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SPECTRUM AND FINE SPECTRUM OF BAND MATRICES GENERATED BY OSCILLATORY SEQUENCES

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Abstract. In this paper, a new class of band matrices is considered where the entries of each non-zero band form a sequence with two limit points. The compact perturbation technique is used to study the spectrum over the ℓ_p, $(1 < p < \infty)$ sequence space. Several spectral subdivisions such as fine spectrum, discrete spectrum, essential spectrum, etc. are obtained. In addition, a few sufficient conditions on the absence of point spectrum over the essential spectrum are also discussed.

1. Introduction

The spectral analysis of infinite matrices, in particular band matrices, defined over sequence spaces has been treated by many researchers worldwide. Over the last few decades, localization of spectrum and various spectral sub-divisions of band matrices over sequences spaces generated by difference equations has evolved into a substantial area of study in the field of spectral theory.

The spectral properties of the double band operator $B(r,s)$ defined by the difference equation with constant coefficient

$$(B(r,s)x)_n = sx_{n-1} + rx_n,\ n \in \mathbb{N}$$

with $x_0 = 0$ were studied by Altay and Başar [6], Furkan et al. [31], Bilgiç and Furkan [15] over the sequence spaces c_0 and c, l_1 and bv, l_p and bv_p $(1 < p < \infty)$ respectively. The case $r = 1 = s$ was explored by Altay and Başar [5], Kayaduman and Furkan [38], Akhmedov and Başar [2] over the sequence spaces c_0, c and l_1, bv and bv_p $(1 \leqslant p < \infty)$ respectively. $B(r,s)$ was further generalised to the triple band operator $B(r,s,t)$ and Furkan et al. [29, 30], Bilgiç and Furkan [14] examined the spectrum and fine spectrum of $B(r,s,t)$. The fine spectrum of the generalised n-band triangular Toeplitz operator was studied by Altun [7], Birbonshi and Srivastava [16]. Band matrices with non-constant band are also been considered in the literature. The spectrum and fine spectrum of the generalised differnece operators $\triangle_v$ and $\triangle_{uv}$ were obtained in [3, 46] and [4, 24, 25, 26, 47] respectively where

$$(\triangle_v\, x)_n = v_n x_n - v_{n-1} x_{n-1},$$
$$(\triangle_{uv}\, x)_n = u_n x_n - v_{n-1} x_{n-1}$$

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with $x_{-1} = 0$, $n \in \mathbb{N}$ and certain assumptions on the sequences u and v. Later on, El-Shabrawy [24, 25] has studied the fine spectrum of the operator $\triangle_{ab}$ on the sequence space l_p $(1 < p < \infty)$ and c_0 respectively. In 2012, the fine spectrum of lower triangular triple band matrix Δ^2_{uvw} on l_1 is studied by Panigrahi and Srivastava [43] and analogously, the upper triangular case is studied by Altundag and Abay [9]. An important class of tridiagonal matrices, known as Jacobi matrices, has been studied by several researchers. In this context, we refer the readers to the recent works done by J. Dombrowski [21, 19, 20]. The generalised $m+1$ banded matrix Δ^m, $m \in \mathbb{N}$ is considered in [22]. The spectrum and fine spectrum of the difference operator $\triangle^r_v$ over the sequence spaces c_0 and l_1 have been studied by Dutta and Baliarsingh (see [10, 23]). Recently Meng and Mei [41, 42] characterise the spectrum of the generalised difference operator $B^{(m)}_v$. Spectra of tridiagonal matrices [13, 27, 37] and symmetric $2n+1$ band matrices [8] are also studied. For a detailed review, one may refer to the survey articles [11, 48] and the references therein.

An interesting problem is to study the spectrum and fine spectrum of banded matrices acting over sequence spaces where the entries of the band forms oscillatory sequences. Recent articles [18, 40, 44] focus in this direction. However, the spectral properties of symmetric band matrices whose bands are generated by oscillatory sequences are not much explored. In this article, we attempt to study the spectral properties of a class of penta-diagonal band matrices defined over the sequence space ℓ_p $(1 < p < \infty)$ where the entries in the non-zero bands form sequences with two limit points. The case where the entries in the non-zero bands form oscillatory sequences is treated separately.

Let ℓ_p represents the Banach space of p-absolutely summable sequences of real or complex numbers with the norm

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{\frac{1}{p}}.$$

Also let $\mathcal{D}_p$ denotes the set of all diagonal operators on ℓ_p. For any operator $T \in \mathcal{D}_p$, $\operatorname{diag}(T)$ represents the sequence in the diagonal of T. In this work we investigate the spectral properties of a class of operators T defined over ℓ_p represented by the following form:

$$T = S_r^2 D_1 + D_2 S_\ell^2 + D_3$$

where $S_r, S_\ell : \ell_p \to \ell_p$, denotes the right shift operator, left shift operator respectively and $D_1, D_2, D_3 \in \mathcal{D}_p$ with $\operatorname{diag}(D_1) = \{c_n\} \subset \mathbb{C} \setminus \{0\}$ $\operatorname{diag}(D_2) = \{b_n\} \subset \mathbb{C} \setminus \{0\}$ and $\operatorname{diag}(D_3) = \{a_n\} \subset \mathbb{C}$. We further assume that the subsequences $\{a_{2n-1}\}$, $\{b_{2n-1}\}$, $\{c_{2n-1}\}$ converges to the non-negative real numbers r_1, s_1, s_1 respectively and $\{a_{2n}\}$, $\{b_{2n}\}$, $\{c_{2n}\}$ converges to the non-negative real numbers r_2, s_2, s_2 respectively where $s_1 \neq 0$ and $s_2 \neq 0$.

Our focus is to investigate the spectral properties of the operator T using compact perturbation technique. Let us consider another operator T_0 over ℓ_p defined by

$$T_0 = S_r^2 D'_1 + D'_2 S_\ell^2 + D'_3$$

where $D_1', D_2', D_3' \in \mathcal{D}_p$ with

$$\operatorname{diag}(D_1') = \operatorname{diag}(D_2') = \{s_1, s_2, s_1, s_2, \cdots\},\ \operatorname{diag}(D_3') = \{r_1, r_2, r_1, r_2, \cdots\}.$$

Using the properties of compact operators, it can be proved that $T - T_0$ is a compact operator over ℓ_p. Both the operators T and T_0 can be represented by the following penta-diagonal matrices

$$T = \begin{pmatrix} a_1 & 0 & b_1 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & b_2 & 0 & \cdots \\ c_1 & 0 & a_3 & 0 & b_3 & \cdots \\ 0 & c_2 & 0 & a_4 & 0 & \cdots \\ 0 & 0 & c_3 & 0 & a_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad T_0 = \begin{pmatrix} r_1 & 0 & s_1 & 0 & 0 & \cdots \\ 0 & r_2 & 0 & s_2 & 0 & \cdots \\ s_1 & 0 & r_1 & 0 & s_1 & \cdots \\ 0 & s_2 & 0 & r_2 & 0 & \cdots \\ 0 & 0 & s_1 & 0 & r_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We obtain the spectrum, fine spectrum and the sets of various spectral subdivisions of the operator T_0. It is interesting to note that the spectrum of T_0 is given by

$$[r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2],$$

which is also its essential spectrum with no eigenvalues. Later we investigate how the spectrums of T and T_0 are related. Few results on the essential spectrum of T (which is identical to the essential spectrum of T_0) being devoid of its eigenvalues are also derived. This helps us to characterize the point spectrum of T. Theory of difference equations plays an important role in our study. We use various results on the asymptotic behaviour of solutions of difference equations to demonstrate the findings of our paper. For more details on difference equations one can refer [28].

The remainder of paper is organized as follows: section 2 is devoted to introduce some terminologies and results which are relevant to our work. Section 3 contains the results on the spectrum and fine spectrum of T_0 over ℓ_p. The spectral properties of T are discussed in section 4.

2. Preliminaries

Let X and Y are Banach spaces and for any operator $A : X \to Y$, $N(A)$ and $R(A)$ denote the null space and range space of A respectively. The operator $A^* : Y^* \to X^*$ is called the adjoint operator and defined by

$$(A^* f)(x) = f(Ax) \quad \text{for all } f \in Y^* \text{ and } x \in X$$

where X^*, Y^* are the dual spaces of X and Y respectively. $B(X)$ denotes the set of all bounded linear operators from X to itself. For any $A \in B(X)$, the resolvent set $\rho(A, X)$ of A is the set of all λ in the complex plane such that $(A - \lambda I)$ has a bounded inverse in X where I is the identity operator defined over X. The complement of resolvent set in the complex plane $\mathbb{C}$ is called the spectrum of A and it is denoted by $\sigma(A, X)$. The spectrum $\sigma(A, X)$ can be partitioned into three disjoint sets which are

(i) the point spectrum, denoted by $\sigma_p(A,X)$, is the set of all such $\lambda \in \mathbb{C}$ for which $(A-\lambda I)^{-1}$ does not exist. An element $\lambda \in \sigma_p(A,X)$ is called an eigenvalue of A,

(ii) the continuous spectrum, denoted by $\sigma_c(A,X)$, is the set of all such $\lambda \in \mathbb{C}$ for which $(A-\lambda I)^{-1}$ is exists, unbounded and $R(A-\lambda I)$ is dense in X but $R(A-\lambda I) \neq X$,

(iii) the residual spectrum, denoted by $\sigma_r(A,X)$, is the set of all such $\lambda \in \mathbb{C}$ for which $(A-\lambda I)^{-1}$ exists (and may be bounded or not) but $R(A-\lambda I)$ is not dense in X.

These three disjoint sets are together known as fine spectrum and their union becomes the whole spectrum. There are some other important subdivisions of the spectrum such as approximate point spectrum $\sigma_{app}(A,X)$, defect spectrum $\sigma_\delta(A,X)$ and compression spectrum $\sigma_{co}(A,X)$, defined by

$$\begin{aligned}
\sigma_{app}(A,X) &= \{\lambda \in \mathbb{C} : \text{ there exists a Weyl sequence for } (A-\lambda I)\},\\
\sigma_\delta(A,X) &= \{\lambda \in \mathbb{C} : (A-\lambda I) \text{ is not surjective}\},\\
\sigma_{co}(A,X) &= \left\{\lambda \in \mathbb{C} : \overline{R(A-\lambda I)} \neq X\right\}.
\end{aligned}$$

The sets which are defined above also forms subdivisions of spectrum of A (which are not necessarily disjoint) as follows [12, p. 178]

$$\begin{aligned}
\sigma(A,X) &= \sigma_{app}(A,X) \cup \sigma_{co}(A,X),\\
\sigma(A,X) &= \sigma_{app}(A,X) \cup \sigma_\delta(A,X).
\end{aligned}$$

An operator $A \in B(X)$ is said to be Fredholm operator if $R(A)$ is closed and $\dim(N(A))$, $\dim(X/R(A))$ are finite. In this case the number

$$\dim(N(A)) - \dim(X/R(A))$$

is called the index of the Fredholm operator A. The essential spectrum of A is defined by the set

$$\sigma_{ess}(A,\ell_p) = \{\lambda \in \mathbb{C} : (A-\lambda I) \text{ is not a Fredholm operator}\}.$$

If A is a Fredholm operator and $K \in B(X)$ is a compact operator then $A+K$ is also a Fredholm operator with same indices. Since compact perturbation does not effect the Fredholmness and index of a Fredholm operator, we have

$$\sigma_{ess}(A,X) = \sigma_{ess}(A+K,X).$$

For any isolated eigenvalue λ of A, the operator P_A which is defined by

$$P_A(\lambda) = \frac{1}{2\pi i}\int_\gamma (\mu I - A)^{-1} d\mu,$$

is called the Riesz projection of A with respect to λ where γ is positively orientated circle centred at λ with sufficiently small radius such that it excludes other spectral values of A. An eigenvalue λ of A is said to be a discrete eigenvalue if it is isolated and the rank of the associated Riesz projection is finite. The rank of the Riesz projection is called the algebraic multiplicity of λ. The set of all such eigenvalues with finite multiplicities is called the discrete spectrum of A and it is denoted by $\sigma_d(A,X)$. This type of eigenvalues sometimes referred as eigenvalues with finite type.

In the following proposition, we mention some inclusion relation of spectrum of a bounded linear operator and its adjoint operator.

PROPOSITION 2.1. [12, p. 195] *If X is a Banach space and $A \in B(X)$, $A^* \in B(X^*)$ then the spectrum and subspectrum of A and A^* are related by the following relations:*

(a) $\sigma(A^*,X^*) = \sigma(A,X)$,

(b) $\sigma_c(A^*,X^*) \subseteq \sigma_{app}(A,X)$,

(c) $\sigma_{app}(A^*,X^*) = \sigma_\delta(A,X)$,

(d) $\sigma_\delta(A^*,X^*) = \sigma_{app}(A,X)$,

(e) $\sigma_p(A^*,X^*) = \sigma_{co}(A,X)$,

(f) $\sigma_{co}(A^*,X^*) \supseteq \sigma_p(A,X)$,

(g) $\sigma(A,X) = \sigma_{app}(A,X) \cup \sigma_p(A^*,X^*) = \sigma_p(A,X) \cup \sigma_{app}(A^*,X^*)$.

Here we record few lemmas related to the boundness of an infinite matrix defined over sequence spaces, which are useful to our research.

LEMMA 2.2. [17, p. 253] *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_1)$ from ℓ_1 to itself if and only if the supremum of ℓ_1 norms of the columns of A is bounded.*

LEMMA 2.3. [17, p. 245] *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_\infty)$ from ℓ_∞ to itself if and only if the supremum of ℓ_1 norms of the rows of A is bounded.*

LEMMA 2.4. [17, p. 254] *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(\ell_p)(1 < p < \infty)$ if $T \in B(\ell_1) \cap B(\ell_\infty)$.*

3. Spectra of T_0

It is already mentioned that we study the spectral properties of T by using the spectral properties of T_0 and compact perturbation technique. In this section we derive the spectrum and fine spectrum of T_0. The notation $\|T\|_p$ denotes the operator norm of an operator $T \in B(\ell_p)$ where $1 \leqslant p \leqslant \infty$.

THEOREM 3.1. *The operator* $T_0 : \ell_p \to \ell_p$ *is a bounded linear operator which satisfies the following inequality*

$$\left(\frac{|r_1|^p+|r_2|^p+|s_1|^p+|s_2|^p}{2}\right)^{\frac{1}{p}} \leqslant \|T_0\|_p \leqslant \left(3^{p-1}\left(|r_1|^p+2|s_1|^p+|r_2|^p+2|s_2|^p\right)\right)^{\frac{1}{p}}.$$

Proof. As linearity of T_0 is trivial, we omit it. Let $e=(1,1,0,0,\ldots)\in\ell_p$. Then $T_0(e)=(r_1,r_2,s_1,s_2,0,\ldots)$ and one can observe that

$$\frac{\|T_0(e)\|_p}{\|e\|_p}=\left(\frac{|r_1|^p+|r_2|^p+|s_1|^p+|s_2|^p}{2}\right)^{\frac{1}{p}}.$$

This proves

$$\left(\frac{|r_1|^p+|r_2|^p+|s_1|^p+|s_2|^p}{2}\right)^{\frac{1}{p}} \leqslant \|T_0\|_p.$$

Also, let $x=\{x_n\}\in\ell_p$ and $x_n=0$ if $n\leqslant 0$. Then,

$$\begin{aligned}
\|T_0(x)\|_p^p &= \sum_{n=1}^{\infty}|s_1x_{2n-3}+r_1x_{2n-1}+s_1x_{2n+1}|^p \\
&\quad+\sum_{n=1}^{\infty}|s_2x_{2n-2}+r_2x_{2n}+s_2x_{2n+2}|^p \\
&\leqslant \sum_{n=1}^{\infty}\left(|s_1x_{2n-3}|+|r_1x_{2n-1}|+|s_1x_{2n+1}|\right)^p \\
&\quad+\sum_{n=1}^{\infty}\left(|s_2x_{2n-2}|+|r_2x_{2n}|+|s_2x_{2n+2}|\right)^p.
\end{aligned}$$

By Jensen's inequality we get,

$$\begin{aligned}
\|T_0(x)\|_p^p \leqslant & 3^{p-1}\sum_{n=1}^{\infty}\left(|s_1x_{2n-3}|^p+|r_1x_{2n-1}|^p+|s_1x_{2n+1}|^p\right) \\
&+3^{p-1}\sum_{n=1}^{\infty}\left(|s_2x_{2n-2}|^p+|r_2x_{2n}|^p+|s_2x_{2n+2}|^p\right) \\
\leqslant & 3^{p-1}\left(|r_1|^p+2|s_1|^p+|r_2|^p+2|s_2|^p\right)\|x\|_p^p.
\end{aligned}$$

This implies,

$$\|T_0\| \leqslant \left(3^{p-1}\left(|r_1|^p+2|s_1|^p+|r_2|^p+2|s_2|^p\right)\right)^{\frac{1}{p}}.$$

This completes the proof. □

The following theorem proves the non-existence of eigenvalues of the operator T_0 in ℓ_p.

THEOREM 3.2. *The point spectrum of T_0 over ℓ_p is given by $\sigma_p(T_0,\ell_p) = \emptyset$.*

Proof. Consider $(T_0 - \lambda I)x = 0$ for $\lambda \in \mathbb{C}$ and $x = \{x_n\} \in \mathbb{C}^{\mathbb{N}}$. This gives the following system of equations

$$\left.\begin{aligned}
(r_1-\lambda)x_1 + s_1x_3 &= 0\\
(r_2-\lambda)x_2 + s_2x_4 &= 0\\
s_1x_1 + (r_1-\lambda)x_3 + s_1x_5 &= 0\\
s_2x_2 + (r_2-\lambda)x_4 + s_2x_6 &= 0\\
&\vdots\\
s_1x_{2n-1} + (r_1-\lambda)x_{2n+1} + s_1x_{2n+3} &= 0\\
s_2x_{2n} + (r_2-\lambda)x_{2n+2} + s_2x_{2n+4} &= 0\\
&\vdots
\end{aligned}\right\}$$

If $x_1 = 0$ then $x_{2n-1} = 0$ for all $n \in \mathbb{N}$. Similarly $x_2 = 0$ implies $x_{2n} = 0$ for all $n \in \mathbb{N}$. Therefore let $(x_1,x_2) \neq (0,0)$ and consider two sequences $\{y_n\}$ and $\{z_n\}$ where $y_n = x_{2n-1}$ and $z_n = x_{2n}$, $n \in \mathbb{N}$ respectively. Then the system of equations of $(T_0 - \lambda I)x = 0$ reduces to

$$y_n + p_1y_{n+1} + y_{n+2} = 0, \tag{3.1}$$

$$z_n + p_2z_{n+1} + z_{n+2} = 0, \tag{3.2}$$

where $p_1 = \frac{r_1-\lambda}{s_1}$, $p_2 = \frac{r_2-\lambda}{s_2}$, $n \in \mathbb{N}\cup\{0\}$ and $y_0 = z_0 = 0$. If $x_1 \neq 0$, the general solution of the difference equation (3.1) is given by ([28, p. 75],

$$y_n = \begin{cases} (c_1+nc_2)(-1)^n, & \text{if } p_1 = 2\\ c_1 + nc_2, & \text{if } p_1 = -2\\ c_1\alpha_1^n + c_2\alpha_2^n, & \text{if } p_1 \notin \{-2,2\} \end{cases} \tag{3.3}$$

where c_1, c_2 are arbitrary constants and α_1, α_2 are the roots of the polynomial

$$y^2 + p_1y + 1 = 0 \tag{3.4}$$

which is called the characteristic polynomial of (3.1). The following two equalities

$$\alpha_1\alpha_2 = 1 \text{ and } \alpha_1 + \alpha_2 = -p_1$$

are useful. Equation (3.3) suggests there are three cases to be considered.

Case 1: If $p_1 = 2$ (i.e., $\lambda = r_1 - 2s_1$). In this case the general solution of (3.1) is

$$y_n = (c_1 + c_2n)(-1)^n,\ n \in \mathbb{N}\cup\{0\}$$

with the initial condition $y_0 = 0$ which gives $c_1 = 0$. This reduces the solution as $y_n = nc_2(-1)^n$. This also implies $c_2 = -y_1$ and the solution in this case is

$$y_n = ny_1(-1)^{n+1},\ \ n \in \mathbb{N}.$$

Case 2: If $p_1 = -2$ (i.e., $\lambda = r_1 + 2s_1$). Similar as Case 1, the solution reduces to

$$y_n = ny_1, \ \ n \in \mathbb{N}.$$

Case 3: If $p_1 \notin \{-2, 2\}$. The general solution of (3.1) is given by

$$y_n = c_1\alpha_1^n + c_2\alpha_2^n.$$

With the help of initial condition $y_0 = 0$ and by using the equalities $\alpha_1\alpha_2 = 1$, $\alpha_1 + \alpha_2 = -p_1$, one can obtain that $c_2 = -c_1$ and the solution reduces to

$$y_n = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2} y_1, \ \ n \in \mathbb{N}.$$

If $y_1 \neq 0$ then $\{y_n\} \notin \ell_p$ in Case 1 and Case 2. In Case 3 $\{y_n\} \in \ell_p$ if and only if $|\alpha_1| < 1$ and $|\alpha_2| < 1$ which can not be the case since $\alpha_1\alpha_2 = 1$. Hence in all the three cases $\{y_n\} \in \ell_p$ if and only if $y_1 = 0$ and this leads to the fact that $x_1 \neq 0$. Hence there is no non-trivial solution of (3.1).

Similarly for the difference equation (3.2), if $x_2 \neq 0$, the general solution $\{z_n\}$ is of the form

$$z_n = \begin{cases} (d_1 + nd_2)(-1)^n, & \text{if } p_2 = 2 \\ d_1 + nd_2, & \text{if } p_2 = -2 \\ d_1\beta_1^n + d_2\beta_2^n, & \text{if } p_2 \notin \{-2, 2\} \end{cases} \tag{3.5}$$

where d_1, d_2 are arbitrary constants and β_1, β_2 are the roots of the polynomial

$$z^2 + p_2 z + 1 = 0 \tag{3.6}$$

which is called the characteristic polynomial of difference equation (3.2). In a similar way, it can be proved that $\{z_n\} \in \ell_p$ if and only if $z_1 = 0$ and this leads to the trivial solution of (3.2). Hence, there does not exist any non-trivial solution of the system $(T_0 - \lambda I)x = 0$ such that $x \in \ell_p$. This proves the required result. □

REMARK 3.3. The solution $x = \{x_n\}$ of the system $Tx = \lambda x$, which are obtained in terms of the sequences $\{y_n\}$ and $\{z_n\}$ in the equations (3.3) and (3.5) respectively, actually depends on the unknown λ. Therefore, instead of writing $x_n(\lambda)$, we write x_n for the sake of brevity throughout this paper except in Theorem 4.8 where the dependency of the solutions on λ is vital.

The adjoint operator of T_0 is T_0^* which is defined over sequence space ℓ_p^* where ℓ_p^* denotes the dual space of ℓ_p which is isomorphic to ℓ_q where $\frac{1}{p} + \frac{1}{q} = 1$.

COROLLARY 3.4. *The point spectrum of adjoint operator T_0^* over the sequence space ℓ_p^* is given by $\sigma_p(T_0^*, \ell_p^*) = \emptyset$.*

Proof. It is well known that the adjoint operator $T_0{}^* : \ell_p^* \to \ell_p^*$, is represented by transpose of the matrix T_0. Since T_0 is represented by a symmetric matrix, using the same argument as Theorem 3.2, it is easy to prove that $\sigma_p(T_0^*, \ell_p^*) = \emptyset$. □

COROLLARY 3.5. *The residual spectrum of T_0 over the sequence space ℓ_p is given by $\sigma_r(T_0, \ell_p) = \emptyset$.*

Proof. We know that the operator T has a dense range if and only if T^* is one to one [12, p. 197]. Using this we have the following relation

$$\sigma_r(T_0, \ell_p) = \sigma_p(T_0^*, \ell_p^*) \setminus \sigma_p(T_0, \ell_p).$$

Hence, $\sigma_r(T_0, \ell_p) = \emptyset$. $\square$

Following that, we obtain the spectrum of T_0.

THEOREM 3.6. *The spectrum of T_0 over ℓ_p is given by*

$$\sigma(T_0, \ell_p) = [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2].$$

Proof. First we prove the inclusion relation

$$\sigma(T_0, \ell_p) \subseteq [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2].$$

Let $\lambda \notin [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2]$. Since point spectrum of T_0 over ℓ_p is empty, $(T_0 - \lambda I)^{-1}$ exists for all $\lambda \in \mathbb{C}$. Consider the characteristic polynomials (3.4) and (3.6) as defined in Theorem 3.2 which has the roots α_1, α_2 and β_1, β_2 respectively. Since $\lambda \in \mathbb{C} \setminus ([r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2])$, either $|\alpha_1| < 1$, $|\alpha_2| > 1$ or $|\alpha_1| > 1$, $|\alpha_2| < 1$. Similar argument also applies for the roots β_1 and β_2. Without loss of generality, we can assume that $|\alpha_1| < 1 < |\alpha_2|$ and $|\beta_1| < 1 < |\beta_2|$. One can check that the infinite matrix $B = (b_{nk})$ mentioned below is the inverse of $(T_0 - \lambda I)$, where

$$b_{nk} = \begin{cases} \frac{1}{s_1(\alpha_1^2 - 1)} \left(\alpha_1^{\frac{n+1}{2} - \frac{k+1}{2} + 1} - \alpha_1^{\frac{n+1}{2} + \frac{k+1}{2} + 1} \right), & \text{if } n,k \text{ both are odd and } n \geqslant k \\ \frac{1}{s_1(\alpha_1^2 - 1)} \left(\alpha_1^{\frac{k+1}{2} - \frac{n+1}{2} + 1} - \alpha_1^{\frac{k+1}{2} + \frac{n+1}{2} + 1} \right), & \text{if } n,k \text{ both are odd and } n < k \\ \frac{1}{s_2(\beta_1^2 - 1)} \left(\beta_1^{\frac{n}{2} - \frac{k}{2} + 1} - \beta_1^{\frac{n}{2} + \frac{k}{2} + 1} \right), & \text{if } n,k \text{ both are even and } n \geqslant k \\ \frac{1}{s_2(\beta_1^2 - 1)} \left(\beta_1^{\frac{k}{2} - \frac{n}{2} + 1} - \beta_1^{\frac{k}{2} + \frac{n}{2} + 1} \right), & \text{if } n,k \text{ both are even and } n < k \\ 0, & \text{otherwise.} \end{cases}$$

From Lemma 2.2 it follows that

$$\left\| (T_0 - \lambda I)^{-1} \right\|_1 = \sup_k \sum_{n=1}^{\infty} |b_{nk}|.$$

Let for each $k \in \mathbb{N}$, S_k denotes the sum $\sum_{n=1}^{\infty} |b_{nk}|$. Thus

$$S_k = \sum_{n=1}^{k-1} |b_{nk}| + \sum_{n=k}^{\infty} |b_{nk}|.$$

Two cases are considered.

Case 1: (k is odd) In this case

$$\sum_{n=1}^{k-1}|b_{nk}| = |b_{1k}|+|b_{3k}|+\cdots+|b_{k-2,k}|$$

as $b_{nk}=0$ when n is even. Putting $n=2q-1$ where q runs over $1,2,\cdots,\frac{k-1}{2}$ we obtain

$$\sum_{n=1}^{k-1}|b_{nk}| = \sum_{q=1}^{\frac{k-1}{2}}|b_{2q-1,k}| = \frac{1}{s_1(\alpha_1^2-1)}\sum_{q=1}^{\frac{k-1}{2}}|\alpha_1^{\frac{k+1}{2}-q+1}-\alpha_1^{\frac{k+1}{2}+q+1}|.$$

Now,

$$\begin{aligned}\sum_{q=1}^{\frac{k-1}{2}}|\alpha_1^{\frac{k+1}{2}-q+1}-\alpha_1^{\frac{k+1}{2}+q+1}| &\leqslant |\alpha_1|^{\frac{k+1}{2}+1}\sum_{q=1}^{\frac{k-1}{2}}|\alpha_1|^{-q}+|\alpha_1|^{\frac{k+1}{2}+1}\sum_{q=1}^{\frac{k-1}{2}}|\alpha_1|^{q}\\ &=|\alpha_1|^{\frac{k+3}{2}}\left(\frac{|\alpha_1|^{\frac{1-k}{2}}-1}{1-|\alpha_1|}+\frac{|\alpha_1|-|\alpha_1|^{\frac{k+1}{2}}}{1-|\alpha_1|}\right)\\ &=\frac{|\alpha_1|^2-|\alpha_1|^{\frac{k+3}{2}}+|\alpha_1|^{\frac{k+5}{2}}-|\alpha_1|^{k+2}}{1-|\alpha_1|}.\end{aligned}$$

As $|\alpha_1|<1$, the above relation gives us

$$\sum_{n=1}^{k-1}|b_{nk}|<\infty.$$

Also

$$\begin{aligned}\sum_{n=k}^{\infty}|b_{nk}| &= \sum_{q=0}^{\infty}|b_{k+2q,,k}|\\ &=\frac{1}{s_1(\alpha_1^2-1)}\sum_{q=0}^{\infty}\left|\alpha_1^{\frac{k+2q+1}{2}-\frac{k+1}{2}+1}-\alpha^{\frac{k+2q+1}{2}+\frac{k+1}{2}+1}\right|\\ &=\frac{1}{s_1(\alpha_1^2-1)}\sum_{q=0}^{\infty}\left|\alpha_1^{q+1}-\alpha_1^{k+q+2}\right|.\end{aligned}$$

The inequality $|\alpha_1^{q+1}-\alpha_1^{k+q+2}|\leqslant|\alpha_1|^{q+1}+|\alpha_1|^{k+q+2}$ and the fact $|\alpha_1|<1$ provides

$$\sum_{q=0}^{\infty}\left|\alpha_1^{q+1}-\alpha_1^{k+q+2}\right|<\infty,$$

which proves $\sum\limits_{n=k}^{\infty}|b_{nk}|<\infty$. Hence we have,

$$\sum_{n=1}^{\infty}|b_{nk}|<\infty, \text{ for odd } k. \tag{3.7}$$

Case 2: (k is even) In this case if n is odd then $b_{nk}=0$. Let n is even and $n=2q$ where $q\in\mathbb{N}$. Then

$$\sum_{n=1}^{k-1}|b_{nk}| = |b_{2k}|+|b_{4k}|+\ldots+|b_{n-2,k}| = \sum_{q=1}^{\frac{k-2}{2}}|b_{2q,k}|$$
$$= \frac{1}{s_2(\beta_1^2-1)}\sum_{q=1}^{\frac{k-2}{2}}\left|\beta_1^{\frac{k}{2}-q+1}-\beta_1^{\frac{k}{2}+q+1}\right|.$$

Since $|\beta_1|<1$,

$$\sum_{q=1}^{\frac{k-2}{2}}\left|\beta_1^{\frac{k}{2}-q+1}-\beta_1^{\frac{k}{2}+q+1}\right| \leqslant |\beta_1|^{\frac{k}{2}+1}\left(\sum_{q=1}^{\frac{k-2}{2}}\left|\beta_1^{-q}\right|+\sum_{q=1}^{\frac{k-2}{2}}|\beta_1^{q}|\right)$$
$$= |\beta_1|^{\frac{k}{2}+1}\left(\frac{|\beta_1|^{1-\frac{k}{2}}-1}{1-|\beta_1|}+\frac{|\beta_1|-|\beta_1|^{\frac{k}{2}}}{1-|\beta_1|}\right)$$
$$= \frac{|\beta_1|^2-|\beta_1|^{\frac{k}{2}+1}+|\beta_1|^{\frac{k}{2}+2}-|\beta_1|^{k+1}}{1-|\beta_1|}<\infty.$$

Therefore,

$$\sum_{n=1}^{k-1}|b_{nk}|<\infty.$$

Now,

$$\sum_{n=k}^{\infty}|b_{nk}| = |b_{kk}|+|b_{k+2,k}|+|b_{k+4,k}|\ldots$$
$$= \sum_{p=0}^{\infty}|b_{k+2p,k}| = \frac{1}{s_2(\beta_1^2-1)}\sum_{p=0}^{\infty}|\beta_1^{p+1}-\beta_1^{p+k+1}|<\infty.$$

Therefore,

$$\sum_{n=k}^{\infty}|b_{nk}|<\infty.$$

Hence we have,

$$\sum_{n=1}^{\infty}|b_{nk}|<\infty,\quad \text{if } k \text{ is even}. \tag{3.8}$$

By using relations (3.7) and (3.8) we get,

$$\left\|(T_0-\lambda I)^{-1}\right\|_1 = \sup_k\sum_{n=1}^{\infty}|b_{nk}| = \sup_k S_k<\infty,\quad \text{for all } k.$$

As the matrix of inverse operator $(T_0-\lambda I)^{-1}$ is symmetric, performing similar calculations to the rows of $B=(b_{nk})$ we get,

$$\left\|(T_0-\lambda I)^{-1}\right\|_\infty<\infty.$$

From Lemma 2.4, it follows

$$(T_0-\lambda I)^{-1}\in B(\ell_p).$$

Hence it is proved that,

$$\sigma(T_0,\ell_p)\subseteq[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2].$$

For the reverse inclusion relation let $\lambda\in[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]$ and $y=(1,1,0,0,\cdots)\in\ell_p$. Define $x=(x_1,x_2,x_3,\cdots)$ by

$$x=(T_0-\lambda I)^{-1}y.$$

This implies,

$$x_n=\begin{cases}\dfrac{-\beta_1^{\frac{n}{2}}}{s_2}, & \text{if } n \text{ is even}\\[2ex] \dfrac{-\alpha_1^{\frac{n+1}{2}}}{s_1}, & \text{if } n \text{ is odd.}\end{cases}$$

Also $\lambda\in[r_1-2s_1,r_1+2s_1]$ implies $|\alpha_1|=1$ which also implies $\dfrac{-\alpha_1^{\frac{n+1}{2}}}{s_1}\not\to 0$ as $n\to\infty$. Similarly $\lambda\in[r_2-2s_2,r_2+2s_2]$ implies $\dfrac{-\beta_1^{\frac{n}{2}}}{s_2}\not\to 0$ as $n\to\infty$. Hence $x\notin\ell_p$ and $\lambda\in\sigma(T_0,\ell_p)$. This proves the following relation

$$[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]\subseteq\sigma(T_0,\ell_p).$$

Hence, we conclude that

$$\sigma(T_0,\ell_p)=[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]. \quad \square$$

COROLLARY 3.7. *The continuous spectrum of T_0 over ℓ_p is given by*

$$\sigma_c(T_0,\ell_p)=[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2].$$

Proof. It is evident that $\sigma(T_0,\ell_p)$ is the disjoint union of $\sigma_p(T_0,\ell_p)$, $\sigma_r(T_0,\ell_p)$ and $\sigma_c(T_0,\ell_p)$, we have

$$\sigma(T_0,\ell_p)=\sigma_c(T_0,\ell_p).$$

Hence, $\sigma_c(T_0,\ell_p)=[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]$. $\square$

COROLLARY 3.8. *Essential spectrum of T_0 defined over ℓ_p is given by*

$$\sigma_{ess}(T_0,\ell_p)=[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2].$$

Proof. It is well-known that $\sigma_c(T_0,\ell_p)\subseteq\sigma_{ess}(T_0,\ell_p)$ and we have

$$\sigma_{ess}(T_0,\ell_p)\subseteq\sigma(T_0,\ell_p)=\sigma_c(T_0,\ell_p)\subseteq\sigma_{ess}(T_0,\ell_p).$$

Hence, the desired result is obvious. □

Using the relations which are mentioned in Proposition 2.1 we can easily obtain the following results.

COROLLARY 3.9. *The compression spectrum, approximate point spectrum and defect spectrum of T_0 over ℓ_p are as follows*

(i) $\sigma_{co}(T_0,\ell_p)=\emptyset$,

(ii) $\sigma_{app}(T_0,\ell_p)=[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]$,

(iii) $\sigma_{\delta}(T_0,\ell_p)=[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]$.

In particular, if $r_1=r_2=r$ and $s_1=s_2=s$ then the operator T_0 reduces to an operator with the following matrix representation

$$\begin{pmatrix} r & 0 & s & 0 & 0 & \cdots \\ 0 & r & 0 & s & 0 & \cdots \\ s & 0 & r & 0 & s & \cdots \\ 0 & s & 0 & r & 0 & \cdots \\ 0 & 0 & s & 0 & r & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The following results can be obtained from the previously proved results in this setting.

COROLLARY 3.10. *The spectrum and other spectral subdivisions over ℓ_p are given by,*

(i) $\sigma(T_0,\ell_p)=\sigma_c(T_0,\ell_p)=\sigma_{app}(T_0,\ell_p)=\sigma_{\delta}(T_0,\ell_p)=[r-2s,r+2s]$,

(ii) $\sigma_p(T_0,\ell_p)=\sigma_r(T_0,\ell_p)=\sigma_p(T_0^*,\ell_p^*)=\emptyset$.

It is interesting to note that the spectrum and fine spectra mentioned in Corollary 3.10 coincide with the spectrum and fine spectrum of the tridiagonal matrix $U(s,r,s)$ defined over ℓ_p which is obtained in [37].

4. Spectra of $T = T_0 + K$

In this section, we focus on the spectral properties of the operator T defined over ℓ_p which can be expressed as $T = T_0 + K$ where K is represented by the following matrix

$$K = \begin{pmatrix} a_1 - r_1 & 0 & b_1 - s_1 & 0 & 0 & \cdots \\ 0 & a_2 - r_2 & 0 & b_2 - s_2 & 0 & \cdots \\ c_1 - s_1 & 0 & a_3 - r_1 & 0 & b_3 - s_1 & \cdots \\ 0 & c_2 - s_2 & 0 & a_4 - r_2 & 0 & \cdots \\ 0 & 0 & c_3 - s_1 & 0 & a_5 - r_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The following result proves the compactness of K on ℓ_p.

THEOREM 4.1. *The operator K is a compact operator on ℓ_p.*

Proof. The operator K on ℓ_p can be represented by the following infinite matrix

$$K = \begin{pmatrix} u_1 & 0 & v_1 & 0 & 0 & \cdots \\ 0 & u_2 & 0 & v_2 & 0 & \cdots \\ w_1 & 0 & u_3 & 0 & v_3 & \cdots \\ 0 & w_2 & 0 & u_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are null sequences, which are defined as follows,

$$u_n = \begin{cases} a_n - r_1, & n \text{ is odd} \\ a_n - r_2, & n \text{ is even}, \end{cases} \qquad v_n = \begin{cases} b_n - s_1, & n \text{ is odd} \\ b_n - s_2, & n \text{ is even} \end{cases}$$

and

$$w_n = \begin{cases} c_n - s_1, & n \text{ is odd} \\ c_n - s_2, & n \text{ is even}. \end{cases}$$

Let $x = \{x_1, x_2, x_3 \dots\} \in \ell_p$. We construct a sequence of compact operators $\{K_n\}$ such that for $i \in \mathbb{N}$,

$$(K_n(x))_i = \begin{cases} (Kx)_i, & i = 1, 2, \dots n \\ 0, & \text{otherwise.} \end{cases}$$

For $n \geqslant 2$,

$$\begin{aligned} \|(K - K_n)x\|_p &= \left(\sum_{k=n-1}^{\infty} |w_k x_k + u_{k+2} x_{k+2} + v_{k+2} x_{k+4}|^p \right)^{\frac{1}{p}} \\ &\leqslant \left(\sup_{k \geqslant n-1} |w_k| \right) \|x\|_p + \left(\sup_{k \geqslant n-1} |u_k| \right) \|x\|_p + \left(\sup_{k \geqslant n-1} |v_k| \right) \|x\|_p . \end{aligned}$$

This implies,

$$\|K-K_n\|_p \leqslant \sup_{k\geqslant n-1}|w_k| + \sup_{k\geqslant n-1}|u_k| + \sup_{k\geqslant n-1}|v_k|.$$

Thus, $\{K_n\}$ converges to K as $n\to\infty$ in operator norm and hence K is a compact operator over ℓ_p. $\square$

Hence, the operator T is a compact perturbation of T_0 and since $T_0\in B(\ell_p)$, K is compact on ℓ_p, the operator T is an bounded linear operator on ℓ_p.

Next we derive an inclusion relation between $\sigma(T_0,\ell_p)$ and $\sigma(T,\ell_p)$ and to prove this result we require the following Lemma.

LEMMA 4.2. [32, p. 373] *Let $T: X\to X$ be an operator with a non-empty resolvent set, and let Ω be an open connected subset of $\mathbb{C}\setminus\sigma_{ess}(T)$. If $\Omega\cap\rho(T)\neq\emptyset$ then $\sigma(T)\cap\Omega$ is a finite or countable set, with no accumulation point in Ω, consisting of eigenvalues of T of finite type.*

THEOREM 4.3. *The spectrum of T over ℓ_p satisfies the following inclusion relation*

$$\sigma(T_0,\ell_p)\subseteq\sigma(T,\ell_p)$$

and $\sigma(T,\ell_p)\setminus\sigma(T_0,\ell_p)$ contains finite or countable number of eigenvalues of T of finite type with no accumulation point in $\sigma(T,\ell_p)\setminus\sigma(T_0,\ell_p)$.

Proof. Suppose $\lambda\notin\sigma(T,\ell_p)=\sigma(T_0+K,\ell_p)$. This implies $(T_0+K-\lambda I)^{-1}$ exists and belongs to $B(\ell_p)$. Then there exists $U\in B(\ell_p)$ such that $(T_0+K-\lambda I)U=I$. Hence,

$$KU-I=-(T_0-\lambda I)U \tag{4.1}$$

and $(KU-I)x=0$ implies $(T_0-\lambda I)Ux=0$. This gives us $Ux\in N(T_0-\lambda I)=\{0\}$ as $\sigma(T_0,\ell_p)=\emptyset$. Therefore $x=0$ and consequently $1\notin\sigma_p(KU)$. As K is compact operator, KU is also a compact operator and it follows that $1\notin\sigma(KU)$. Hence, $(KU-I)$ is invertible and consequently $(T_0-\lambda I)$ is invertible by equation (4.1). This implies, $\lambda\notin\sigma(T_0)$ and $\sigma(T_0)\subseteq\sigma(T)$.

For the second part we have, $\rho(T)$ is non-empty as T is a bounded linear operator and $\sigma_{ess}(T,\ell_p)=\sigma_{ess}(T_0,\ell_p)$. Let $\Omega=\mathbb{C}\setminus\sigma(T_0,\ell_p)$ then $\Omega\cap\rho(T,\ell_p)\neq\emptyset$. The set Ω is open connected subset of $\mathbb{C}\setminus\sigma_{ess}(T_0,\ell_p)=\mathbb{C}\setminus\sigma_{ess}(T,\ell_p)$. Then by using Lemma 4.2 we have, $\sigma(T,\ell_p)\cap\Omega$ is a finite or countable set with no accumulation point in Ω consisting eigenvalues of finite type. $\square$

COROLLARY 4.4. $\sigma_{ess}(T,\ell_p)=\sigma(T_0,\ell_p)=[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]$.

Proof. Since a compact perturbation does not effect the Fredholmness and index of a Fredholm operator, it follows $\sigma_{ess}(T_0,\ell_p)=\sigma_{ess}(T,\ell_p)$. Hence, by using Corollary 3.8, we have

$$\sigma_{ess}(T,\ell_p)=\sigma(T_0,\ell_p)=[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]. \quad\square$$

We now focus on the point spectrum of T on ℓ_p. First we analyze the eigenvalues of T lying in $\sigma_{ess}(T,\ell_p)=\sigma(T_0,\ell_p)$, in particular we derive sufficient conditions for the absence of point spectrum on $\sigma_{ess}(T,\ell_p)$. In Theorem 4.5, sufficient conditions are provided in terms of the rate of convergence of the sequences $\{a_{2n-1}\}$, $\{a_{2n}\}$, $\{b_{2n-1}\}$, $\{b_{2n}\}$, $\{c_{2n-1}\}$ and $\{c_{2n}\}$. Sufficient conditions of absence of point spectrum on $\sigma_{ess}(T,\ell_p)$ are also provided in Theorem 4.6 in terms of the entries of the matrix T.

THEOREM 4.5. *If the convergence of the sequences* $\{a_{2n-1}\}$, $\{a_{2n}\}$, $\{b_{2n-1}\}$, $\{b_{2n}\}$, $\{c_{2n-1}\}$ *and* $\{c_{2n}\}$ *are exponentially fast then*

$$\sigma_{ess}(T,\ell_p)\cap\sigma_p(T,\ell_p)=\emptyset.$$

Proof. Let $\lambda\in\sigma_{ess}(T,\ell_p)$. The equation $Tx=\lambda x$ for some $\lambda\in\mathbb{C}$ reduces to the following system

$$\left.\begin{aligned} a_1x_1+b_1x_3 &= \lambda x_1\\ a_2x_2+b_2x_4 &= \lambda x_2\\ c_1x_1+a_3x_3+b_3x_5 &= \lambda x_3\\ c_2x_2+a_4x_4+b_4x_6 &= \lambda x_4\\ &\vdots \end{aligned}\right\}.$$

If we separate the odd and even terms of the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$, the above system of equations can also be expressed as

$$\left.\begin{aligned} c_{2n-1}x_{2n-1}+(a_{2n+1}-\lambda)x_{2n+1}+b_{2n+1}x_{2n+3} &= 0,\\ c_{2n}x_{2n}+(a_{2n+2}-\lambda)x_{2n+2}+b_{2n+2}x_{2n+4} &= 0, \end{aligned}\right\} \tag{4.2}$$

where $n\in\mathbb{N}$ with the initial conditions

$$\left.\begin{aligned} a_1x_1+b_1x_3 &= \lambda x_1,\\ a_2x_2+b_2x_4 &= \lambda x_2. \end{aligned}\right\} \tag{4.3}$$

Introducing two sequences $\{y_n\}$ and $\{z_n\}$ such that $y_n=x_{2n-1}$ and $z_n=x_{2n}$ for $n\in\mathbb{N}$, the system (4.2) with the initial conditions (4.3) reduces to

$$c_{2n-1}y_n+(a_{2n+1}-\lambda)y_{n+1}+b_{2n+1}y_{n+2}=0, \tag{4.4}$$

$$c_{2n}z_n+(a_{2n+2}-\lambda)z_{n+1}+b_{2n+2}z_{n+2}=0, \tag{4.5}$$

where $n\in\mathbb{N}\cup\{0\}$ with $y_0=z_0=0$. Using the assumed convergence of the sequences $\{a_{2n-1}\}$, $\{a_{2n}\}$, $\{b_{2n-1}\},\{b_{2n}\}$, $\{c_{2n-1}\}$, and $\{c_{2n}\}$, the characteristic polynomials of the difference equations (4.4) and (4.5) are

$$t^2+p_1t+1=0, \tag{4.6}$$

$$t^2+p_2t+1=0, \tag{4.7}$$

where $p_1 = \frac{r_1-\lambda}{s_1}$ and $p_2 = \frac{r_2-\lambda}{s_2}$. Let μ_1, μ_2 and γ_1, γ_2 are the pair of roots of equations (4.6) and (4.7) respectively. Also we have

$$\lambda \in \sigma_{ess}(T,\ell_p) = \sigma(T_0,\ell_p) = [r_1-2s_1, r_1+2s_1] \cup [r_2-2s_2, r_2+2s_2].$$

Let $\lambda \in [r_1-2s_1, r_1+2s_1]$ i.e., $p_1 \in [-2,2]$. Then the roots μ_1, μ_2 satisfies $|\mu_1| = 1$ and $|\mu_2| = 1$. Since the convergence of $\{a_{2n-1}\}, \{b_{2n-1}\}, \{c_{2n-1}\}$ are exponentially fast, if $\{y_n\}$ is a solution of equation (4.4) then by Theorem 2.3 [1], it can be deduced that, either $y_n = 0$ for large n or there exists $\rho \in (0,1)$ such that

$$y_n = y_n' + O((1-\rho)^n), \qquad \text{for large } n \tag{4.8}$$

where $\{y_n'\}$ is a solution of limiting equation of (4.4) which is given by

$$y_n + p_1 y_{n+1} + y_{n+2} = 0,$$

and the solution of this limiting equation is already obtained in Case 1, Case 2, Case 3 of Theorem 3.2. Based on equation (4.8), there exists an $M > 0$ such that

$$|y_n - y_n'| \leqslant M(1-\rho)^n.$$

Thus,

$$|y_n'|^p \leqslant (|y_n| + M(1-\rho)^n)^p.$$

Applying Jensen's inequality we get,

$$|y_n'|^p \leqslant 2^{(p-1)}(|y_n|^p + M^p(1-\rho)^{np}).$$

As $|\mu_1| = 1$ and $|\mu_2| = 1$, from Theorem 3.2 we have $\{y_n'\} \notin \ell_p$. Also $0 < 1-\rho < 1$ implies, $\{y_n\} \notin \ell_p$. Hence, $\lambda \notin \sigma_p(T,\ell_p)$. In a similar way, if $\lambda \in [r_2-2s_2, r_2+2s_2]$ we can obtain that $\lambda \notin \sigma_p(T,\ell_p)$. Hence the desired result is proved. □

In the next theorem, we apply transfer matrix approach as discussed in [35, 36]. This enables us to examine the sufficient condition for the absence of point spectrum in essential spectrum of T in terms of the entries of matrix T.

THEOREM 4.6. *If $\lambda \in \sigma_{ess}(T,\ell_p)$ satisfies either of the following conditions*

(i) $\sum_{n=1}^{\infty} \prod_{j=1}^{n} \left[\frac{1}{2}\left(P_j(\lambda) - \sqrt{P_j(\lambda)^2 - \left|\frac{2c_{2j-1}}{b_{2j+1}}\right|^2}\right)\right]^{\frac{p}{2}} = +\infty$

or

(ii) $\sum_{n=1}^{\infty} \prod_{j=1}^{n} \left[\frac{1}{2}\left(Q_j(\lambda) - \sqrt{Q_j(\lambda)^2 - \left|\frac{2c_{2j}}{b_{2j+2}}\right|^2}\right)\right]^{\frac{p}{2}} = +\infty$

where,

$$P_j(\lambda) = \left|\frac{c_{2j-1}}{b_{2j+1}}\right|^2 + \left|\frac{a_{2j+1}-\lambda}{b_{2j+1}}\right|^2 + 1, \;\; Q_j(\lambda) = \left|\frac{c_{2j}}{b_{2j+2}}\right|^2 + \left|\frac{a_{2j+2}-\lambda}{b_{2j+2}}\right|^2 + 1,$$

then $\lambda \notin \sigma_p(T,\ell_p)$.

Proof. (i) Let $\lambda \in \sigma_{ess}(T,\ell_p) = \sigma(T_0,\ell_p)$. Using $Tx = \lambda x$, we have the following difference equation

$$\left.\begin{aligned} c_{2n-1}x_{2n-1} + (a_{2n+1}-\lambda)x_{2n+1} + b_{2n+1}x_{2n+3} &= 0 \\ c_{2n}x_{2n} + (a_{2n+2}-\lambda)x_{2n+2} + b_{2n+2}x_{2n+4} &= 0, \end{aligned}\right\} \tag{4.9}$$

where $n \in \mathbb{N}$ with the initial conditions

$$\left.\begin{aligned} a_1x_1 + b_1x_3 &= \lambda x_1, \\ a_2x_2 + b_2x_4 &= \lambda x_2. \end{aligned}\right\} \tag{4.10}$$

The first equation of (4.9) can be written in the following form

$$\begin{pmatrix} x_{2n+1} \\ x_{2n+3} \end{pmatrix} = B_n(\lambda) \begin{pmatrix} x_{2n-1} \\ x_{2n+1} \end{pmatrix}, \quad n \in \mathbb{N} \cup \{0\}$$

where

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ \frac{-c_{2n-1}}{b_{2n+1}} & \frac{-(a_{2n+1}-\lambda)}{b_{2n+1}} \end{pmatrix},$$

with $x_{-1} = 0$ when $n = 0$. This includes the initial condition

$$a_1x_1 + b_1x_3 = \lambda x_1.$$

In this setting, we have

$$\begin{pmatrix} x_{2n+1} \\ x_{2n+3} \end{pmatrix} = B_n(\lambda)B_{n-1}(\lambda)\cdots B_1(\lambda)y \tag{4.11}$$

where, $y = \begin{pmatrix} x_1 \\ \frac{(\lambda-a_1)}{b_1}x_1 \end{pmatrix}$. Also

$$\begin{aligned} \|B_nB_{n-1}\cdots B_1y\|_p^p &\geqslant \max\left\{2^{\frac{1}{p}-\frac{1}{2}},1\right\}\left(\|B_n\cdots B_1y\|_2^2\right)^{p/2} \\ &= \max\left\{2^{\frac{1}{p}-\frac{1}{2}},1\right\}(\langle B_n\cdots B_1y, B_n\cdots B_1y\rangle)^{\frac{p}{2}} \\ &= M_1\left(\frac{\langle B_n\cdots B_1y, B_n\cdots B_1y\rangle}{\|y\|^2}\right)^{\frac{p}{2}}, \end{aligned} \tag{4.12}$$

where $M_1 = \|y\|^p \max\left\{2^{\frac{1}{p}-\frac{1}{2}},1\right\}$. Using the singular value analog of the famous Courant-Fischer theorem [33, Theorem 7.3.8] and the result

$$\sigma_{\min}(AB) \geqslant \sigma_{\min}(A)\ \sigma_{\min}(B)$$

where $\sigma_{\min}(A)$ denotes the smallest singular value of a matrix A, it follows from the relation (4.12) that

$$\begin{aligned} \|B_nB_{n-1}\cdots B_1y\|_p^p &\geqslant M_1\sigma_{\min}^p(B_nB_{n-1}\ldots B_1) \\ &\geqslant M_1\sigma_{\min}^p(B_n)\,\sigma_{\min}^p(B_{n-1})\ldots\sigma_{\min}^p(B_1), \quad n \in \mathbb{N}. \end{aligned}$$

From equation (4.11) we obtain that

$$|x_{2n+1}|^p + |x_{2n+3}|^p \geqslant M_1 \sigma_{\min}^p (B_n)\, \sigma_{\min}^p (B_{n-1}) \cdots \sigma_{\min}^p (B_1).$$

Taking summation over n, the above relation reduces to

$$2\left[\sum_{n=1}^{\infty} |x_{2n+1}|^p\right] \geqslant M_1 \sum_{n=1}^{\infty}\prod_{j=1}^{n} \sigma_{\min}^p (B_j).$$

This implies,

$$\left[\sum_{n=1}^{\infty} |x_{2n+1}|^p\right] \geqslant M_1' \sum_{n=1}^{\infty}\prod_{j=1}^{n} \sigma_{\min}^p (B_j) \tag{4.13}$$

for some constant $M_1' > 0$. The lowest singular value of B_j is given by

$$\sigma_{\min}(B_j) = \left[\frac{1}{2}\left(\left|\frac{c_{2j-1}}{b_{2j+1}}\right|^2 + \left|\frac{a_{2j+1}-\lambda}{b_{2j+1}}\right|^2 + 1 - \sqrt{\left(\left|\frac{c_{2j-1}}{b_{2j+1}}\right|^2 + \left|\frac{a_{2j+1}-\lambda}{b_{2j+1}}\right|^2 + 1\right)^2 - \left|\frac{2c_{2j-1}}{b_{2j+1}}\right|^2}\right)\right]^{\frac{1}{2}}$$

where we assume $\sqrt{a^2} = a$ for some positive a. Hence, if

$$\sum_{n=1}^{\infty}\prod_{j=1}^{n}\left[\frac{1}{2}\left(P_j(\lambda) - \sqrt{P_j(\lambda)^2 - \left|\frac{2c_{2j-1}}{b_{2j+1}}\right|^2}\right)\right]^{\frac{p}{2}} = +\infty$$

where,

$$P_j(\lambda) = \left|\frac{c_{2j-1}}{b_{2j+1}}\right|^2 + \left|\frac{a_{2j+1}-\lambda}{b_{2j+1}}\right|^2 + 1,$$

then $\{x_n\} \notin \ell_p$ and consequently $\lambda \notin \sigma_p(T,\ell_p)$. This proves the first part of the result.

(ii) As similar to part (i) the second equation of (4.9) can be written as

$$\begin{pmatrix} x_{2n+2} \\ x_{2n+4} \end{pmatrix} = C_n(\lambda) \begin{pmatrix} x_{2n} \\ x_{2n+2} \end{pmatrix}, \quad n \in \mathbb{N}$$

where

$$C_n(\lambda) = \begin{pmatrix} 0 & 1 \\ \frac{-c_{2n}}{b_{2n+2}} & \frac{-(a_{2n+2}-\lambda)}{b_{2n+2}} \end{pmatrix},$$

with $x_0 = 0$ when $n = 0$. This includes the initial condition

$$a_2x_2 + b_2x_4 = \lambda x_2.$$

We can obtain the desired result by using the same argument as in first part. □

REMARK 4.7. Instead of calculating the singular value $\sigma_{\min}(B_j)$ in the relation (4.13), various lower bounds for the same can be used to obtain a less complicated expression than $\sigma_{\min}(B_j)$. Several researchers have been working to refine the lower bound of lowest singular value. Some of the recent works for the lower bound of smallest singular value of a matrix can be found in [45, 34, 50, 49, 39].

Now we focus our study on the point spectrum of T. Under the sufficient conditions as mentioned in previous two results, we have $\sigma_p(T,\ell_p)\cap\sigma(T_0,\ell_p)=\emptyset$. In this case, all the eigenvalues of T are lying outside the set $\sigma(T_0,\ell_p)$. To characterize the eigenvalues, let $Tx=\lambda x$, $x\in\mathbb{C}^{\mathbb{N}}$ and $\lambda\in\sigma(T_0,\ell_p)^c$ where $\sigma(T_0,\ell_p)^c$ denotes the complement of $\sigma(T_0,\ell_p)$. From equations (4.4) and (4.5) in Theorem 4.5, we have the following system

$$c_{2n-1}y_n+(a_{2n+1}-\lambda)y_{n+1}+b_{2n+1}y_{n+2}=0, \tag{4.14}$$

$$c_{2n}z_n+(a_{2n+2}-\lambda)z_{n+1}+b_{2n+2}z_{n+2}=0, \tag{4.15}$$

where $n\in\mathbb{N}\cup\{0\}$ with $y_0=z_0=0$ and $y_n=x_{2n-1}$, $z_n=x_{2n}$. Clearly each of the difference equations (4.14) and (4.15) have two fundamental solutions. Let $\{y_n^{(1)}(\lambda), y_n^{(2)}(\lambda)\}$ and $\{z_n^{(1)}(\lambda),z_n^{(2)}(\lambda)\}$ are the sets of fundamental solutions of the equations (4.14) and (4.15) respectively. Under this setting we have the following result.

THEOREM 4.8. *If either of the sufficient conditions mentioned in Theorem 4.5 and Theorem 4.6 hold true then the point spectrum of T over ℓ_p is given by*

$$\sigma_p(T,\ell_p)=\left\{\lambda\in\mathbb{C}:y_0^{(1)}(\lambda)=0\right\}\cup\left\{\lambda\in\mathbb{C}:{z_0}^{(1)}(\lambda)=0\right\}.$$

Proof. As $\sigma_{ess}(T,\ell_p)\cap\sigma_p(T,\ell_p)=\emptyset$, we restrict our search for point spectrum outside the set $[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]$. Let μ_1, μ_2 and γ_1, γ_2 are the pair of roots of equations (4.6) and (4.7) respectively which are the characteristic polynomials of equations (4.14) and (4.15) respectively. Since $\lambda\notin[r_1-2s_1,r_1+2s_1]\cup[r_2-2s_2,r_2+2s_2]$, we have $p_1\notin[-2,2]$, and without loss of generality we assume that $|\mu_1|<1$ and $|\mu_2|>1$. By Perron's First Theorem [28, p. 344] it can be deduced that

$$\lim_{n\to\infty}\frac{y_{n+1}^{(1)}(\lambda)}{y_n^{(1)}(\lambda)}=\mu_1,\quad \lim_{n\to\infty}\frac{y_{n+1}^{(2)}(\lambda)}{y_n^{(2)}(\lambda)}=\mu_2.$$

Hence $\{y_n^{(1)}(\lambda)\}\in\ell_p$ but $\{y_n^{(2)}(\lambda)\}\notin\ell_p$ and the general solution of the difference equation (4.14), which is the linear combination of the fundamental solutions, is given by

$$y_n(\lambda)=c_1y_n^{(1)}(\lambda)+c_2y_n^{(2)}(\lambda),\quad n\in\mathbb{N}\cup\{0\}$$

where c_1 and c_2 are arbitrary constants. In a similar way, we can assume that $|\gamma_1|<1$ and $|\gamma_2|>1$ and by Perron's First Theorem we have

$$\lim_{n\to\infty}\frac{z_{n+1}^{(1)}(\lambda)}{z_n^{(1)}(\lambda)}=\gamma_1,\quad \lim_{n\to\infty}\frac{z_{n+1}^{(2)}(\lambda)}{z_{(n)}^{(2)}(\lambda)}=\gamma_2.$$

This implies $\{z_n^{(1)}(\lambda)\}\in\ell_p$ and $\{z_n^{(2)}(\lambda)\}\notin\ell_p$ and the general solution of the difference equation (4.15) is given by

$$z_n(\lambda)=d_1z_n^{(1)}(\lambda)+d_2z_n^{(2)}(\lambda),\quad n\in\mathbb{N}\cup\{0\}$$

where d_1 and d_2 are arbitrary constants and consequently the general solution of the system $Tx = \lambda x$ is given by $x_n(\lambda)$ where

$$x_{2n-1}(\lambda) = c_1 y_n^{(1)}(\lambda) + c_2 y_n^{(2)}(\lambda), \;\; n \in \mathbb{N}$$
$$x_{2n}(\lambda) = d_1 z_n^{(1)}(\lambda) + d_2 z_n^{(2)}(\lambda), \;\; n \in \mathbb{N}$$

with $x_{-1}(\lambda) = x_0(\lambda) = 0$. Consider

$$S_1 = \left\{\lambda \in \mathbb{C} : y_0^{(1)}(\lambda) = 0\right\} \cup \left\{\lambda \in \mathbb{C} : z_0^{(1)}(\lambda) = 0\right\}.$$

Let $\lambda \in S_1$, then $y_0^{(1)}(\lambda) = 0$ or $z_0^{(1)}(\lambda) = 0$. If $y_0^{(1)}(\lambda) = 0$, we can construct a non-trivial solution $x_n(\lambda)$ of the system $Tx = \lambda x$ in the following way.

Let $c_2 = 0$ and $d_1 = d_2 = 0$. In this case we have $y_n(\lambda) = y_n^{(1)}(\lambda)$ and $z_n(\lambda) = 0$ for all n. Since, $\{y_n^{(1)}(\lambda)\} \in \ell_p$, we have $x_n(\lambda)$ is a non-trivial solution of $Tx = \lambda x$ and $\{x_n(\lambda)\} \in \ell_p$. If $z_0^{(1)}(\lambda) = 0$ then in a similar way we can construct a non-trivial solution $x_n(\lambda)$ of $Tx = \lambda x$ where $x_n(\lambda) = 0$, if n is odd and $x_n(\lambda) = z_n^{(1)}(\lambda)$, if n is even. Hence, $\lambda \in \sigma_p(T, \ell_p)$ and consequently $S_1 \subseteq \sigma(T, \ell_p)$. Now, suppose $\lambda \notin S$. Then $y_0^{(1)}(\lambda) \neq 0$ and $z_0^{(1)}(\lambda) \neq 0$. Clearly $\lambda \in \sigma_p(T, \ell_p)$ if and only if $c_2 = 0$ and $d_2 = 0$. Now we consider following cases with the assumption $c_2 = d_2 = 0$.

Case 1: If $c_2 = d_2 = 0$ and $c_1 = 0$, we have

$$y_n(\lambda) = 0 \;\forall n \quad \text{and} \quad z_n(\lambda) = d_1 z_n^{(1)}(\lambda) \;\forall n.$$

Using the initial condition $z_0(\lambda) = 0$, we have $d_1 z_0^{(1)}(\lambda) = 0$. If $d_1 = 0$ then we get a trivial solution and if $d_1 \neq 0$ then $z_0^{(1)}(\lambda) = 0$ and this is a contradiction.

Case 2: If $c_2 = d_2 = 0$ and $c_1 \neq 0$ we have

$$y_n(\lambda) = c_1 y_n^{(1)}(\lambda) \;\forall n.$$

Using the initial condition $y_0(\lambda) = 0$ we have $c_1 y_0^{(1)}(\lambda) = 0$, this implies $y_0^{(1)}(\lambda) = 0$ which is a contradiction. Hence there are no solution of the difference equation (4.14). By Case 1 and Case 2, we can deduce that no non-trivial solution exists for the system $Tx = \lambda x$. Hence $\lambda \notin \sigma_p(T, \ell_p)$. Thus,

$$\sigma_p(T, \ell_p) = \left\{\lambda \in \mathbb{C} : y_0^{(1)}(\lambda) = 0\right\} \cup \left\{\lambda \in \mathbb{C} : {z_0}^{(1)}(\lambda) = 0\right\}. \quad \square$$

REMARK 4.9. The adjoint operator $T^* : \ell_p^* \to \ell_p^*$, is represented by transpose of the matrix T and dual of ℓ_p is isomorphic to ℓ_q where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < q < \infty$. Similar as T, the operator T^* can also be written as

$$T^* = T_0 + K^t,$$

where K^t denotes the transpose of K and K^t is also a compact operator. Since $\sigma(T,\ell_p) = \sigma(T^*,\ell_p^*)$, Theorem 4.3 implies

$$\sigma(T_0,\ell_p) \subseteq \sigma(T^*,\ell_p^*),$$

and using similar argument of the proof of Theorem 4.3 it can be obtain that $\sigma(T^*,\ell_p^*) \setminus \sigma(T_0,\ell_p)$ contains finite or countable number of eigenvalues of T^* of finite type with no accumulation point in $\sigma(T^*,\ell_p^*) \setminus \sigma(T_0,\ell_p)$. Assuming similar hypothesis on the rate of convergence of sequences in Theorem 4.5, we can prove that

$$\sigma_{ess}(T^*,\ell_p^*) \cap \sigma_p(T^*,\ell_p^*) = \emptyset$$

and this implies, the point spectrum of T^* is lying outside of the region

$$[r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2].$$

Now similar as Theorem 4.8, let $\{g_n^{(1)}(\lambda), g_n^{(2)}(\lambda)\}$ and $\{h_n^{(1)}(\lambda), h_n^{(2)}(\lambda)\}$ are the sets of fundamental solutions of the following difference equations respectively

$$\begin{aligned} b_{2n-1}g_n + (a_{2n+1} - \lambda)g_{n+1} + c_{2n+1}g_{n+2} &= 0, \\ b_{2n}h_n + (a_{2n+2} - \lambda)h_{n+1} + c_{2n+2}h_{n+2} &= 0, \end{aligned}$$

which are obtained from $T^*f = \lambda f$, $f \in \ell_p^*$ and $g_n(\lambda) = f_{2n-1}(\lambda)$, $h_n(\lambda) = f_{2n}(\lambda)$. Also, $g_0(\lambda) = h_0(\lambda) = 0$. This leads us to the following result

$$\sigma_p(T^*,\ell_p^*) = \left\{\lambda \in \mathbb{C} : g_0^{(1)}(\lambda) = 0\right\} \cup \left\{\lambda \in \mathbb{C} : h_0^{(1)}(\lambda) = 0\right\}.$$

Eventually, we obtain that

$$\sigma(T^*,\ell_p^*) = [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2] \cup S_2$$

where,

$$S_2 = \left\{\lambda \in \mathbb{C} : g_0^{(1)}(\lambda) = 0\right\} \cup \left\{\lambda \in \mathbb{C} : h_0^{(1)}(\lambda) = 0\right\}.$$

Since, $\sigma(T,\ell_p) = \sigma(T^*,\ell_p^*)$ and S_1, S_2 both sets are disjoint from $[r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2]$ we have, $S_1 = S_2$.

Using the observations in Remark 4.9 and Proposition 2.1, we can summarize all the results of spectrum and various spectral subdivisions of the operator T in the following theorem.

THEOREM 4.10. *If the convergence of the sequences* $\{a_{2n-1}\}$, $\{a_{2n}\}$, $\{b_{2n-1}\}$, $\{b_{2n}\}$, $\{c_{2n-1}\}$ *and* $\{c_{2n}\}$ *are exponentially fast and*

$$S_1 = \left\{\lambda \in \mathbb{C} : y_0^{(1)}(\lambda) = 0\right\} \cup \left\{\lambda \in \mathbb{C} : z_0{}^{(1)}(\lambda) = 0\right\},$$

then we have the following results.

(i) The spectrum of T on ℓ_p is

$$\sigma(T,\ell_p) = [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2] \cup S_1.$$

(ii) The point spectrum of T on ℓ_p is

$$\sigma_p(T,\ell_p) = S_1.$$

(iii) The residual spectrum of T on ℓ_p is

$$\sigma_r(T,\ell_p) = \emptyset.$$

(iv) The continuous spectrum of T on ℓ_p is

$$\sigma_c(T,\ell_p) = [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2].$$

(v) The essential spectrum of T on ℓ_p is

$$\sigma_{ess}(T,\ell_p) = [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2].$$

(vi) The discrete spectrum of T on ℓ_p is

$$\sigma_d(T,\ell_p) = S_1.$$

(vii) The compression spectrum of T on ℓ_p is

$$\sigma_{co}(T,\ell_p) = S_1.$$

(viii) The approximate spectrum of T on ℓ_p is

$$\sigma_{app}(T,\ell_p) = [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2] \cup S_1.$$

(ix) The defect spectrum of T on ℓ_p is

$$\sigma_\delta(T,\ell_p) = [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2] \cup S_1.$$

Proof. The proofs of the above statements are given below.

(i) It is well known that $\sigma_p(T,\ell_p) \subseteq \sigma(T,\ell_p)$ and $\sigma(T_0,\ell_p) \subseteq \sigma(T,\ell_p)$. This implies,

$$[r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2] \cup S_1 \subseteq \sigma(T,\ell_p).$$

Also by using Theorem (4.3) we get,

$$\sigma(T,\ell_p) \subseteq [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2] \cup S_1.$$

Hence,

$$\sigma(T,\ell_p) = [r_1 - 2s_1, r_1 + 2s_1] \cup [r_2 - 2s_2, r_2 + 2s_2] \cup S_1.$$

(ii) Result has been proved in Theorem 4.8.

(iii) We already aware of that $\sigma_r(T,\ell_p) = \sigma_p(T^*,\ell_p^*) \setminus \sigma_p(T,\ell_p)$. Hence,

$$\sigma_r(T,\ell_p) = \emptyset.$$

(iv) Spectrum of an operator is the disjoint union of point spectrum, residual spectrum and continuous spectrum. By using this result we can obtain the desired result.

(v) The required result has been proved in Corollary 4.4.

(vi) We already proved that the point spectrum of T is disjoint from $[r_1-2s_1, r_1+2s_1] \cup [r_2-2s_2, r_2+2s_2]$, and by Theorem 4.3 we have, every element of $\sigma_p(T,\ell_p)$ is of finite type. Hence,

$$\sigma_d(T,\ell_p) = \left\{\lambda \in \mathbb{C} : y_0^{(1)}(\lambda) = 0\right\} \cup \left\{\lambda \in \mathbb{C} : {z_0}^{(1)}(\lambda) = 0\right\}.$$

(vii) From part (e) of Proposition 2.1, the desired result is obvious.

(viii) Clearly,

$$\sigma_{app}(T,\ell_p) \subseteq \sigma(T,\ell_p) = [r_1-2s_1, r_1+2s_1] \cup [r_2-2s_2, r_2+2s_2] \cup S_1.$$

Also, we know that point spectrum is always a subset of approximate point spectrum. By using this fact and with the help of part (g) of Proposition 2.1, we have

$$[r_1-2s_1, r_1+2s_1] \cup [r_2-2s_2, r_2+2s_2] \cup S_1 \subseteq \sigma_{app}(T,\ell_p).$$

Hence,

$$\sigma_{app}(T,\ell_p) = \sigma(T,\ell_p) = [r_1-2s_1, r_1+2s_1] \cup [r_2-2s_2, r_2+2s_2] \cup S_1.$$

(ix) From part (c) of Proposition 2.1, we have

$$\sigma_{app}(T^*,\ell_p^*) = \sigma_\delta(T,\ell_p).$$

Clearly,

$$\sigma_{app}(T^*,\ell_p^*) \subseteq \sigma(T^*,\ell_p^*) = [r_1-2s_1, r_1+2s_1] \cup [r_2-2s_2, r_2+2s_2] \cup S_1.$$

Then $S_1 \subseteq \sigma_{app}(T^*,\ell_p^*)$, follows from the fact that $\sigma_p(T^*,\ell_p^*) \subseteq \sigma_{app}(T^*,\ell_p^*)$. By using this fact and with the help of part (g) of Proposition 2.1, we have

$$[r_1-2s_1, r_1+2s_1] \cup [r_2-2s_2, r_2+2s_2] \cup S_1 \subseteq \sigma_{app}(T^*,\ell_p^*).$$

Therefore, $\sigma_{app}(T^*,\ell_p^*) = \sigma(T,\ell_p)$. Hence,

$$\sigma_\delta(T,\ell_p) = [r_1-2s_1, r_1+2s_1] \cup [r_2-2s_2, r_2+2s_2] \cup S_1. \quad \square$$

REMARK 4.11. One interesting observation of the above theorem is the relation $\sigma_p(T,\ell_p) = \sigma_d(T,\ell_p) = \sigma_{co}(T,\ell_p) = S_1$ holds. In other words, all the eigenvalues are of finite type and range of $T-\lambda I$ is not dense in ℓ_p for any eigenvalue λ.

Declarations

Conflict of Interest. All the authors declare that they have no conflict of interest.

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