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EIGENVALUES OF INDEFINITE q–STURM–LIOUVILLE PROBLEM WITH q–COUPLED BOUNDARY CONDITION

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Abstract. The present paper deals with non-real eigenvalues of indefinite q-Sturm-Liouville problems with coupled boundary condition

$$\begin{cases} -\dfrac{1}{q}D_{q^{-1}}D_q y(x) + v(x)y(x) = \lambda w(x)y(x), \\ \begin{pmatrix} y(1) \\ D_{q^{-1}}y(1) \end{pmatrix} = K \begin{pmatrix} y(0) \\ D_{q^{-1}}y(0) \end{pmatrix}. \end{cases}$$

The upper bounds on the imaginary and real parts of non-real eigenvalues for this indefinite q-Sturm-Liouville problem are obtained in terms of the coefficients v, w and the q-coupled boundary conditions. This is a challenging open problem according to the regular indefinite Sturm-Liouville problems and there has been little research on this problems so far. A priori bounds on the non-real eigenvalues in this paper can of course be combined with other estimates of the indefinite q-Sturm-Liouville problems under the assumption in this paper and the methods partly inspired by the estimates for nonlocal regular indefinite Sturm-Liouville problems with nonlocal coupled boundary conditions.

1. Introduction

It's well known that q-difference equations arise q-analogues of differential equations and this subject has developed into a multidisciplinary subject. The q-difference equations and their related problems appears in several physical models involving q-derivatives, q-integrals, q-exponential function, q-trigonometric function, q-Taylor formula, q-Beta(Gamma) functions (see [4, 7, 8, 9, 11, 12]). This paper is to study a basic analogue of Sturm-Liouville systems when the differential operator is replaced by the q-difference operator D_q, where

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \ \ x \in [0,1]/\{0\}, \ \ q \in (\mathbb{R}, (0,1)).$$

And the q-derivative at zero is defined by

$$D_q f(0) := \lim_{n \to \infty} \frac{f(xq^n) - f(0)}{xq^n}, \ \ x \in (0,1),$$

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if the limit exists and does not depend on x. The basic q-Sturm-Liouville system and classic Sturm-Liouville problems are defined in [9, Chapter 5] [10] and [20, Chapter 4] respectively, and hence the indefinite q-Sturm-Liouville problem with q-coupled boundary conditions is defined as

$$\begin{cases} -\dfrac{1}{q}D_{q^{-1}}D_q y(x) + v(x)y(x) = \lambda w(x)y(x), \\ \begin{pmatrix} y(1) \\ D_{q^{-1}}y(1) \end{pmatrix} = K \begin{pmatrix} y(0) \\ D_{q^{-1}}y(0) \end{pmatrix}, \end{cases} \tag{1}$$

where the functions q, v, w are real-valued which posses appropriate q-derivatives, $v \in L_q^1(0,1)$ and $w \in L_q^2(0,1)$ changes sign on $[0,1]$, which means that

$$\mathrm{mes}\{x \in [0,1] : w(x) > 0\} > 0, \quad \mathrm{mes}\{x \in [0,1] : w(x) < 0\} > 0$$

and $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$, $k_{ij} \in \mathbb{R}$, $i, j = 1,2$ and $\det K = 1$. A complex number λ is called an eigenvalue of boundary value problem (1) if there is a nontrivial solution y satisfying the boundary conditions and in this case the corresponding solution y is called an eigenfunction of λ. Since the sign change of the weight function w, the indefinite q-Sturm-Liouville eigenvalues problem (1) is not self-adjoint in a Hilbert space but it can be interpreted as self-adjoint with indefinite inner product. Jackson in [14] introduced an integral denoted by $\int_a^b f(t)\mathrm{d}_q t$ as a right inverse of the q-derivative. It is defined by

$$\int_a^b f(t)\mathrm{d}_q t := \int_0^b f(t)\mathrm{d}_q t - \int_0^a f(t)\mathrm{d}_q t,$$

where

$$\int_0^x f(t)\mathrm{d}_q t := (1-q)\sum_{n=0}^{\infty} xq^n f(xq^n) \tag{2}$$

provided that the series at the right-hand side of (2) converges at $x = a$ and b. The rule of q-integration by parts is

$$\int_0^1 f(x)D_q g(x)\mathrm{d}_q x = f(x)g(x) - \lim_{n\to\infty} f(q^n)g(q^n) - \int_0^1 g(qx)D_q f(x)\mathrm{d}_q x,$$

and the non-symmetric q-product rule is

$$D_q(fg)(x) = D_q f(x)g(x) + f(qx)D_q g(x).$$

For $w \geqslant 0$ in (1), this basic q-Sturm-Liouville eigenvalue problem have been proved that all eigenvalues of this system are real in in [9, pp. 164–170] and the eigenfunctions satisfy an orthogonality relation in [9, Eq. (5.1.5)]. Annaby and Mansour in [3] investigated a self-adjoint q-Sturm-Liouville operator in a Hilbert space and discussed the properties of eigenvalues and associated eigenfunctions. The singular q-Sturm-Liouville problem have been studied in [1, 2, 5] including the q-Titchmarsh-Weyl theory, q-limit-point and q-limit-circle singularities, spectral problems of non-self-adjoint singular q-Sturm-Liouville problem with an eigenparameter in the boundary condition. Since w changes sign, i.e., the q-Sturm-Liouville problem is indefinite, and the indefinite nature is that non-real spectral points may appear. The classic indefinite Sturm-Liouville equation $-y''(x)+v(x)y(x)=\lambda w(x)y(x)$ with suitable boundary condition were carried out by Haupt [13] and Richardson [17] which pointed out the non-real eigenvalues may exist. Determining a priori bounds of these non-real eigenvalues in terms of the coefficients and the boundary conditions is an interesting and difficult problems in Sturm-Liouville theory. Recently, these open problems have been solved by Qi et al. in [6, 15, 16, 19]. However, as far as we know, there is no study of the indefinite q-Sturm-Liouville problem (1) as we do in the present setting. Only in the very recent past first results in this direction of q-Sturm-Liouville problem with q-Dirichlet boundary conditions were obtained in [18]. In this paper we investigate the q-Sturm-Liouville equation with q-coupled self-adjoint boundary conditions and we prove explicit bounds on the real and imaginary parts of these eigenvalues only restriction on the coefficients of the differential expression. The techniques in the proof of our main results are inspired by the methods in the estimate of non-real eigenvalues for regular indefinite Sturm-Liouville problems with arbitrary self-adjoint boundary conditions in [6, 16]. The present paper will focus on the indefinite q-Sturm-Liouville eigenvalue problem (1), a priori bounds on real and imaginary parts of non-real eigenvalues for this problem are obtained in Theorem 1 of the following Section 2.

2. Preliminary knowledge and bounds on non-real eigenvalues

In this section we provide a priori bounds on the non-real eigenvalues of (1) in Theorem 1 (see the below). The following constants will be incorporated into these bounds. Setting

$$\begin{aligned} &\widehat{\mathscr{K}} := \max\begin{cases} \dfrac{|k_{22}|+|k_{11}|+2}{|k_{12}|}, & k_{12}\neq 0,\\ |k_{11}||k_{21}|, & k_{12}=0, \end{cases} \\ &\mathscr{K}_{k,v} := \widehat{\mathscr{K}} + \|v_-\|_1, \quad \mathscr{K} := \sqrt{\mathscr{K}_{k,v}(1+\mathscr{K}_{k,v})}+\mathscr{K}_{k,v}, \end{aligned} \tag{3}$$

and $v_-(x)=\min\{0,v(x)\}$, $x\in(0,1)$. Note that the constants $\widehat{\mathscr{K}}$, $\mathscr{K}_{k,v}$ and $\mathscr{K}$ do not depend on the weight function w. The norm of $L^2_q(0,1)$ will be denoted by $\|\cdot\|_2$. As usual the L^1-norm and L^∞-norm will be denoted by $\|\cdot\|_1$ and $\|\cdot\|_\infty$, respectively. The following lemmas are the estimates on the $\int_0^1|D_qf(x)|^2\mathrm{d}_qx$ and $\|f\|_\infty$ which play a key role in the proof of the eigenvalue estimates in this paper. Let λ be a non-real eigenvalue of (1) and f be a corresponding eigenfunction. It is no restriction to assume that $\int_0^1|f(x)|^2\mathrm{d}_qx=1$ in the following discussion.

LEMMA 1. *Let f be the eigenfunction of (1), then we have*

$$D_{q^{-1}}f(1)\overline{f(1)}-D_{q^{-1}}f(0)\overline{f(0)}\leqslant \widehat{\mathscr{K}}\max\{|f(1)|^2,|f(0)|^2\},$$

where $\widehat{\mathscr{K}}$ is defined in (3).

Proof. It follows from

$$\begin{pmatrix} f(1) \\ D_{q^{-1}}f(1) \end{pmatrix} = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \begin{pmatrix} f(0) \\ D_{q^{-1}}f(0) \end{pmatrix} \tag{4}$$

that

$$f(1) = k_{11}f(0) + k_{12}D_{q^{-1}}f(0).$$

From (4) and $\det K = 1$ one sees that

$$\begin{pmatrix} k_{22} & -k_{12} \\ -k_{21} & k_{11} \end{pmatrix} \begin{pmatrix} f(1) \\ D_{q^{-1}}f(1) \end{pmatrix} = \begin{pmatrix} f(0) \\ D_{q^{-1}}f(0) \end{pmatrix},$$

which

$$k_{22}f(1) - k_{12}D_{q^{-1}}f(1) = f(0).$$

And hence

$$\begin{aligned} &k_{12}D_{q^{-1}}f(1)\overline{f(1)} - k_{12}D_{q^{-1}}f(0)\overline{f(0)} \\ &= [k_{22}f(1) - f(0)]\overline{f(1)} - [f(1) - k_{11}f(0)]\overline{f(0)} \\ &= k_{22}|f(1)|^2 + k_{11}|f(0)|^2 - 2\operatorname{Re}[f(0)\overline{f(1)}]. \end{aligned}$$

If $k_{12} \neq 0$, then

$$\begin{aligned} &D_{q^{-1}}f(1)\overline{f(1)} - D_{q^{-1}}f(0)\overline{f(0)} \\ &= k_{12}^{-1}\{k_{22}|f(1)|^2 + k_{11}|f(0)|^2 - 2\operatorname{Re}[f(0)\overline{f(1)}]\} \\ &\leqslant |k_{12}^{-1}|\{|k_{22}||f(1)|^2 + |k_{11}||f(0)|^2 + 2|f(0)||\overline{f(1)}|\} \\ &\leqslant \frac{|k_{22}| + |k_{11}| + 2}{|k_{12}|}\max\{|f(1)|^2, |f(0)|^2\}. \end{aligned} \tag{5}$$

If $k_{12} = 0$, then from $f(1) = k_{11}f(0)$, $D_{q^{-1}}f(1) = k_{21}f(0) + k_{22}D_{q^{-1}}f(0)$, $k_{22}f(1) = f(0)$ and $K = k_{11}k_{22} = 1$, we obtain

$$\begin{aligned} &D_{q^{-1}}f(1)\overline{f(1)} - D_{q^{-1}}f(0)\overline{f(0)} \\ &= [k_{21}f(0) + k_{22}D_{q^{-1}}f(0)]\overline{f(1)} - k_{22}D_{q^{-1}}f(0)\overline{f(1)} \\ &= k_{21}f(0)\overline{f(1)} = k_{11}k_{21}f(0)\overline{f(0)} = k_{11}k_{21}|f(0)|^2 \\ &\leqslant |k_{11}||k_{21}|\max\{|f(1)|^2, |f(0)|^2\}. \end{aligned} \tag{6}$$

It follows from (5) and (6) that this Lemma holds immediately. $\square$

LEMMA 2. *Let λ and f be the non-real eigenvalue and corresponding eigenfunction defined as above, then the following estimates hold*

$$\int_0^1 |D_qf(x)|^2 d_qx \leqslant \mathscr{K}^2, \quad \|f\|_\infty^2 \leqslant 2\mathscr{K} + 1,$$

where $\mathscr{K}$ is defined in (3).

Proof. Multiplying both sides of (1) by $\overline{f}$ and integrating over the interval $[0,1]$

$$\begin{aligned}
\lambda\int_0^1 w(x)|f(x)|^2\mathrm{d}_qx &= -\int_0^1 \frac{1}{q}D_{q^{-1}}D_qf(x)\overline{f(x)}\mathrm{d}_qx+\int_0^1 v(x)f(x)\overline{f(x)}\mathrm{d}_qx\\
&= \Big\langle -\frac{1}{q}D_{q^{-1}}D_qf(x),f(x)\Big\rangle+\int_0^1 v(x)|f(x)|^2\mathrm{d}_qx\\
&= -D_qf(q^{-1})\overline{f(1)}+\lim_{n\to\infty}[D_qf(q^{n-1})\overline{f(q^n)}]+\langle D_qf(x),D_qf(x)\rangle+\int_0^1 v(x)|f(x)|^2\mathrm{d}_qx\\
&= -D_{q^{-1}}f(1)\overline{f(1)}+D_{q^{-1}}f(0)\overline{f(0)}+\int_0^1 |D_qf(x)|^2\mathrm{d}_qx+\int_0^1 v(x)|f(x)|^2\mathrm{d}_qx\\
&= -K^2D_{q^{-1}}f(0)\overline{f(0)}+D_{q^{-1}}f(0)\overline{f(0)}+\int_0^1 |D_qf(x)|^2\mathrm{d}_qx+\int_0^1 v(x)|f(x)|^2\mathrm{d}_qx\\
&= \int_0^1 |D_qf(x)|^2\mathrm{d}_qx+\int_0^1 v(x)|f(x)|^2\mathrm{d}_qx.
\end{aligned}$$

This together with $\operatorname{Im}\lambda\neq 0$ yields that $\int_0^1 w(x)|f(x)|^2\mathrm{d}_qx=0$, i.e.,

$$D_{q^{-1}}f(0)\overline{f(0)}-D_{q^{-1}}f(1)\overline{f(1)}+\int_0^1 |D_qf(x)|^2\mathrm{d}_qx+\int_0^1 v(x)|f(x)|^2\mathrm{d}_qx=0.$$

By Lemma 1 and hence we find

$$\begin{aligned}
\int_0^1 |D_qf(x)|^2\mathrm{d}_qx &= D_{q^{-1}}f(1)\overline{f(1)}-D_{q^{-1}}f(0)\overline{f(0)}-\int_0^1 v(x)|f(x)|^2\mathrm{d}_qx\\
&\leqslant D_{q^{-1}}f(1)\overline{f(1)}-D_{q^{-1}}f(0)\overline{f(0)}+\int_0^1 v_-(x)|f(x)|^2\mathrm{d}_qx\\
&\leqslant \widehat{\mathscr{K}}\max\{|f(1)|^2,|f(0)|^2\}+\int_0^1 v_-(x)|f(x)|^2\mathrm{d}_qx\\
&\leqslant \|f\|_\infty^2(\widehat{\mathscr{K}}+\|v_-\|_1).
\end{aligned}\tag{7}$$

For every $x,y\in[0,1]$, $x<y$, we obtain

$$\begin{aligned}
|f(y)|^2-|f(x)|^2 &= \int_x^y D_q(|f(t)|^2)\mathrm{d}_qt\\
&= \int_x^y \overline{f(t)}D_qf(t)\mathrm{d}_qt+\int_x^y f(qt)\overline{D_qf(t)}\mathrm{d}_qt\\
&\leqslant \left(\int_0^1 |f(t)|^2\mathrm{d}_qt\right)^{1/2}\left(\int_0^1 |D_qf(t)|^2\mathrm{d}_qt\right)^{1/2}\\
&\quad+\left(\int_0^1 |f(qt)|^2\mathrm{d}_qt\right)^{1/2}\left(\int_0^1 |\overline{D_qf(t)}|^2\mathrm{d}_qt\right)^{1/2}\\
&\leqslant 2\left(\int_0^1 |f(t)|^2\mathrm{d}_qt\right)^{1/2}\left(\int_0^1 |D_qf(t)|^2\mathrm{d}_qt\right)^{1/2}\\
&= 2\left(\int_0^1 |D_qf(t)|^2\mathrm{d}_qt\right)^{1/2}.
\end{aligned}$$

Integrating the above inequality over $[0,1]$ with respect to x gives

$$\int_0^1 |f(y)|^2 \mathrm{d}_q x - \int_0^1 |f(x)|^2 \mathrm{d}_q x \leqslant 2\left(\int_0^1 |D_q f(t)|^2 \mathrm{d}_q t\right)^{1/2} \int_0^1 \mathrm{d}_q x,$$

this implies that

$$\begin{aligned} |f(y)|^2 &\leqslant 2\left(\int_0^1 |D_q f(t)|^2 \mathrm{d}_q t\right)^{1/2} \int_0^1 \mathrm{d}_q x + \int_0^1 |f(x)|^2 \mathrm{d}_q x \\ &= 2\left(\int_0^1 |D_q f(t)|^2 \mathrm{d}_q t\right)^{1/2} + 1. \end{aligned}$$

Hence it follows that

$$\|f\|_\infty^2 \leqslant 2\left(\int_0^1 |D_q f(t)|^2 \mathrm{d}_q t\right)^{1/2} + 1. \tag{8}$$

Therefore we obtain from (7)

$$\int_0^1 |D_q f(x)|^2 \mathrm{d}_q x \leqslant \left[2\left(\int_0^1 |D_q f(t)|^2 \mathrm{d}_q t\right)^{1/2} + 1\right] \mathscr{K}_{k,v}.$$

This yields

$$\left[\left(\int_0^1 |D_q f(x)|^2 \mathrm{d}_q x\right)^{1/2} - \mathscr{K}_{k,v}\right]^2 \leqslant \mathscr{K}_{k,v}(1+\mathscr{K}_{k,v}),$$

and hence

$$\left(\int_0^1 |D_q f(x)|^2 \mathrm{d}_q x\right)^{1/2} \leqslant \sqrt{\mathscr{K}_{k,v}(1+\mathscr{K}_{k,v})} + \mathscr{K}_{k,v},$$

so that the first estimate in this Lemma is proved. The second estimate follows from the first estimate result and (8) implies

$$\|f\|_\infty^2 \leqslant 2\left[\sqrt{\mathscr{K}_{k,v}(1+\mathscr{K}_{k,v})} + \mathscr{K}_{k,v}\right] + 1,$$

which completes the proof of Lemma 2. □

Since w changes sign on $[0,1]$ and $w(x) \neq 0$ *a.e.* $x \in [0,1]$, we choose $\theta > 0$ so small such that

$$\mathscr{I} = \{x \in [0,1] : w^2(x) \leqslant \theta\}, \quad 0 < \mu(\theta) = \text{mes } \mathscr{I} \leqslant \frac{1}{2(2\mathscr{K}+1)}. \tag{9}$$

Then we can state the result of a priori bounds on the non-real eigenvalues for this indefinite q-Sturm-Liouville problem (1) as follows.

THEOREM 1. *Assume that* $w \in L_q^2(0,1)$. *If there exists* $\mathscr{W}_k > 0$ *such that* $|w(x)| \leqslant \mathscr{W}_k$ *a.e.on* $[0,1]$ *and* (9) *holds. Then for the non-real eigenvalue* λ *of indefinite* q*-Sturm-Liouville problem* (1)*, it holds that*

$$
\begin{aligned}
|\Im\lambda| &\leqslant \frac{2}{\theta}\sqrt{2\mathscr{K}+1}\left[\mathscr{W}_k\widehat{\mathscr{K}}\sqrt{2\mathscr{K}+1}+\mathscr{W}_q\mathscr{K}\right],\\
|\Re\lambda| &\leqslant \frac{2}{\theta}\left\{\mathscr{W}_k\left[(2\mathscr{K}+1)\left(\widehat{\mathscr{K}}+\|v\|_1\right)+\mathscr{K}^2\right]+\mathscr{W}_q\mathscr{K}\sqrt{2\mathscr{K}+1}\right\},
\end{aligned}
$$

where $\mathscr{W}_q = \left(\int_0^1 |D_q w(x)|^2 d_q x\right)^{1/2}$ *and* $\mathscr{K}$, $\widehat{\mathscr{K}}$ *are defined in* (3).

Proof. Multiplying both sides of (1) by $w(x)\overline{f(x)}$ and integrating over the interval $[0,1]$, then

$$
-\int_0^1 \frac{1}{q}D_{q^{-1}}D_q f(x)w(x)\overline{f(x)}\mathrm{d}_q x+\int_0^1 w(x)v(x)|f(x)|^2\mathrm{d}_q x=\lambda\int_0^1 w^2(x)|f(x)|^2\mathrm{d}_q x.
$$

Using q-integration by parts and non-symmetric q-product rule we obtain

$$
\begin{aligned}
\lambda\int_0^1 w^2(x)|f(x)|^2\mathrm{d}_q x &= w(0)D_{q^{-1}}f(0)\overline{f(0)}-w(1)D_{q^{-1}}f(1)\overline{f(1)}\\
&\quad+\int_0^1 w(qx)|D_q f(x)|^2\mathrm{d}_q x+\int_0^1 w(x)v(x)|f(x)|^2\mathrm{d}_q x\\
&\quad+\int_0^1 D_q w(x)D_q f(x)\overline{f(x)}\mathrm{d}_q x.
\end{aligned}
\tag{10}
$$

It follows from $|w(x)| \leqslant \mathscr{W}_k$, Lemma 1 and 2 that

$$
\begin{aligned}
& w(0)D_{q^{-1}}f(0)\overline{f(0)}-w(1)D_{q^{-1}}f(1)\overline{f(1)}\\
&\quad+\int_0^1 w(qx)|D_q f(x)|^2\mathrm{d}_q x+\int_0^1 w(x)v(x)|f(x)|^2\mathrm{d}_q x\\
&\leqslant \mathscr{W}_k\left[\left|D_{q^{-1}}f(0)\overline{f(0)}-D_{q^{-1}}f(1)\overline{f(1)}\right|+\int_0^1|D_q f(x)|^2\mathrm{d}_q x+\int_0^1 v(x)|f(x)|^2\mathrm{d}_q x\right]\\
&\leqslant \mathscr{W}_k\left[\widehat{\mathscr{K}}\max\{|f(1)|^2,|f(0)|^2\}+\int_0^1|D_q f(x)|^2\mathrm{d}_q x+\int_0^1 v(x)|f(x)|^2\mathrm{d}_q x\right]\\
&\leqslant \mathscr{W}_k\|f\|_\infty^2\left(\widehat{\mathscr{K}}+\|v\|_1\right)+\mathscr{W}_k\mathscr{K}^2\\
&\leqslant \mathscr{W}_k(2\mathscr{K}+1)\left(\widehat{\mathscr{K}}+\|v\|_1\right)+\mathscr{W}_k\mathscr{K}^2.
\end{aligned}
\tag{11}
$$

By $w \in L_q^2(0,1)$ and $\mathscr{W}_q = \left(\int_0^1 |D_q w(x)|^2 \mathrm{d}_q x\right)^{1/2}$ which yields

$$
\begin{aligned}
\int_0^1 D_q w(x)D_q f(x)\overline{f(x)}\mathrm{d}_q x &\leqslant \|f\|_\infty\left(\int_0^1|D_q w(x)|^2\mathrm{d}_q x\right)^{1/2}\left(\int_0^1|D_q f(x)|^2\mathrm{d}_q x\right)^{1/2}\\
&\leqslant \mathscr{W}_q\mathscr{K}\sqrt{2\mathscr{K}+1}.
\end{aligned}
\tag{12}
$$

The integral at the left-hand side of (10) satisfies

$$\begin{aligned}\int_0^1 w^2(x)|f(x)|^2 \mathrm{d}_q x &\geqslant \int_{[0,1]\setminus\mathscr{I}} w^2(x)|f(x)|^2 \mathrm{d}_q x \\ &\geqslant \theta\left(\int_0^1 |f(x)|^2 \mathrm{d}_q x - \int_{\mathscr{I}} |f(x)|^2 \mathrm{d}_q x\right) \\ &\geqslant \theta\left[1-\|f\|_\infty^2 \mu(\theta)\right] \\ &\geqslant \theta\left[1-(2\mathscr{K}+1)\,\mu(\theta)\right] \\ &\geqslant \theta/2.\end{aligned} \tag{13}$$

This fact together with (10)-(12) can lead

$$\begin{aligned}\frac{\theta}{2}|\Re\lambda| &\leqslant |\Re\lambda|\int_0^1 w^2(x)|f(x)|^2 \mathrm{d}_q x \\ &\leqslant \mathscr{W}_k\left[(2\mathscr{K}+1)\left(\widehat{\mathscr{K}}+\|v\|_1\right)+\mathscr{K}^2\right]+\mathscr{W}_q\mathscr{K}\sqrt{2\mathscr{K}+1}.\end{aligned}$$

Note that

$$\begin{aligned}\Im\lambda\int_0^1 w^2(x)|f(x)|^2 \mathrm{d}_q x = {}&\Im\left[w(0)D_{q^{-1}}f(0)\overline{f(0)}-w(1)D_{q^{-1}}f(1)\overline{f(1)}\right] \\ &+\Im\left[\int_0^1 D_q w(x) D_q f(x)\overline{f(x)}\mathrm{d}_q x\right]\end{aligned}$$

by (10). Hence, (11), (12), (13), Lemma 1 and 2 lead us to

$$\begin{aligned}\frac{\theta}{2}|\Im\lambda| &\leqslant |\Im\lambda|\int_0^1 w^2(x)|f(x)|^2 \mathrm{d}_q x \\ &\leqslant \mathscr{W}_k\widehat{\mathscr{K}}(2\mathscr{K}+1)+\mathscr{W}_q\mathscr{K}\sqrt{2\mathscr{K}+1}.\end{aligned}$$

Thus the inequation in Theorem 1 is established and the proof is complete. □

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