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REDUCING SUBSPACES OF SKEW SYMMETRIC OPERATORS

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Abstract. An operator T on a complex, separable Hilbert space $\mathscr{H}$ is said to be skew symmetric if there exists a conjugation C on $\mathscr{H}$ such that $CTC = -T^*$. This paper aims to describe reducing subspaces of skew symmetric operators from the view point of approximation. In particular, given a skew symmetric operator T, $1 \leqslant n \leqslant \aleph_0$ and $\varepsilon > 0$, it is proved that there exists a compact operator K with $\|K\| < \varepsilon$ such that $T+K$ is skew symmetric and has exactly n minimal reducing subspaces.

1. Introduction

In this paper, let $\mathscr{H}$ be a complex, separable Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle$. Denote by $\mathscr{B}(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$ and $\mathscr{K}(\mathscr{H})$ the ideal of all compact operators in $\mathscr{B}(\mathscr{H})$.

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *skew symmetric* if $CTC = -T^*$ for some conjugation C on $\mathscr{H}$; in this case, we say T is C-*skew-symmetric*. Recall that a conjugate-linear map $C : \mathscr{H} \to \mathscr{H}$ is called a *conjugation* if C is invertible with $C^{-1} = C$ and $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathscr{H}$. Denote by *SSO* the set of all skew symmetric operators on $\mathscr{H}$. The term skew symmetric stems from the fact that T is a skew symmetric operator if and only if T can be written as a skew symmetric matrix (i.e., those matrices M satisfying $M + M^{tr} = 0$, where M^{tr} denotes the transpose of M) relative to some orthonormal basis of $\mathscr{H}$ [11, Lemma 2.11].

There are several motivations for studying skew symmetric operators. In particular, skew symmetric operators have many applications in algebra and geometry.

Skew symmetric operators provide an important example of Lie algebra of operators. Recall that, for a conjugation C on $\mathscr{H}$, the *orthogonal Lie algebra* of operators $\mathscr{O}_C$ is the set of all C-skew-symmetric operators on $\mathscr{H}$. That is,

$$\mathscr{O}_C = \left\{ T \in \mathscr{B}(\mathscr{H}) : CTC = -T^* \right\}.$$

de La Harpe [9] discussed in detail many elementary aspects of $\mathscr{O}_C$ (and some other classical Lie algebras of operators). Topics treated include Lie derivations, Lie ideals,

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automorphisms, real forms and so on. Recently, [3] classified Lie ideals of $\mathscr{O}_C$ and determined their dual spaces.

Skew symmetric operators play an important role in algebraic geometry. We remark that $\mathscr{O}_C$ is also called a symplectic type Cartan factor. The Cartan factors derive from Cartan's classification of bounded symmetric domains [5, Theorem 2.5.9]. Moreover, $\mathscr{O}_C$ is a concrete example of JB*-triples which plays a significant role in geometry and analysis. Kaup [18] proved that the category of all bounded symmetric domains with base point is equivalent to the category of JB*-triples.

Recently there has been growing interest in the study of skew symmetric operators. There are many interesting results being obtained in [1, 2, 19, 24, 26, 28]. In particular, several special classes of skew symmetric operators are classified, such as normal operators, compact operators, partial isometries and weighted shifts [20, 21, 27]. Also we remark that skew symmetric operator is closely related to complex symmetric operator, which is an important class of operators. The reader is referred to [29] for more details.

If $T \in \mathscr{B}(\mathscr{H})$ and $\mathscr{M}$ is a closed subspace of $\mathscr{H}$, say that $\mathscr{M}$ is an *invariant subspace* of T if $T(\mathscr{M}) \subset \mathscr{M}$; $\mathscr{M}$ is a *reducing subspace* of T if both $\mathscr{M}$ and $\mathscr{M}^\perp$ are invariant subspaces of T. An operator $T \in \mathscr{B}(\mathscr{H})$ is *reducible* if it has a nontrivial reducing subspace; otherwise, T is called *irreducible*. The study of invariant subspaces and reducing subspaces for various classes of linear operators has inspired much deep research and promoted many interesting problems. There are many nice work on describing the structures of reducing subspaces for Toeplitz operators with finite Blaschke products symbols on the Bergman space over the unit disk can be found in [8, 12, 14, 25].

In [27], Zhu described the block structure of skew symmetric operators. More precisely, each skew symmetric operator can be written as the direct sum of three kinds of elementary skew symmetric operators (some of which may be absent), that is, completely reducible ones, irreducible ones and operators of form $A \oplus (-CA^*C)$, where A is irreducible, not a skew symmetric operator and C is a conjugation. Let $T \in \mathscr{B}(\mathscr{H})$ and $\mathscr{M}$ be a nonzero reducing subspace of T. Recall that $\mathscr{M}$ is called a *minimal reducing subspace* of T if $T|_{\mathscr{M}}$ is irreducible; and T is said to be *completely reducible* if T does not admit any minimal reducing subspace.

In [27], some concrete examples of completely reducible and irreducible skew symmetric operators were provided. In a recent paper [4], the first author and Zhu proved that each skew symmetric operator is a small compact perturbation of irreducible skew symmetric operators.

Inspired by the preceding results, the present paper aims to study reducing subspaces of skew symmetric operators from the view point of approximation. This is partly because the topic of skew symmetric operators mainly falls into the category of abstract operator theory, although skew symmetric operators also have many concrete examples.

We pay more attention on minimal reducing subspaces. For $n = 0, 1, 2\ldots$, set

$$SSO_n = \{T \in SSO : T \text{ has } n \text{ minimal reducing subspaces}\}.$$

In addition, we denote

$$SSO_{\aleph_0} = \{T \in SSO : T \text{ has countably infinitely many minimal reducing subspaces}\}$$

and

$$SSO_{\infty} = \{T \in SSO : T \text{ has uncountably many minimal reducing subspaces}\}.$$

Then SSO_1 is the set of all irreducible ones in SSO, and SSO_0 is the set of all completely reducible ones in SSO. By [4, Thm. 1.3], $\overline{SSO_1} = \overline{SSO}$.

The main result of this paper is the following theorem, which describes minimal reducing subspaces of skew symmetric operators from the view point of approximation.

THEOREM 1.1. *(i)* SSO_0 *is nowhere dense in* $\overline{SSO}$.

(ii) $\overline{SSO_n} = \overline{SSO}$ *for* $1 \leqslant n \leqslant \aleph_0$.

(iii) $\overline{SSO_\infty} = \overline{SSO}$.

REMARK 1. There some similar results for complex symmetric operators in the literature. [22, 23] showed that those complex symmetric operators having n reducing subspaces are norm dense in the class of complex symmetric operators. It can be concluded that the skew symmetric operator and the complex symmetric operator are consistent in terms of reducibility.

The rest of this paper is organized as follows. In Subsect.2.1, we shall prove that reducible skew symmetric operators are norm dense in the class of SSOs by $\mathscr{L}$-normality. In Subsect.2.2 we will give some results mainly concerning the irreducible compact perturbations of skew symmetric operators. The Proof of Theorem 1.1 shall be provided in Subsect. 2.3.

2. Proof of main result

This section is devoted to proving Theorem 1.1.

2.1. $\mathscr{L}$-normal operators

DEFINITION 2.1. An operator $T \in \mathscr{B}(\mathscr{H})$ is called $\mathscr{L}$-normal if it satisfies

$$\|p(T,T^*)\| = \|\tilde{p}(-T^*,-T)\|$$

for any polynomial $p(x,y)$ in two free variables. Here $\tilde{p}(x,y)$ is obtained from $p(x,y)$ by conjugating each coefficient.

Note that each skew symmetric operator is $\mathscr{L}$-normal. In fact, if $T \in \mathscr{B}(\mathscr{H})$ is skew symmetric, there exists a conjugation C on $\mathscr{H}$ such that $CTC = -T^*$. Since C is conjugate-linear and isomeric, it is easy to verify that T is $\mathscr{L}$-normal. Noting that $\mathscr{L}$-normality is defined in terms of a norm equality, it implies a C^*-algebra approach to skew symmetric operators. [26] classifies up to approximate unitary equivalence some skew symmetric operators by $\mathscr{L}$-normal.

We denote by $\pi : \mathscr{B}(\mathscr{H}) \to \mathscr{A}(\mathscr{H}) := \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ the canonical projection of $\mathscr{B}(\mathscr{H})$ onto the Calkin algebra and by $\hat{T}$ the imagine $\pi(T)$ of T. An operator $T \in \mathscr{B}(\mathscr{H})$ is called *essentially* $\mathscr{L}$*-normal* if $\hat{T}$ is a $\mathscr{L}$-normal element of $\mathscr{A}(\mathscr{H})$.

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be *semi-Fredholm*, if $\operatorname{ran}T$ is closed and either $\dim\ker T$ or $\dim\ker T^*$ is finite; in this case, we say $\operatorname{ind}T := \dim\ker T - \dim\ker T^*$ is the Fredholm index of T. In particular, if $\operatorname{ind}T$ is finite, then T is called a *Fredholm operator*. The *Wolf spectrum* $\sigma_{lre}(T)$ and the essential spectrum $\sigma_e(T)$ of T are defined, respectively, as

$$\sigma_{lre}(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not semi-Fredholm}\}$$

and

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is not Fredholm}\}.$$

In the following, the notion $\cong$ denotes unitary equivalence, and $\cong_a$ denotes approximate unitary equivalence. Two operators A_1 and A_2 are said to be *approximately unitarily equivalent* if there exist unitary operators $\{U_n\}$ from $\mathscr{H}_1$ onto $\mathscr{H}_2$ such that $U_n A_1 U_n^* \to A_2$ as $n \to \infty$.

The following result can be verified directly by [16, Prop. 4.21(iv)].

LEMMA 2.2. *Let $A, B \in SSO$, $A \cong_a B$ and $\varepsilon > 0$, then there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that $A + K \cong B$.*

Given a unital C^*-algebra $\mathscr{A}$ and $a \in \mathscr{A}$, let $C^*(a)$ be the C^*-subalgebra of $\mathscr{A}$ generate by a and the identity. For $T \in \mathscr{B}(\mathscr{H})$, denote $\sigma_{lr}(T) = \sigma_l(T) \cap \sigma_r(T)$, where $\sigma_l(T), \sigma_r(T)$ are the left spectrum and right spectrum of T respectively.

LEMMA 2.3. *If $T \in \mathscr{B}(\mathscr{H})$ is essentially $\mathscr{L}$-normal, then $T \cong_a T \oplus R$ for some skew symmetric operator R with $R \cong_a R \oplus R$,*

$$\sigma(R) = \sigma_e(R) = \sigma_e(T) \text{ and } \sigma_{lre}(R) = \sigma_{lre}(T).$$

Proof. There exists a unital, faithful $*$-representation ρ of $C^*(\hat{T})$ on $\mathscr{H}_\rho$. Denote $A = \rho(\hat{T})$ and $B = A^{(\infty)}$, where $A^{(\infty)}$ denotes the direct sum of $\aleph_0$ copies of A. Noting that ρ is faithful and $\hat{T}$ is $\mathscr{L}$-normal, both A and B are $\mathscr{L}$-normal. By [16, Prop. 4.21(ii)], we have $T \cong_a T \oplus B$. Since $C^*(B)$ contains no nonzero compact operator, by [26, Thm. 2.4], B is approximately unitarily equivalent to a skew symmetric operator R. Thus

$$T \cong_a T \oplus B \cong_a T \oplus R.$$

Since $B \cong B \oplus B$, it is clear that $R \cong_a R \oplus R$.

Since $R \cong_a B$, we have $\sigma(R) = \sigma(B), \sigma_e(R) = \sigma_e(B)$ and $\sigma_{lr}(R) = \sigma_{lr}(B)$. It follows from $B = A^{(\infty)}$ that $\sigma(A) = \sigma(B)$ and $\sigma_{lr}(A) = \sigma_{lr}(B)$. If B is a Fredholm operator, it is easily verified that A is invertible. Thus $\sigma(A) \subset \sigma_e(B)$. Hence $\sigma_e(B) =$

$\sigma(A) = \sigma(B)$. This implies that $\sigma_e(R) = \sigma(A) = \sigma(R)$. Noting that ρ is faithful, so $\sigma(A) = \sigma(\hat{T}) = \sigma_e(T)$. Therefore $\sigma_e(R) = \sigma(A) = \sigma(R) = \sigma_e(T)$.

Next we shall prove $\sigma_{lre}(R) = \sigma_{lre}(T)$. If $\lambda \notin \sigma_{lre}(R)$, since $R \cong_a B$, we have $\lambda \notin \sigma_{lre}(B)$. That is, $B - \lambda$ is a semi-Fredholm operator. It follows from $B = A^{(\infty)}$ that $A - \lambda$ is a semi-Fredholm operator. We claim that $\lambda \notin \sigma_{lr}(A)$. If $\lambda \in \sigma_{lr}(A)$, then $\dim\ker(A-\lambda) > 0$ and $\dim\ker(A-\lambda)^* > 0$. This implies that $\dim\ker(B-\lambda) = \dim\ker(B-\lambda)^* = \infty$, a contradiction. Thus $\sigma_{lr}(A) \subset \sigma_{lre}(R)$. Since $\sigma_{lr}(A) = \sigma_{lr}(B) = \sigma_{lr}(R)$, we have $\sigma_{lr}(R) = \sigma_{lre}(R)$. Noting that ρ is faithful, so $\sigma_{lr}(A) = \sigma_{lr}(\hat{T}) = \sigma_{lre}(T)$. Thus $\sigma_{lre}(R) = \sigma_{lre}(T)$. $\square$

THEOREM 2.4. *If $T \in \mathscr{B}(\mathscr{H})$ is skew symmetric, then T is approximately unitarily equivalent to a reducible skew symmetric operator.*

Proof. Assume that there is a conjugation C on $\mathscr{H}$ such that $CTC = -T^*$. In view of Lemma 2.3, it suffices to prove that T is essentially $\mathscr{L}$-normal. Fix a polynomial $p(x,y)$ in two free variables x,y. Then

$$\begin{aligned}\|\tilde{p}(\hat{T},\hat{T}^*)\| &= \inf\{\|\tilde{p}(T,T^*)+K\| : K \in \mathscr{K}(\mathscr{H})\}\\ &= \inf\{\|Cp(-T^*,-T)C+K\| : K \in \mathscr{K}(\mathscr{H})\}\\ &= \inf\{\|p(-T^*,-T)+CKC\| : K \in \mathscr{K}(\mathscr{H})\}\\ &= \inf\{\|p(-T^*,-T)+K\| : K \in \mathscr{K}(\mathscr{H})\}\\ &= \|p(-\hat{T}^*,-\hat{T})\|.\end{aligned}$$

Thus T is essentially $\mathscr{L}$-normal. $\square$

By Lemma 2.2 and Theorem 2.4, the following result is clear.

COROLLARY 2.5. *Those reducible ones in SSO constitute a norm dense subset of $\overline{SSO}$.*

2.2. Irreducible compact perturbations

In this subsection we give an auxiliary result.

THEOREM 2.6. *Let $T \in SSO$, $n \in \mathbb{N}$ and $\varepsilon > 0$. Then there exist $K_1,\ldots,K_n \in \mathscr{K}(\mathscr{H})$ with $\sum_{i=1}^n \|K_i\| < \varepsilon$ such that $T+K_1,\cdots,T+K_n$ are irreducible skew symmetric operators with pairwise distinct spectra.*

Before giving the proof of Theorem 2.6, we first introduce some notations and definitions.

Let $T \in \mathscr{B}(\mathscr{H})$ and Λ be a clopen subset of $\sigma(T)$. Then there exists an analytic Cauchy domain Ω such that $\Lambda \subset \Omega$ and $[\sigma(T)\setminus\Lambda] \cap \overline{\Omega} = \emptyset$. Recall that the *Riesz idempotent* of T corresponding to Λ is

$$E(\Lambda;T) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma (z-T)^{-1}dz,$$

where $\Gamma = \partial\Omega$ positively oriented with respect to Ω; in this case, we denote $\mathscr{H}(\Lambda;T) = \operatorname{ran} E(\Lambda;T)$.

Let λ be an isolated point of $\sigma(T)$. The element λ is called a *normal eigenvalue* of T if $\dim \mathscr{H}(\{\lambda\};T) < \infty$. We denoted by $\sigma_0(T)$ the set of all normal eigenvalues of T. The reader is referred to [16, Chapter 1] for more details.

LEMMA 2.7. ([17, Lem. 3.2.6]) *Let $T \in \mathscr{B}(\mathscr{H})$ and suppose $\emptyset \neq \Delta \subset \sigma_{lre}(T)$. Then, given $\varepsilon > 0$, there exists a compact operator K with $\|K\| < \varepsilon$ such that*

$$T + K = \begin{bmatrix} N & * \\ 0 & A \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \end{matrix},$$

where N is a diagonal normal operator of uniformly infinite multiplicity with $\sigma(N) = \overline{\Delta}$ and $\sigma(T) = \sigma(A)$.

LEMMA 2.8. ([6, page 366]) *If $T \in \mathscr{B}(\mathscr{H})$, then $\partial\sigma(T) \subset [\sigma_0(T) \cup \sigma_{lre}(T)]$.*

If Γ is a subset of $\mathbb{C}$, we denote by $\operatorname{iso}\Gamma$ the set of all isolated points of Γ. For $r > 0$ and $\lambda \in \mathbb{C}$, we set $B(\lambda, r) := \{z \in \mathbb{C} : |z - \lambda| < r\}$.

Proof of Theorem 2.6. In view of Lemma 2.2 and Lemma 2.3, it suffices to prove the result for $W := T \oplus R \oplus R$.

By Lemma 2.8, there exists $\lambda \in \partial\sigma(T) \cap \sigma_{lre}(T)$. Lemma 2.3 implies that $\lambda \in \sigma_{lre}(R)$. Thus, given $\varepsilon > 0$, by Lemma 2.7, there is a compact operator F with $\|F\| < \frac{\varepsilon}{2}$ such that

$$R + F = \begin{bmatrix} \lambda & B \\ 0 & A \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^{\perp} \end{matrix},$$

where $e \in \mathscr{H}$ is a unit vector and $\sigma(A) = \sigma(R)$. Since $\lambda \in \partial\sigma(T)$, there are pairwise distinct points $\lambda_1, \ldots, \lambda_n \in \mathbb{C} \setminus \sigma(T)$ such that $\sum_{i=1}^{n} |\lambda_i - \lambda| < \frac{\varepsilon}{4}$. Thus we can find compact operator G_i with $\sum_{i=1}^{n} \|G_i\| < \frac{\varepsilon}{4}$ such that

$$R_i := R + F + G_i = \begin{bmatrix} \lambda_i & B \\ 0 & A \end{bmatrix} \begin{matrix} \mathbb{C}e \\ (\mathbb{C}e)^{\perp} \end{matrix}.$$

Since $\sigma(A) = \sigma(R) \subset \sigma(T)$, we have $\lambda_i \notin \sigma(A)$. Thus $\lambda_i \in \sigma_0(R_i)$ for $i = 1, \ldots, n$.

Noting that R is a skew symmetric operator, there exists a conjugation C on $\mathscr{H}$ such that $CRC = -R^*$. For $1 \leqslant i \leqslant n$, set

$$K_{1,i} := \begin{bmatrix} 0 & & \\ & F + G_i & \\ & & -C(F^* + G_i^*)C \end{bmatrix} \begin{matrix} \mathscr{H} \\ \mathscr{H} \\ \mathscr{H} \end{matrix}.$$

Then $K_{1,i}$ is compact with $\|K_{1,i}\| < \frac{3\varepsilon}{4}$ and

$$W_i := W + K_{1,i} = \begin{bmatrix} T & & \\ & R_i & \\ & & R - C(F^* + G_i^*)C \end{bmatrix} = \begin{bmatrix} T & & \\ & R_i & \\ & & -CR_i^*C \end{bmatrix}.$$

It follows from $\lambda_i \notin \sigma(T)$ that $\lambda_i \in \sigma_0(W_i) \setminus \sigma(W_j)$ whenever $i \neq j$.

Note that $R \oplus (-CR^*C)$ is a skew symmetric operator relative to the following conjugation

$$\begin{bmatrix} 0 & C \\ C & 0 \end{bmatrix} \begin{matrix} \mathscr{H} \\ \mathscr{H} \end{matrix}.$$

Hence W_i is skew symmetric and $\sigma_0(W_i) \neq \emptyset$.

Since $\lambda_i \in \sigma_0(W_i) \setminus \sigma(W_j)$ whenever $i \neq j$, there is a $r > 0$ such that $\overline{B(\lambda_i, r)} \cap \sigma(W_i) = \{\lambda_i\}$ and $\overline{B(\lambda_i, r)} \cap \sigma(W_j) = \emptyset$ whenever $i \neq j$. Noting that $\lambda_1, \ldots, \lambda_n$ are pairwise distinct, it can be required in addition that $\{B(\lambda_i, r)\}$ are pairwise disjoint. It follows from the upper semi-continuity of spectrum that there exists $\delta > 0$ such that

$$\overline{B(\lambda_i, r)} \cap \sigma(W_j + S) = \emptyset \tag{1}$$

whenever $i \neq j$ and $S \in \mathscr{B}(\mathscr{H})$ with $\|S\| < \delta$.

For $1 \leqslant i \leqslant n$, since $\lambda_i \in \mathrm{iso}\,\sigma(W_i)$, it follows by [16, Cor 1.6] that there exists $\delta_i > 0$ such that

$$\sigma(W_i + S) \cap B(\lambda_i, r) \neq \emptyset \tag{2}$$

for all $S \in \mathscr{B}(\mathscr{H})$ with $\|S\| < \delta_i$.

For $1 \leqslant i \leqslant n$, by [4, Thm. 1.3], there exists a compact operator $K_{2,i}$ with $\|K_{2,i}\| < \min\{\frac{\varepsilon}{4}, \delta, \delta_i\}$ such that $W_i + K_{2,i}$ is an irreducible skew symmetric operator. In view of (1) and (2), $\{\sigma(W_i + K_{2,i})\}_{i=1}^n$ are pairwise distinct. Set $K_i = K_{1,i} + K_{2,i}$. Then K_i is compact with $\|K_i\| < \varepsilon$ and $\{W + K_i\}_{i=1}^n$ are irreducible skew symmetric operators with pairwise distinct spectra. □

From the proof of Theorem 2.6, one can get the following result.

COROLLARY 2.9. *Given $T \in SSO$ and $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that $T + K \in SSO$ and $\sigma_0(T + K) \neq \emptyset$.*

Next we shall prove the following result, which implies Theorem 1.1(i).

THEOREM 2.10. *Given $A \in SSO$ and $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that $A + K \in SSO \setminus \overline{SSO_0}$.*

Proof. By Corollary 2.9, given $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that $A + K \in SSO$ and $\sigma_0(A + K) \neq \emptyset$. So it remains to prove that $T := A + K \notin \overline{SSO_0}$.

For $\lambda \in \sigma_0(T)$, there is a $r > 0$ such that $\overline{B(\lambda, r)} \cap \sigma_0(T) = \{\lambda\}$. Define an analytic function f over a neighborhood of $\sigma(T)$ by setting

$$f(z) = \begin{cases} 1, & |z - \lambda| < r, \\ 0, & |z - \lambda| > r. \end{cases}$$

Then $f(T) = E(\{\lambda\};T)$. Since $\lambda \in \sigma_0(T)$ and $\dim \operatorname{ran} E(\{\lambda\},T) < \infty$, one can get that $f(T)$ is a nonzero, finite-rank and idempotent operator.

It follows from [16, Prop. 1.7] that there exists $\delta > 0$ such that

$$\|f(T) - f(S)\| < \frac{1}{2(\|f(T)\| + 2)}$$

for all $S \in \mathscr{B}(\mathscr{H})$ satisfying $\|T - S\| < \delta$.

Claim. $f(S)$ is a nonzero, finite-rank and idempotent operator for $S \in \mathscr{B}(\mathscr{H})$ with $\|T - S\| < \delta$.

Suppose $S \in \mathscr{B}(\mathscr{H})$ with $\|T - S\| < \delta$. Set $A = f(S)$ and $P = f(T)$. Obviously, A and P are both idempotent. Set $W = PA + (I - P)(I - A)$. Then

$$W = I - (P - PA) + (PA - A) = I - P(P - A) - (P - A)A.$$

We can deduce that

$$\begin{aligned} \|P(P-A)\| + \|(P-A)A\| &\leqslant \frac{\|P\|}{2(\|P\|+2)} + \frac{\|A\|}{2(\|P\|+2)} \\ &\leqslant \frac{\|P\|}{2(\|P\|+2)} + \frac{\|P\| + \|P-A\|}{2(\|P\|+2)} \\ &\leqslant \frac{\|P\|}{2(\|P\|+2)} + \frac{\|P\|+1}{2(\|P\|+2)} < 1. \end{aligned}$$

Thus W is invertible and $PW = PA = WA$. Hence A is similar to P. This proves the claim.

Let $S \in \mathscr{B}(\mathscr{H})$ with $\|T - S\| < \delta$. By the above claim, $f(S)$ is a nonzero compact operator and belongs to the unital C^*-algebra $C^*(S)$ generated by S. Thus, by [10, Lem. 2.5], S is not completely reducible. This completes the proof. □

2.3. Irreducible summands

In this subsection we give the following result, which implies Theorem 1.1(ii).

THEOREM 2.11. *Given $T \in SSO$, $1 \leqslant n \leqslant \aleph_0$ and $\varepsilon > 0$, there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that $A + K \in SSO_n$.*

The proof of the following results can be referred to [15, Prop. 2.3] and [13, Prop. 7.4].

LEMMA 2.12. *Let $T \in \mathscr{B}(\mathscr{H})$.*

(i) If $T = \oplus_{i=1}^{n} T_i$ where $1 \leqslant n \leqslant \aleph_0$ and T_i's are irreducible with distinct spectra, then $\{\mathscr{H}_i : 1 \leqslant i \leqslant n\}$ are all minimal reducing subspaces of T.

(ii) $T = A \oplus B$, where A, B are irreducible and $A \cong B$, then T has uncountably many minimal reducing subspaces.

Proof of Theorem 2.11. It is sufficient to give the proof in the case that $n = \aleph_0$. The proof is a minor modification of that for $n \in \mathbb{N}$.

By Lemma 2.3, there exists $R \in SSO$ with $R \cong_a R \oplus R$ such that $T \cong_a T \oplus R$. By [7, Cor. II.5.5], we have $R \cong_a \oplus_{i=1}^{\infty} R$. So $T \cong_a T \oplus (\oplus_{i=1}^{\infty} R)$ Hence, by Lemma 2.2, there exists $K_0 \in \mathscr{K}(\mathscr{H})$ with $\|K_0\| < \frac{\varepsilon}{2}$ such that

$$T + K_0 \cong \bigoplus_{i=1}^{\infty} T_i$$

where $T_i \in \mathscr{B}(\mathscr{H}_i)$ is skew symmetric and $\dim \mathscr{H}_i = \infty, i = 1, 2, \ldots$.

For each $i \geqslant 1$, it follows from Lemma 2.6 that there exist $\{K_{i,j} : 1 \leqslant j \leqslant i\} \subset \mathscr{K}(\mathscr{H}_i)$ with $\|K_{i,j}\| < \frac{\varepsilon}{2^{i+1}}$ such that $\{T_i + K_{i,j} : 1 \leqslant j \leqslant i\}$ are irreducible skew symmetric operators with pairwise distinct spectra.

Denote $K_1 = K_{1,1}$ and $A_1 = T_1 + K_1$.

Since $\sigma(T_2 + K_{2,1}) \neq \sigma(T_2 + K_{2,2})$, we obtain that either $\sigma(T_2 + K_{2,1}) \neq \sigma(A_1)$ or $\sigma(T_2 + K_{2,2}) \neq \sigma(A_1)$. It may be assumed without loss of generality that the former holds. In this case, set $K_2 = K_{2,1}$ and $A_2 = T_2 + K_2$.

Noting that $\{\sigma(T_3 + K_{3,j}) : 1 \leqslant j \leqslant 3\}$ are pairwise distinct, then there exists j such that $\sigma(A_1), \sigma(A_2), \sigma(T_3 + K_{3,j})$ are pairwise distinct. Set $K_3 = K_{3,j}$ and $A_3 = T_3 + K_3$.

Using this method, we can find there exists $\{K_i : i \geqslant 1\} \subset \mathscr{K}(\mathscr{H})$ with $\|K_i\| < \frac{\varepsilon}{2^{i+1}}$ such that $A_i := T_i + K_i$ is an irreducible skew symmetric operator; moreover, $\{\sigma(A_i) : i \geqslant 1\}$ are pairwise distinct. Thus $A_i \ncong A_j$ whenever $i \neq j$.

Set

$$K = K_0 + \left(\bigoplus_{i=1}^{\infty} K_i\right).$$

Then $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| \leqslant \|K_0\| + \max_{i \geqslant 1} \|K_i\| < \varepsilon$ and

$$T + K \cong \bigoplus_{i=1}^{\infty} A_i,$$

where $\{A_i : i \geqslant 1\}$ are irreducible skew symmetric operators and $A_i \ncong A_j$ whenever $i \neq j$. Thus, by Lemma 2.12, $T + K \in SSO_{\aleph_0}$. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. (i) The result is a direct consequence of Theorem 2.10.

(ii) The result is a direct consequence of Theorem 2.11.

(iii) Let $T \in SSO$, by Lemma 2.3, there exists $R \in SSO$ such that $T \cong_a T \oplus R \oplus R$. For any $\varepsilon > 0$, it follows from Theorem 2.6 that there exists $F \in \mathscr{B}(\mathscr{H})$ with $\|F\| < \frac{\varepsilon}{2}$ such that $A := R + F$ is an irreducible skew symmetric operator. Thus, by Lemma 2.2, there is $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that

$$T + K \cong T \oplus A \oplus A.$$

By Lemma 2.12, it is easy to see that $T + K \in SSO$ has uncountably many minimal reducing subspaces. Therefore $T \in \overline{SSO_\infty}$. □

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