T|_{E\mathcal{H}}) = \{\lambda\}$ and $\sigma(T|_{(1-E)\mathcal{H}}) = \sigma(T) \setminus \{\lambda\}$, where the integral is taken by the positive direction and D_λ is a closed disk with center λ and small enough radius r such as $D_\lambda \cap \sigma(T) = \{\lambda\}$. In [31], Uchiyama proved that for every paranormal operator T and each isolated point λ of $\sigma(T)$ the Riesz idempotent E_λ satisfies that

$$\begin{aligned} E_0 &= \ker T \\ E_\lambda &= \ker(T-\lambda) = \ker(T-\lambda)^* \text{ and } E_\lambda \text{ is self-adjoint if } \lambda \neq 0. \end{aligned}$$

We shall show that for every class A_n operator T and each isolated point $\lambda \in \sigma(T)$ the Riesz idempotent E_λ of T with respect to λ is self-adjoint with the property that $E_\lambda \mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$.

If $T \in \mathcal{B}(\mathcal{H})$, we denote $\ker T$ and $\operatorname{ran} T$ for the kernel of T and the range of T respectively. We also denote the spectrum of T, the point spectrum of T, the Weyl spectrum of T and the set of all eigenvalues of T with finite multiplicity which are isolated in the spectrum by $\sigma(T)$, $\sigma_p(T)$, $w(T)$ and $\pi_{00}(T)$ respectively. An operator $T \in \mathcal{B}(\mathcal{H})$ is called to be Fredholm if $\operatorname{ran} T$ is closed and both of $\ker T$ and $\ker T^*$ are finite dimensional subspaces. For arbitrary Fredholm operator T, the index of T is definded by

$$\operatorname{ind}(T) := \dim\ker T - \dim\ker T^*.$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is called to be Weyl if and only if T is a Fredholm operator with $\operatorname{ind}(T) = 0$. And the Weyl spectrum of T is defined by

$$w(T) := \{\lambda \in \mathbb{C} : T-\lambda \text{ is not Weyl}\}.$$

We say that the Weyl's theorem holds for an operator $T \in \mathcal{B}(\mathcal{H})$ if

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

2. Complementary results and definitions

We introduce several ideas and notations in this part that the study will build upon (see [25, 27, 33]).

DEFINITION 2.1. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be belongs to class A_n operators (briefly, $T \in \mathcal{A}_n$) if

$$|T^{n+1}|^{\frac{2}{n+1}} \geqslant |T|^2$$

for some positive integer n.

REMARK 2.2. It follows from the definition that

(i) If $n = 1$, then class A_n and class A operators are coincides.

(ii) If T belongs to a class A_n, then T is n-paranormal.

THEOREM 2.3. *For $T \in \mathcal{B}(\mathcal{H})$. Then the following assertions hold.*

(i) If T belongs to a class A_n, then T is normaloid [27].

(ii) If T belongs to class A_n and $\lambda \neq 0$, then for unit vectors $\{x_m\}$, $(T-\lambda)x_m \to 0$ implies $(T-\lambda)^* x_m \to 0$, $\sigma_p(T)\setminus\{0\} = \sigma_{jp}(T)\setminus\{0\}$ and $\sigma_a(T)\setminus\{0\} = \sigma_{ja}(T)\setminus\{0\}$ [33].

(iii) If T belongs to class A_n, then $T-\lambda$ has finite ascent for all complex number λ and T has SVEP [33].

(iv) Any restriction $T|_{\mathcal{M}}$ of T to an arbitrary T-invariant subspace $\mathcal{M}$ also belongs to class A_n [27].

(v) T is isoloid, i.e., every isolated point of $\sigma(T)$ is an eigen value of T [27].

The following results are very important in the sequel.

LEMMA 2.4. (Hölder-McCarthy Inequality) *Let $T \geqslant 0$. Then the following assertions hold.*

(i) $\langle T^r x, x\rangle \geqslant \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $r > 1$ and $x \in \mathcal{H}$.

(ii) $\langle T^r x, x\rangle \leqslant \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ for $r \in [0,1]$ and $x \in \mathcal{H}$.

THEOREM 2.5. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \mathcal{A}_n$ if and only if*

$$\|Tx\|^2 \leqslant \left\|T^{n+1}x\right\|^{\frac{2}{n+1}} \|x\|^{\frac{2n}{n+1}}$$

for all $x \in \mathcal{H}$.

Proof. For each a vector $x \in \mathcal{H}$, we have

$$\begin{aligned} T \in \mathcal{A}_n \Longleftrightarrow \|Tx\|^2 &= \langle Tx, Tx \rangle = \left\langle |T|^2 x, x \right\rangle \\ &\leqslant \left\langle |T^{n+1}|^{\frac{2}{n+1}} x, x \right\rangle \\ &\leqslant \left\langle |T^{n+1}|^2 x, x \right\rangle^{\frac{1}{n+1}} \|x\|^{\frac{2n}{n+1}} \quad \text{(by Lemma 2.4)} \\ &= \left\|T^{n+1} x\right\|^{\frac{2}{n+1}} \|x\|^{\frac{2n}{n+1}}. \quad \square \end{aligned}$$

COROLLARY 2.6. *Suppose that T belongs to class A_n and $\alpha, \beta \in^(T)$ with $\alpha \neq \beta$. Then* $\ker(T-\alpha) \perp \ker(T-\beta)$.

Proof. Let $x \in \ker(T-\alpha)$ and $y \in \ker(T-\beta)$. Then $Tx = \alpha x$ and $Ty = \beta y$. Therefore

$$\alpha \langle x, y \rangle = \langle \alpha x, y \rangle = \langle Tx, y \rangle = \langle x, T^* y \rangle = \left\langle x, \overline{\beta} y \right\rangle = \beta \langle x, y \rangle .$$

Hence $\alpha \langle x, y \rangle = \beta \langle x, y \rangle$ and so $(\alpha - \beta) \langle x, y \rangle = 0$. But $\alpha \neq \beta$, hence $\langle x, y \rangle = 0$. Consequently $\ker(T-\alpha) \perp \ker(T-\beta)$. $\square$

The following lemma is very useful in the sequel.

LEMMA 2.7. ([16]) *If $A, B \in \mathcal{B}(\mathcal{H})$ satisfying $A \geqslant 0$ and $\|B\| \leqslant 1$, then*

$$(B^*AB)^\gamma \geqslant B^*A^\gamma B \qquad \textit{for all } \gamma \in (0,1].$$

THEOREM 2.8. *Let $T \in \mathcal{B}(\mathcal{H})$ belongs to class A_n. If $\mathcal{M}$ is an invariant subspace of T and $T|_{\mathcal{M}}$ is a normal operator with $0 \notin \sigma_p(T|_{\mathcal{M}})$, then $\mathcal{M}$ reduces T.*

Proof. (a). Decompose T into

$$T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix} \quad \text{on} \quad \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$$

and let $S = T|_{\mathcal{M}}$ be a normal operator. Let Q be the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. We remark that $TQ = QTQ$ and

$$\begin{aligned} Q|T^{n+1}|^2 Q = QT^{*(n+1)}T^{n+1}Q &= (QT^*Q)^{n+1}(QTQ)^{n+1} \\ &= \begin{pmatrix} S^{*(n+1)}S^{n+1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |S|^{2(n+1)} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

since S is normal. Then

$$\begin{pmatrix} |S|^2 & 0 \\ 0 & 0 \end{pmatrix} = (Q(|T^{n+1}|^2 Q)^{\frac{1}{n+1}} \geqslant Q|T^{n+1}|^{\frac{2}{n+1}} Q \geqslant Q|T|^2 Q = \begin{pmatrix} |S|^2 & 0 \\ 0 & 0 \end{pmatrix}$$

by Lemma 2.7. We can write $|T^{n+1}|^{\frac{2}{n+1}} = \begin{pmatrix} |S|^2 & C \\ C^* & D \end{pmatrix}$. Since

$$\begin{pmatrix} |S|^4 & 0 \\ 0 & 0 \end{pmatrix} = (Q|T^{n+1}|^2 Q)^{\frac{2}{n+1}} \geqslant Q|T^{n+1}|^{\frac{2}{n+1}}|T^{n+1}|^{\frac{2}{n+1}} Q$$
$$= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} |S|^2 & C \\ C^* & D \end{pmatrix} \begin{pmatrix} |S|^2 & C \\ C^* & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} |S|^4 + CC^* & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $CC^* = 0$ and so $C = 0$. Therefore,

$$|T^{n+1}|^{\frac{2}{n+1}} = \begin{pmatrix} |S|^2 & 0 \\ 0 & D \end{pmatrix}.$$

Since $|T^{n+1}|^{\frac{2}{n+1}} \geqslant |T|^2 = T^*T$,

$$|T^{n+1}|^{\frac{2}{n+1}} - T^*T = \begin{pmatrix} |S^2| - |S^2| & -S^*A \\ -A^*S & D - (A^*A + B^*B) \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -S^*A \\ -A^*S & D - (A^*A + B^*B) \end{pmatrix} \geqslant 0.$$

This implies that $S^*A = 0$. Since S is normal and $\ker S = \{0\}$, $\ker S^* = \ker S = \{0\}$ and hence $A = 0$. Therefor $\mathcal{M}$ reduces T and B belongs to class A_n. □

COROLLARY 2.9. *If T belongs to class A_n and $\sigma(T) \subset S^1 := \{z \in \mathbb{C} : |z| = 1\}$, then T is unitary.*

Proof. If T belongs to class A_n such that $\sigma(T) \subset S^1$, then $r(T) = r(T^{-1}) = 1$. Hence $\|T\| = r(T) = 1$ and $1 = r(T^{-1}) \leqslant \|T^{-1}\| \leqslant r(T^{-1})^{\frac{n(n-1)}{2}} r(T)^{\frac{(n+1)(n-2)}{2}} = 1$ implies $\|T^{-1}\| = 1$. It follows that T is invertible and an isometry because

$$\|x\| = \|T^{-1}Tx\| \leqslant \|Tx\| \leqslant \|x\|$$

for all $x \in \mathcal{H}$, so T is unitary. □

COROLLARY 2.10. *Let $T \in \mathcal{B}(\mathcal{H})$ be a class A_n operator. Then* $\ker(T) \cap \operatorname{ran}(T^k) = \{0\}$ *for some positive integer k.*

Proof. If $T \in \mathcal{A}_n$, then it follows from Theorem 2.3 (iii) that T has finite ascent and so $\ker(T) = \ker(T^k)$ for some positive integer k. If $y \in \ker(T) \cap \operatorname{ran}(T^k)$, then $Ty = 0$ and $y = T^k x$ for some $x \in \mathcal{H}$. This implies that $T^{k+1}x = Ty = 0$. Since $x \in \ker(T^{k+1}) = \ker(T)$, we have $y = T^k x = 0$. Hence $\ker(T) \cap \operatorname{ran}(T^k) = \{0\}$. □

Halmos showed in [14] that a partial isometry is subnormal if and only if it is the direct sum of an isometry and zero. We generalize this theorem to the case of a class A_n operator.

PROPOSITION 2.11. *A partial isometry T is quasinormal (i.e., $T^*T^2 = TT^*T$) if and only if T belongs to class A_n operators.*

Proof. Suppose that T is partial isometry and a class A_k operator. Then

$$T^{*2}T^2 = T^*(T^*T)T = T^*T(T^*T) = (T^*T)^2 = T^*T \quad (\therefore T^*T \text{ is a projection})$$
$$T^{*3}T^3 = T^{*2}T^2 = T^*T, \cdots, T^{*m}T^m = T^*T, \quad (T^{*m}T^m)^{1/m} = T^*T \quad ({}^{\forall}m \geqslant 1).$$

Hence, T is class A_k for all $k \geqslant 1$. Suppose that T is partial isometry and a class A_n operator. Then T is contraction, $TT^*T = T$ and

$$T^*T \leqslant \left(T^{*(n+1)}T^{n+1}\right)^{\frac{1}{n+1}} \leqslant (T^*T)^{\frac{1}{n+1}} = T^*T,$$
$$\therefore\ T^{*(n+1)}T^{n+1} = (T^*T)^{n+1} = T^*T.$$
$$T^*T \geqslant T^{*2}T^2 \geqslant T^{*(n+1)}T^{n+1} = T^*T, \quad \therefore\ T^{*2}T^2 = T^*T.$$

Hence $(TT^*)(T^*T-1)(TT^*) = T\{T^{*2}T^2 - T^*T\}T^* = 0$ and $T^*TTT^* = TT^*$. This implies that $TT^* \leqslant T^*T$ and

$$(T^*T)T = (T^*T)(TT^*T) = (T^*T)(TT^*)T = (TT^*)T = T(T^*T).$$

Therefore, T is quasinormal. □

3. Weyl's theorem and the self-adjointness of any Riesz idempotent with respect to an arbitrary isolated point of $\sigma(T)$

THEOREM 3.1. *Let T be a class A_n and λ is an isolated point of $\sigma(T)$ then the Riesz idempotent E_λ satisfies the followings;*

(i) $E_0(\mathcal{H}) = \ker T$ $(\lambda = 0)$

(ii) $E(\mathcal{H}) = \ker(T-\lambda) = \ker(T-\lambda)^*$, $E_\lambda = E_\lambda^*$ $(\lambda \neq 0)$

for each $n \geqslant 2$.

Proof. (i) Both of $E_0\mathcal{H}$ and $(1-E_0)\mathcal{H}$ are T-invariant closed subspaces which satisfy that $\sigma(T|_{E_0\mathcal{H}}) = \{0\}$ and $\sigma(T|_{(1-E_0)\mathcal{H}} = \sigma(T) \setminus \{0\}$. Since, $T \in A_n$ the restrictions $T|_{E_0\mathcal{H}}, T|_{(1-E_0)\mathcal{H}} \in A_n$ and $\|T|_{E_0\mathcal{H}}\| = r(T|_{E_0\mathcal{H}}) = 0$ by Theorem 2.3(i) and hence $T|_{E_0\mathcal{H}} = 0$. This implies that $E_0\mathcal{H} \subset \ker T$. Conversely, let $x = y + z \in \ker T$ be arbitrary where $y \in E_0\mathcal{H}$ and $z \in (1-E_0)\mathcal{H}$. Since $T|_{E_0\mathcal{H}} = 0$ and $T|_{(1-E_0)\mathcal{H}}$ is invertible,

$$0 = Tx = Ty + Tz = T|_{E_0\mathcal{H}}x + T|_{(1-E_0)\mathcal{H}}y = T|_{(1-E_0)\mathcal{H}}z$$

implies $z = 0$ and hence $x = y \in E_0\mathcal{H}$. Therefore $E_0\mathcal{H} = \ker T$ holds.

(ii) Both of $E_\lambda\mathcal{H}$ and $(1-E_\lambda)\mathcal{H}$ are T-invariant closed subspaces which satisfy that $\sigma(T|_{E_\lambda\mathcal{H}}) = \{\lambda\}$ and $\sigma(T|_{(1-E_\lambda)\mathcal{H}} = \sigma(T) \setminus \{\lambda\}$. Since, $T \in A_n$ the restrictions

$T|_{E_\lambda \mathcal{H}}, T|_{(1-E_\lambda)\mathcal{H}} \in A_n$ and $\|T|_{E_\lambda \mathcal{H}}\| = r(T|_{E_\lambda \mathcal{H}}) = |\lambda|$ by Theorem 2.3(i) and also $|\lambda|^{-1} \leqslant \left\| \left(T|_{E_0\mathcal{H}}\right)^{-1} \right\| \leqslant |\lambda|^{-\frac{n(n-1)}{2}+\frac{(n+1)(n-2)}{2}} = |\lambda|^{-1}$ by [30, Theorem 1]. Hence $U = \frac{1}{\lambda} T|_{E_\lambda \mathcal{H}}$ is invertible isometry with the spectrum $\sigma(U) = \{1\}$, so U is unitary and $U = 1$ on $E_\lambda \mathcal{H}$. This implies that $T|_{E_\lambda} = \lambda E_\lambda$ and $(T-\lambda)E_\lambda = 0$. It follows that $(T-\lambda)^* E_\lambda = 0$ by Theorem 2.3(ii), and hence $E_\lambda \mathcal{H}$ is a reducing subspace of T. Since $(z-T)^* E_\lambda = (\bar{z} - \overline{\lambda})E_\lambda$ and $(z-T)^{-1*}E_\lambda = \overline{\left(\frac{1}{z-\lambda}\right)} E_\lambda$, it follows that

$$
\begin{aligned}
0 \leqslant E_\lambda^* E_\lambda &= -\frac{1}{2\pi i} \int_{|z-\lambda|=r} (z-T)^{*-1} E_\lambda \, d\bar{z} \\
&= -\frac{1}{2\pi i} \int_{|z-\lambda|=r} \overline{\left(\frac{1}{z-\lambda}\right)} E_\lambda \, d\bar{z} = \overline{\left(\frac{1}{2\pi i} \int \frac{1}{z-\lambda} \, dz\right)} E_\lambda = E_\lambda .
\end{aligned}
$$

Hence $E_\lambda = E_\lambda^*$. Thus T is of the form $T = \lambda \oplus T'$ on $\mathcal{H} = E_\lambda \mathcal{H} \oplus (1-E_\lambda)\mathcal{H}$ with $\lambda \notin \sigma(T')$. Therefore the assertion $E_\lambda \mathcal{H} = \ker(T-\lambda) = \ker(T-\lambda)^*$ holds. □

THEOREM 3.2. *Weyl's theorem hold for any class A_n operators.*

Proof. We first show that $\sigma(T) \setminus w(T) \subset \pi_{00}(T)$. Let $\lambda \in \sigma(T) \setminus w(T)$ be arbitrary. Then $T - \lambda$ is not invertible Fredholm operator with the index $\operatorname{ind}(T-\lambda) = 0$.

Case (i). $\lambda = 0$. Then $\ker T \neq \{0\}$ is finite dimension and $\operatorname{ran} T$ is closed, thus T is of the form

$$
T = \begin{pmatrix} 0 & A \\ 0 & T' \end{pmatrix} \text{ on } \ker T \oplus \operatorname{ran} T^* .
$$

Since A is a finite rank operator, it follows that T' is Fredholm with the index $\operatorname{ind}(T') = \operatorname{ind}(T) = \{0\}$. Let $x \in \ker T'$ be arbitrary. Then $T^2(0 \oplus x) = T(Ax \oplus T'x) = T(Ax \oplus 0) = 0 \oplus 0 = 0$, so $T^n(0 \oplus x) = 0$. Since T is class A_n, $\ker T^n = \ker T$ and hence $x \in \ker T \cap \operatorname{ran} T^* = \{0\}$. Therefore T' is Weyl with $\ker T' = \{0\}$, so it is invertible. This implies that 0 is isolated in $\sigma(T) = \{0\} \cup \sigma(T')$ and $0 \in \pi_{00}(T)$.

Case (ii). $\lambda \neq 0$. Then $\ker(T-\lambda)$ is finite dimensional subspace which reduces T and $\operatorname{ran}(T-\lambda)$ is closed, and hence T is of the form $T = \lambda \oplus T'$ on $\mathcal{H} = \ker(T-\lambda) \oplus \operatorname{ran}(T-\lambda)^*$. Since $T' - \lambda$ is Fredholm with the index $\operatorname{ind}(T'-\lambda) = 0$ and $\ker(T'-\lambda) = \{0\}$, it follows that $T' - \lambda$ is invertible and hence λ is isolated in $\sigma(T) = \{\lambda\} \cup \sigma(T')$. Therefore $\lambda \in \pi_{00}(T)$. Thus $\sigma(T) \setminus w(T) \subset \pi_{00}(T)$ holds.

Next, we show that $\pi_{00}(T) \subset \sigma(T) \setminus w(T)$.

Let $\lambda \in \pi_{00}(T)$ be arbitraray. Then λ is isolated in $\sigma(T)$ and $\ker(T-\lambda) \neq \{0\}$ is finite dimension.

Case (i). $\lambda = 0$. Then the Riesz idempotent E_0 with respect to 0 for T satisfies that $T|_{E_0\mathcal{H}} = 0$ and $T' := T|_{(1-E_0)\mathcal{H}}$ is invertible (so, it is Weyl) and $T' \in A_n$. And $T = 0 + T'$ on $\mathcal{H} = E_0\mathcal{H} + (1-E_0)\mathcal{H}$ is also Weyl. Therefore $0 \in \sigma(T) \setminus w(T)$.

Case (ii). $\lambda \neq 0$. Then $\ker(T-\lambda)$ is finite dimensional subspace which reduces T and $T = \lambda \oplus T'$ on $\mathcal{H} = \ker(T-\lambda) \oplus \operatorname{ran}(T-\lambda)^*$, where T' is a class A_n (hence $T' \in A_n$). If $\lambda \in \sigma(T')$ then λ is isolated in $\sigma(T')$ and $\lambda \in \sigma_p(T')$. This contradicts the fact that $\ker(T'-\lambda) \subset \operatorname{ran}(T-\lambda)^* \cap \ker(T-\lambda) = \{0\}$. Thus $T'-\lambda$ is invertible and $T-\lambda = 0 \oplus (T'-\lambda)$ implies that $T-\lambda$ is Fredholm with the index $\operatorname{ind}(T-\lambda) = \operatorname{ind}(T'-\lambda) = 0$, so $T-\lambda$ is Weyl. Therefore $\lambda \in \sigma(T) \setminus w(T)$ holds. □

PROPOSITION 3.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a class A_n operator and T^n be a compact operator for some $n \in \mathbb{N}$. Then T is also compact and normal.*

Proof. Assume that T is a class A_n operator. Hence it follows from Theorem 2.5 that

$$\|Tx\|^2 \leqslant \left\|T^{n+1}x\right\|^{\frac{2}{n+1}} \|x\|^{\frac{2n}{n+1}} \quad \text{for every } x \in \mathcal{H}. \tag{3.1}$$

Let $\{x_m\} \in \mathcal{H}$ be weakly convergent sequence with limit 0 in $\mathcal{H}$. From the compactness of T^n and the relation (3.1) we get the following relation:

$$\|Tx_m\|^2 \to 0,\ m \to \infty.$$

From the last relation it follows that T is compact. Since T is compact $\sigma(T)$ is finite set or countable infinite with 0 as the unique limit point of it. Let $\sigma(T) \setminus \{0\} = \{\lambda_n\}$ with

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_n| \geqslant |\lambda_{n+1}| \geqslant \cdots \geqslant 0, \text{ and } \lambda_n \to 0\ (n \to \infty).$$

By the compactness of T or isoloidness of T, $\lambda_n \in \sigma_p(T)$ and $\dim\ker(T-\lambda_n) < \infty$ for all n. Since $\ker(T-\lambda_n) \subset \ker(T-\lambda_n)^*$, $\mathcal{M} := \bigoplus_{n=1}^{\infty} \ker(T-\lambda_n)$ reduces T, and T is of the form

$$T = \left(\bigoplus_{n=1}^{\infty} \lambda_n\right) \oplus T' \text{ on } \mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}.$$

By the construction, $T' \in \mathcal{A}_n$ and $\sigma(T') = \{0\}$ hence $T' = 0$. This shows that

$$T = \left(\bigoplus_{n=1}^{\infty} \lambda_n\right) \oplus 0$$

and it is normal. □

THEOREM 3.4. *If $T \in \mathcal{B}(\mathcal{H})$ belongs to class A_n with $w(T) = \{0\}$, then T is a compact normal operator.*

Proof. By Theorem 3.2, T satisfies Weyl's theorem and this implies that each element in $\sigma(T) \setminus \sigma_w(T) = \sigma(T) \setminus \{0\}$ is an eigenvalue of T with finite multiplicity, and is isolated in $\sigma(T)$. Hence $\sigma(T) \setminus \{0\}$ is a finite set or a countable set with 0 as its only accumulation point. Put $\sigma(T) \setminus \{\lambda_n\}$, where $\lambda_n \neq \lambda_m$ whenever $n \neq m$ and $\{|\lambda_n|\}$ is a non-increasing sequence. Since T is normaloid, we have $|\lambda_1| = \|T\|$. By Theorem

2.3(ii), we have $(T-\lambda_1)x=0$ implies $(T-\lambda_1)^*x=0$. Hence $\ker(T-\lambda_1)$ is a reducing subspace of T. Let E_1 be the orthogonal projection onto $\ker(T-\lambda_1)$. Then $T=\lambda_1\oplus T_1$ on $\mathcal{H}=E_1\mathcal{H}\oplus(1-E_1)\mathcal{H}$. Since $T_1\in A_n$ by Theorem 2.3 (ii) and ${}^{(}T)={}^{(}T_1)\cup\{\lambda_1\}$, we have $\lambda_2\in{}^{(}T_1)$. By the same argument as above, $\ker(T-\lambda_2)=\ker(T_1-\lambda_2)$ is a finite dimensional reducing subspace of T which is included in $(1-E_1)\mathcal{H}$. Put E_2 be the othogonal projection onto $\ker(T-\lambda_2)$. Then $T=\lambda_1E_1\oplus\lambda_2E_2\oplus T_2$ on $\mathcal{H}=E_1\mathcal{H}\oplus E_2\mathcal{H}\oplus(1-E_1-E_2)\mathcal{H}$. By repeating above argument, each $\ker(T-\lambda_n)$ is a reducing subspace of T and $\left\|T-\bigoplus_{k=1}^{n}\lambda_kE_k\right\|=\|T_n\|=|\lambda_{n+1}|\to 0$ as $n\to\infty$. Here E_k is the orthogonal projection onto $\ker(T-\lambda_k)$ and $T=(\bigoplus_{k=1}^{n}\lambda_kE_k)\oplus T_n$ on $\mathcal{H}=\bigoplus_{k=1}^{n}E_k\mathcal{H}\oplus(1-\sum_{k=1}^{n}E_k)\mathcal{H}$. Hence $T=\bigoplus_{k=1}^{\infty}\lambda_kE_k$ is compact and normal because each E_k is a finite rank orthogonal projection which satisfies $E_kE_l=0$ whenever $k\neq l$ by Corollary 2.6 and $\lambda_n\to 0$ as $n\to\infty$. □

COROLLARY 3.5. *Let $T\in\mathcal{A}_n$. Then T can be written as*

$$T=A\oplus S,$$

where A is normal and S is a class A_n with $w(S)\setminus\{0\}=\sigma(S)\setminus\{0\}$.

Proof. By Theorem 3.2, $\sigma(T)\setminus w(T)=\pi_{00}(T)$. Let N be the closed linear subspace of $\mathcal{H}$ generated by $\bigcup_{\lambda_j\in\pi_{00}(T)\setminus\{0\}}\ker(T-\lambda_j)$. Then N is reduced by T. The decomposition $\mathcal{H}=N\oplus N^{\perp}$ gives $T=A\oplus S$, where A is normal and $S\in\mathcal{A}_n$. One can see that $\sigma(S)\setminus\{0\}=w(S)\setminus\{0\}$. □

THEOREM 3.6. *If T belongs to class A_n with a single limit point of the spectrum, then T is normal.*

Proof. We first show the Case (i) the limit point is zero. By hypothesis, every non-zero point of the spectrum being isolated is an eigenvalue. class A_n operator T implies that each eigenspace of T is reducing and T is normal on that eigenspace. Let $\mathcal{M}$ be the closed linear span of $\mathcal{H}$ generated by $\bigcup\ker(T-\lambda_j)$, where λ_j runs over non-zero values in $\sigma(T)$. $\mathcal{M}$ is thus a closed linear subspace of $\mathcal{H}$ reducing T and $T|_{\mathcal{M}}$ is normal. But then by the decomposition $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^{\perp}$ we get $T|_{\mathcal{M}^{\perp}}$ to be class A_n quasinilpotent operator and hence is zero. So, T is normal.

We show the Case (ii) the limit point is $\lambda_0\neq 0$. Let $\mathcal{M}=\bigoplus_{\lambda\in\sigma(T)\setminus\{0\}}\ker(T-\lambda)$. Here, $\ker(T-\lambda)$ reduces T for all $\lambda\in\sigma(T)\setminus\{0,\lambda_0\}$ and $\ker(T-\lambda_0)$ is $\{0\}$ or non-zero reducing subspace of T. If $\mathcal{M}\neq\mathcal{H}$ then $\sigma(T|_{\mathcal{M}^{\perp}})=\{0\}$ and hence $T|_{\mathcal{M}^{\perp}}=$

0 since $T|_{\mathcal{M}^\perp}$ is normaloid. Thus $T = \left(\bigoplus_{\lambda\in\sigma(T)\setminus\{0\}} \lambda\right) \oplus 0$ on $\mathcal{H} = \mathcal{M}\oplus\mathcal{M}^\perp$ is normal. □

THEOREM 3.7. *If T belongs to class A_n with only a finite number of limits points in its spectrum, then T is normal.*

Proof. Let $z_1, z_2, \cdots, z_k$ be all limit points of $\sigma(T)$ with $z_k = 0$ if 0 is a limit point of $\ker(T)$ and choose a simple closed curve G which does not intersect $\sigma(T)$ and contains only one limit point $z_1 \neq 0$ in its interior.

$$E_1 = \int_G \frac{1}{z-T}\,dz.$$

Then E_1 is a non-zero projection on $\mathcal{H}$ such that $E_1\mathcal{H}$ is invariant under T. Also then

$$\sigma(T|_{E_1\mathcal{H}}) = \sigma(T)\cap G^\circ,$$

where G° denotes the interior of G. Hence $T|_{E_1\mathcal{H}}$ can have only one limit point and therefore is normal by Theorem 3.6. Hence T is reduced by $E_1\mathcal{H}$ by Theorem 2.8. Now considering T on $(E_1\mathcal{H})^\perp$ and continuing the same process we conclude that $E_1, E_2, \cdots, E_{k-1}$ are non-zero projections which satisfy $E_iE_j = 0$ $(i \neq j)$ and T being a direct sum of normal operators on $\mathcal{M} = (E_1\mathcal{H}\oplus E_2\mathcal{H}\oplus\cdots\oplus E_{k-1}\mathcal{H})$, so it is normal. By the construction, the limit point of $\sigma(T|_{\mathcal{M}^\perp})$ is only a single point hence $T|_{\mathcal{M}^\perp}$ is normal by Theorem 3.6. Thus $T = T|_{\mathcal{M}}\oplus T|_{\mathcal{M}^\perp}$ is normal. □

THEOREM 3.8. *If T and S are belong to class A_n, then*

$$S,T \ : Weyl \Longleftrightarrow ST : \ Weyl.$$

Proof. If S and T are Weyl, then S and T are Fredholm and $\mathrm{ind}(S) = \mathrm{ind}(T) = 0$. By [8], ST is Weyl and by the index product theorem, $\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T) = 0$. Hence ST is Weyl.

Conversely, if ST is Weyl, then ST is Fredholm and $\mathrm{ind}(ST) = 0$. Put $T = \begin{pmatrix} 0 & A \\ 0 & T' \end{pmatrix}$ on $\ker T\oplus \mathrm{ran}\, T^*$. Then $\ker T' = \{0\}$. Because if $x\in\ker T'$ then $T^2(0\oplus x) = T(Ax\oplus T'x) = T(Ax\oplus 0) = 0$ and $T(0\oplus x) = 0$ since T is class A_n, hence $x\in\ker T\cap \mathrm{ran}\, T^* = \{0\}$. Since A is a finite rank operator, it is compact, and hence

$$\mathrm{ind}(T) = \dim\,\ker T - \dim\,\ker T^* = 0 - \dim\,\ker T^* \leqslant 0.$$

We also have $\mathrm{ind}(S) \leqslant 0$. Hence

$$0 = \mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T) \leqslant 0$$

implies that $\mathrm{ind}(S) = \mathrm{ind}(T) = 0$. Thus T and S are Weyl. □

A contraction is an operator T such that $\|T\| \leqslant 1$; equivalently, $\|Tx\| \leqslant \|x\|$ for every $x \in \mathcal{H}$. A contraction T is said to be a proper contraction if $\|Tx\| < \|x\|$ for every nonzero $x \in \mathcal{H}$. A strict contraction is an operator T such that $\|T\| < 1$. A strict contraction is a proper contraction, but a proper contraction is not necessarily a strict contraction, although the concepts of strict and proper contraction coincide for compact operators. A contraction T is of class $C_{0.}$ if $\|T^n x\| \to 0$ when $n \to \infty$ for every $x \in \mathcal{H}$ (i.e., T is a strongly stable contraction) and it is said to be of class $C_{1.}$ if $\lim\limits_{n\to\infty} \|T^n x\| > 0$ for every nonzero $x \in \mathcal{H}$. Classes $C_{.0}$ and $C_{.1}$ are defined by considering T^* instead of T and we define the class $C_{\alpha\beta}$ for $\alpha, \beta = 0,1$ by $C_{\alpha.} \cap C_{.\beta}$. An isometry is a contraction for which $\|Tx\| = \|x\|$ for every $x \in \mathcal{H}$.

THEOREM 3.9. *Let T be a contraction of class A_n operators for a positive integer n. Then T is the direct sum of a unitary and a $C_{.0}$ completely nonunitary contraction.*

Proof. Since T is a contraction, then the sequence $\{T^k T^{*k}\}$ is a decreasing sequence of self-adjoint operators, converging strongly to a contraction.

Let $A = \left(\lim\limits_{k\to\infty} T^k T^{*k}\right)^{\frac{1}{2}}$. A is self-adjoint and $0 \leqslant A \leqslant I$ and $TA^2T^* = A^2$. By [9] we have that there exists an isometry $V : \overline{\operatorname{ran}(A)} \to \overline{\operatorname{ran}(A)}$ such that $VA = AT^*$ on $\operatorname{ran}(A)$. V can be extended to a bounded linear operator on $\mathcal{H}$; we still denote it by V. We shall show that A is the orthogonal projection onto $\overline{\operatorname{ran}(A)}$ if T is class A_n. Let $x_k = AV^k x$, $k \in \mathbb{N} \cup \{0\}$. Then for all nonnegative integers m,

$$T^m x_{m+k} = T^m A V^{m+k} x = AV^{*m} V^{k+m} x = AV^k x = x_k. \tag{3.2}$$

So we have, for all $m \leqslant k$, $T^m x_k = x_{k-m}$. The sequence $\{\|x_m\|\}$ is a bounded above increasing sequence.

Firstly we prove that $\{\|x_k\|\}$ is a constant sequence. Suppose that T is a class A_n operator for a positive integer n. Then, for all $k \geqslant 1$ and nonzero $x \in \overline{\operatorname{ran}(A)}$,

$$\begin{aligned} \|x_k\|^2 = \|Tx_{k+1}\|^2 &\leqslant \left\|T^{n+1} x_{k+1}\right\|^{\frac{2}{n+1}} \|x_{k+1}\|^{\frac{2n}{n+1}} \\ &= \left\|x_{k+1-(n+1)}\right\|^{\frac{2}{n+1}} \|x_{k+1}\|^{\frac{2n}{n+1}} \\ &= \|x_{k-n}\|^{\frac{2}{n+1}} \|x_{k+1}\|^{\frac{2n}{n+1}}, \end{aligned} \tag{3.3}$$

and so

$$\|x_k\| \leqslant \|x_{k-n}\|^{\frac{1}{n+1}} \|x_{k+1}\|^{\frac{n}{n+1}} \leqslant \frac{1}{n+1} \left(\|x_{k-n}\| + n\|x_{k+1}\|\right). \tag{3.4}$$

Therefore

$$\begin{aligned} n\left(\|x_{k+1}\| - \|x_k\|\right) &\geqslant \|x_k\| - \|x_{k-n}\| \\ &= \left(\|x_k\| - \|x_{k-1}\|\right) + \|x_{k-1}\| - \|x_{k-2}\| + \cdots + \|x_{k-n+1}\| - \|x_{k-n}\|. \end{aligned} \tag{3.5}$$

If we let $a_k = \|x_k\| - \|x_{k-1}\|$, then we have

$$na_{k+1} \geqslant a_k + a_{k-1} + \cdots + a_{k-n+1} \tag{3.6}$$

where $a_k \geqslant 0$ and $a_k \to 0$ as $k \to \infty$. Suppose that there exists an integer $j \geqslant 1$ such that $a_j > 0$; then $a_{j+1}, a_{j+2}, \cdots, a_{j+n} \geqslant \frac{a_j}{n} > 0$ by (3.6), and $a_{j+n+1} > \frac{1}{n}(a_{j+n} + a_{j+n-1} + \cdots + a_{j+1}) > \frac{a_j}{n}$ by (3.6). We have that $a_k \geqslant \frac{a_j}{n} > 0$, for all $k > j$ by an induction argument. This is contradictory with the fact that $a_k \to 0$ as $k \to \infty$. Consequently, we have that $a_k = 0$ for all k, which implies that $\|x_{k-1}\| = \|x_k\|$ for all $k \geqslant 1$. This means that for all $x \in \overline{\text{ran}(A)}$, $\left\|AV^k x\right\| = \|Ax\| = \|x\|$. So we have that $A^2 = I$ on $\overline{\text{ran}(A)}$ and hence $A = I$ on $\overline{\text{ran}(A)}$. Therefore, we $A = I \oplus 0$ on $\mathcal{H} = \overline{\text{ran}(A)} \oplus \ker(A)$. Hence A is a projection. By [19], we have that if A is a projection, then T has a decomposition:

$$T = T_u \oplus T_c, \;\; T_c = S^* \oplus T_0, \tag{3.7}$$

where T_u is unitary and the completely nonunitary part T_c of T is the direct sum of backward unilateral shift S^* and a $C_{.0}$-contraction T_0. We will prove that S^* is missing from the direct sum. It is well known that an operator $Q = Q_1 \oplus Q_2$ has SVEP at a point λ if and only if Q_1 and Q_2 have SVEP at the point λ. Since class A_n operators have SVEP by Theorem 2.3 (iii), it follows that if S^* is present in the direct sum of T, then it has SVEP. This contradicts the fact that the backward unilateral shift does not have SVEP anywhere on its spectrum except for the boundary point of its spectrum. Therefore, $T = T_u \oplus T_0$. So, the proof is achieved. □

4. Hyperinvariant subspace for class A_n operators

Let $\sigma_T(x) \subseteq \mathbb{C}$ denote the local spectral of T at the point $x \in \mathcal{H}$, i.e., the complement of the set $\rho_T(x)$ of all $\lambda \in \mathbb{C}$ for which there exists an open neighborhood U of λ in $\mathbb{C}$ and an analytic function $f : U \to \mathcal{H}$ such that $(T-\mu)f(\mu) = x$ holds for all $\mu \in U$. Moreover, $\sigma_T(x) \subseteq \sigma(T)$. For every closed subset F of $\mathbb{C}$, let

$$\mathcal{H}_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subseteq F\}$$

denote the corresponding analytic spectral subspace of T.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be decomposable if, for any open covering $\{U, V\}$ of the complex plane $\mathbb{C}$ there are two closed T-invariant subspaces Y and Z of $\mathcal{H}$ such that $\mathcal{H} = Y + Z$, $\sigma(T|_Y) \subseteq U$ and $\sigma(T|_Z) \subseteq V$. For every decomposable operator T the identity $\mathcal{H} = \mathcal{H}_T(\overline{U}) + \mathcal{H}_T(\overline{V})$ holds for every open cover $\{U, V\}$ of $\mathbb{C}$ [20, Theorem 1.2.23].

An operator $Q \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is called quasi-affine if it has trivial kernel and has dense range. An operator $L \in \mathcal{B}(\mathcal{H})$ is said to be a quasi-affine transform of $T \in \mathcal{B}(\mathcal{K})$ if there exists a quasi-affine $Q \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $QL = TQ$.

THEOREM 4.1. *Let $T \in \mathcal{B}(\mathcal{H})$ be a class A_n operator such that $T \neq zI$ for all $z \in \mathbb{C}$. If L is a decomposable quasi-affine transform of T, then T has a nontrivial hyperinvariant subspace.*

Proof. If L is a decomposable quasi-affine transform of T, then there exists a quasi-affine Q such that $QL = TQ$, where L is decomposable. Assume that T has no nontrivial hyperinvariant subspace. It follows From [21, Lemma 3.6.1] that $\sigma_p(T) = \emptyset$ and $\mathcal{H}_T(F) = \{0\}$ for each closed set F proper in $\sigma(T)$. Let $\{U,V\}$ be an open cover of $\mathbb{C}$ such that $\sigma(T)\setminus\overline{U} \neq \emptyset$ and $\sigma(T)\setminus\overline{V} \neq \emptyset$.

Now, if $x \in \mathcal{H}_L(\overline{U})$, then $\sigma_L(x) \subset \overline{U}$. Hence there exists an analytic $\mathcal{H}$-valued function f defined on $\mathbb{C}\setminus\overline{U}$ such that $(L-z)f(z) = x$ for all $z \in \mathbb{C}\setminus\overline{U}$. So $(T-z)Qf(z) = Q(L-z)f(z) = Qx$. Hence $\mathbb{C}\setminus\overline{U} \subset \rho_T(Qx)$, this implies $Qx \in \mathcal{H}_T(\overline{U})$. Thus $Q(\mathcal{H}_L(\overline{U})) \subseteq \mathcal{H}_T(\overline{U})$. A similar argument shows $Q(\mathcal{H}_L(\overline{V})) \subseteq \mathcal{H}_T(\overline{V})$.

Therefore, since L is decomposable then $\mathcal{H} = \mathcal{H}_L(\overline{U}) + \mathcal{H}_L(\overline{V})$, and finally

$$\begin{aligned} Q(\mathcal{H}) &= Q(\mathcal{H}_L(\overline{U})) + Q(\mathcal{H}_L(\overline{V})) \subseteq \mathcal{H}_T(\overline{U}) + \mathcal{H}_T(\overline{V}) \\ &= \mathcal{H}_T(\overline{U}\cap\sigma(T)) + \mathcal{H}_T(\overline{V}\cap\sigma(T)) = \{0\}. \end{aligned}$$

This is a contradiction. Hence, T has a nontrivial hyperinvariant subspace. □

THEOREM 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be a class A_n operator. If there exists a nonzero vector $x \in \mathcal{H}$ such that $\sigma_T(x) \subsetneq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.*

Proof. Assume that $W = \mathcal{H}_T(\sigma_T(x)) = \{y \in \mathcal{H} : \sigma_T(y) \subseteq \sigma_T(x)\}$. Then Theorem 1.2.16 of [20] implies that W is a T-hyperinvariant subspace. Since $x \in W$, $W \neq \{0\}$. Suppose that $W = \mathcal{H}$. Since $T \in \mathcal{A}_n$, T has SVEP by Corollary 2.18 of [27] and hence it follows from [20, Theorem 1.3.2] that

$$\sigma(T) = \bigcup\{\sigma_T(y) : y \in \mathcal{H}\} \subseteq \sigma_T(x) \subsetneq \sigma(T)$$

which is contradiction. Hence W is a nontrivial T-hyperinvariant subspace. □

THEOREM 4.3. *Let $T \in \mathcal{B}(\mathcal{H})$ be a class A_n operator with $T \neq \alpha I$ for any $\alpha \in \mathbb{C}$. If there exists $x \in \mathcal{H}\setminus\{0\}$ such that $\|T^n x\| \leqslant Cr^n$ for all positive integers n, where $C > 0$ and $0 < r < r(T)$ are constants, then T has a nontrivial hyperinvariant subspace.*

Proof. Put $f(z) := -\sum_{n=0}^{\infty} z^{-(n+1)}T^n x$ which is analytic for $|z| > r$; in fact, $\nu = z^{-1}$ for $|z| > r$, then $f(\nu) = -\sum_{n=0}^{\infty} \nu^{n+1}T^n x$ for $0 < |\nu| < 1/r$. Since the hypothesis implies that $\limsup_{n\to\infty} \|T^n x\| \leqslant r$, the radius of convergence for the power series $\sum_{n=0}^{\infty} \nu^{n+1}T^n x$ is at least $1/r$. Setting $f(0) := 0$, we get that $f(\nu)$ is analytic for $|\nu| < 1/r$ i.e., $f(z)$ is analytic for $|z| > r$. Since

$$(T-z)f(z) = -\sum_{n=0}^{\infty} z^{-(n+1)}T^n x + \sum_{n=0}^{\infty} z^{-n}T^n x = x$$

for all $z \in \mathbb{C}$ with $|z| > r$, we have $\rho_T(x) \supset \{z \in \mathbb{C} : |z| > r\}$, i.e.,

$$\sigma_T(x) \subset \{z \in \mathbb{C} : |z| \leqslant r\}.$$

Since $r < r(T)$, it holds that $\sigma_T(x) \subsetneqq \sigma(T)$. Thus, we conclude from Theorem 4.2 that T has a nontrivial hyperinvariant subspace. □

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