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MULTIPLE–TERM REFINEMENTS FOR UPPER BOUND OF THE HERMITE–HADAMARD INEQUALITY AND APPLICATIONS

TRAN DINH PHUNG, NGUYEN NGOC HUE* AND DUONG QUOC HUY

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Abstract. In this work, we propose various new upper bounds of the Hermite-Hadamard inequality for convex and radical convex functions via the linear and quadratic interpolations of convex functions. Some applications of the newly introduced results to operator inequalities, numerical radius and unitarily invariant norm inequalities are also given.

1. Introduction

The classical Hermite-Hadamard inequality [7, 9] states that for a convex function $f:[a,b]\to\mathbb{R}$

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(t)dt\leqslant\frac{f(a)+f(b)}{2}. \tag{1.1}$$

It is known that the right-hand side of the Hermite-Hadamard inequality can be refined as follows (see e.g. [8, 26])

$$\frac{1}{b-a}\int_a^b f(t)dt\leqslant\frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]. \tag{1.2}$$

It follows immediately that the residual (a distance between two sides) in the right inequality of (1.1) is larger than in the left one. It is important to stress that regardless of applications, we always wishful to obtain an inequality with smaller error. There have been recently many works on the refinement of the right side of (1.1) like in [8, 13, 14, 17, 18, 26].

Notice that, the inequality (1.2) is really easy to prove. Indeed, by applying the right-hand side of (1.1) to the intervals $[a,\frac{a+b}{2}]$, $[\frac{a+b}{2},b]$ and add the obtained inequalities one obtain the inequality (1.2). However, the inequality above can also follow from a nice geometrical proof based on the idea of the linear interpolation of a convex function. That is, we can show that

$$2r_0(\nu)\left(\frac{f(a)+f(b)}{2}-f\left(\frac{a+b}{2}\right)\right)\leqslant(1-\nu)f(a)+\nu f(b)-f((1-\nu)a+\nu b) \tag{1.3}$$

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* Corresponding author.

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for all $\nu \in [0,1]$, where $r_0(\nu) = \min\{\nu, 1-\nu\}$. Integrating the above inequality from 0 to 1, we obtain the inequality (1.2).

The inequality (1.3) is the refinement of Jensen's inequality proposed by Dragomir [5] in 2006 (see also [15]). In 2017, Choi, Krnić, Pečarić [4] developed this idea via defining recursively the functions $r_n(\nu)$ as

$$r_0(\nu) = \min\{\nu, 1-\nu\}, \quad r_n(\nu) = \min\{2r_{n-1}(\nu), 1-2r_{n-1}(\nu)\}, \quad \nu \in [0,1]$$

and using them to establish a refinement of Jensen's inequality as follows: If N is a nonnegative integer and f is a convex function defined on $[0,1]$, then

$$f(\nu) + \sum_{n=0}^{N-1} r_n(\nu) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \leqslant (1-\nu)f(0) + \nu f(1) \tag{1.4}$$

here

$$\Delta_f(n,k) = f\left(\frac{k-1}{2^n}\right) + f\left(\frac{k}{2^n}\right) - 2f\left(\frac{2k-1}{2^{n+1}}\right)$$

and χ_I is the characteristic function of the interval I given by $\chi_I(\nu) = 1$ if $\nu \in I$ and $\chi_I(\nu) = 0$ otherwise. The inequality (1.4) is a further refinement of (1.3) in the case $a = 0$, $b = 1$. Recently, Huy, Quang, Van [10] proposed a new refinement of (1.4) by utilizing the following quadratic interpolation

$$\psi_N(\nu) = (1-\nu)f(0) + \nu f(1) - \sum_{n=0}^{N-1} \tau_n(\nu) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu),$$

where τ_n are quadratic functions via the functions r_n.

In [24], Sababheh and Moradi proved that radical convex functions have a better estimate in the right-hand side of Hermite-Hadamard inequality as follows: If f is a 2-radical convex function and $b > a > 0$ then

$$\frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_0^{\frac{b-a}{2}} \frac{4tf(t)}{\sqrt{(b-a)^2-4t^2}}dt \leqslant \frac{f(a)+f(b)}{2}.$$

The main objective of this article is to provide a class of refinements for upper bound of the Hermite-Hadamard inequality (1.2) for convex and radical convex functions. For this purpose, we will need the interpolation results of convex functions was recently introduced in [4, 10]. Our method yields a considerably closer upper bound for the integral mean value of convex function than the classical one. The residuals of obtained inequalities are strictly smaller than in the classical Hermite-Hadamard inequality.

The notion of convex function plays an increasingly important role in the development of operator and norm inequalities. In [6], Dragomir established an operator version of (1.1) by using a simple and smart convexity argument. Sababheh and Moradi [20, 23, 24] established various general inequalities governing numerical radius inequalities for operators and matrices using convex functions. In several works [3, 12, 19, 22, 25], based on convexity of certain functions, the authors provided several improvements and refinements of norm and numerical radius inequalities of bounded linear operators on a complex Hilbert space. Inspired by these works, in this paper we also study some operator norm inequalities and numerical radius inequalities for matrices. We use some newly obtained results for convex functions to improve on existing numerical radius inequalities as well as norm inequalities. We would like to mention that our approach is capable of producing uniform refinements for various norm operator inequalities.

2. Linear and quadratic interpolations for convex functions

Throughout this paper, for $n=0,1,2,\ldots$, we denote by r_n the multipart functions defined by

$$r_n(\nu)=\begin{cases}2^n\nu-k+1, & \frac{k-1}{2^n}\leqslant\nu\leqslant\frac{2k-1}{2^{n+1}}\\ k-2^n\nu, & \frac{2k-1}{2^{n+1}}<\nu\leqslant\frac{k}{2^n}\end{cases}.$$

For simplicity of notation, we will write

$$\begin{aligned}\Delta_f^{a,b}(n,k)=&f\left(\frac{2^n-k+1}{2^n}a+\frac{k-1}{2^n}b\right)+f\left(\frac{2^n-k}{2^n}a+\frac{k}{2^n}b\right)\\ &-2f\left(\frac{2^{n+1}-2k+1}{2^{n+1}}a+\frac{2k-1}{2^{n+1}}b\right)\end{aligned}$$

for any function f defined on $[a,b]$ and $1\leqslant k\leqslant 2^n$. We also write $\Delta_f(n,k)$ instead of $\Delta_f^{0,1}(n,k)$.

In this section, we recall some basic facts on linear and quadratic interpolations which are useful in improving Jensen inequality for convex and radical convex functions. We recall first the definition of the linear interpolation.

LEMMA 2.1. [4] *Let f be a function defined on $[0,1]$. For a nonnegative integer N, define $\varphi_N(\nu)$ by*

$$\varphi_N(\nu)=(1-\nu)f(0)+\nu f(1)-\sum_{n=0}^{N-1}r_n(\nu)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)$$

and the summation is assumed to be zero if $N=0$. Then, $\varphi_N(\nu)$ is the linear interpolation of $f(\nu)$ at $\nu=k/2^N$, $k=0,1,\ldots,2^N$.

Notice that the functions r_n are continuous and linear on intervals $\left(\frac{k-1}{2^{n+1}},\frac{k}{2^{n+1}}\right)$ for $1\leqslant k\leqslant 2^{n+1}$. The linear interpolations $\varphi_N(\nu)$ provides a refinement of the Jensen inequality for a convex function defined on the interval $[0,1]$. In fact, we have

THEOREM 2.2. [4] *Let N be a nonnegative integer. If f is convex on $[0,1]$, then*

$$f(\nu)+\sum_{n=0}^{N-1}r_n(\nu)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\leqslant(1-\nu)f(0)+\nu f(1). \tag{2.1}$$

In [10], Huy, Quang, Van defined the quadratic functions τ_n via the functions r_n and get a quadratic interpolation in the following.

LEMMA 2.3. [10] *Let f be a function defined on $[0,1]$. For a nonnegative integer N, define $\psi_N(\nu)$ by*

$$\psi_N(\nu) = (1-\nu)f(0) + \nu f(1) - \sum_{n=0}^{N-1} \tau_n(\nu) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)$$

here the quadratic functions τ_n are given by

$$\tau_n(\nu) = r_n(\nu) + \alpha\left(\frac{1}{2}r_{N-1}(\nu) - r_{N-1}^2(\nu)\right) \tag{2.2}$$

with the arbitrary constant $\alpha \neq 0$. Then, ψ_N is the quadratic interpolation of f at points $\nu = k/2^N$, $k = 0,1,\ldots,2^N$.

Applying this interpolation, the authors provided a refinement of Jensen type inequalities for twice differentiable convex functions.

THEOREM 2.4. [10] *Let N be a positive integer. If f is a twice differentiable convex function defined on $[0,1]$ satisfying that $0 < m < f'' < M < \infty$, then*

$$f(\nu) + \sum_{n=0}^{N-1} \tau_n(\nu) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \leqslant (1-\nu)f(0) + \nu f(1), \tag{2.3}$$

where the quadratic functions τ_n are defined as in (2.2) with $\alpha = \frac{6m}{(4^N-1)M}$.

Next, we briefly recall the definition and related basic facts on radical convex functions introduced by Sababheh and Moradi [24].

DEFINITION 2.5. [24] Let $f : [0,\infty) \to [0,\infty)$ be a continuous function with $f(0) = 0$, and let $p \geqslant 1$ be a fixed number. If the function $g(t) = f(t^{\frac{1}{p}})$ is convex on $[0,\infty)$, we say that f is p-radical convex.

REMARK 2.6. [24] Let f be p-radical convex, for some $p \geqslant 1$.

(1) f is increasing and convex.

(2) f is q-radical convex, for all $1 \leqslant q \leqslant p$.

(3) If g is q-radical convex for some $q \geqslant 1$, then $f+g$ is $\min\{p,q\}$-radical convex.

(4) If g is increasing convex, then the composite function $g \circ f$ is p-radical convex.

THEOREM 2.7. [24] *Let f be a 2-radical convex function and $a,b \geqslant 0$. If $0 \leqslant \nu \leqslant 1$, then*

$$f((1-\nu)a + \nu b) + f\left(\sqrt{\nu(1-\nu)}|a-b|\right) \leqslant (1-\nu)f(a) + \nu f(b).$$

THEOREM 2.8. [22] *Let f be a 2-radical convex function and $a,b \geqslant 0$. If $0 \leqslant \nu \leqslant 1$, then*

$$\begin{aligned} f((1-\nu)a+\nu b) + f\left(\sqrt{\nu(1-\nu)}|a-b|\right) \\ + 2\min\{1-\nu,\nu\}\left(\frac{f(a)+f(b)}{2} - f\left(\sqrt{\frac{a^2+b^2}{2}}\right)\right) \\ \leqslant (1-\nu)f(a)+\nu f(b). \end{aligned}$$

The following theorem gives a refinement of the Jensen inequality for radical convex functions.

THEOREM 2.9. *Let N,N_1 be two nonnegative integers. Let f be a p-radical convex function for $p \geqslant 2$ and $a,b \geqslant 0$. Then for any $0 \leqslant \nu \leqslant 1$, we have*

$$\begin{aligned} f((1-\nu)a+\nu b) &+ \sum_{n=0}^{N_1-1} f\left(\left(r_n(\nu)\sum_{k=1}^{2^n}\mathscr{P}^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)^{\frac{1}{p}}\right) \\ &+ \sum_{n=0}^{N-1} r_n(\nu)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^p,b^p}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \\ \leqslant f((1-\nu)a+\nu b) \\ &+ f\left(\left(\sum_{n=0}^{N_1-1} r_n(\nu)\sum_{k=1}^{2^n}\mathscr{P}^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)^{\frac{1}{p}}\right) \\ &+ \sum_{n=0}^{N-1} r_n(\nu)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^p,b^p}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \\ \leqslant f([(1-\nu)a^p+\nu b^p]^{\frac{1}{p}}) \\ &+ f\left(\left(\sum_{n=0}^{N_1-1} r_n(\nu)\sum_{k=1}^{2^n}\mathscr{P}^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)^{\frac{1}{p}}\right) \\ &+ \sum_{n=0}^{N-1} r_n(\nu)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^p,b^p}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \\ \leqslant (1-\nu)f(a)+\nu f(b), \end{aligned} \tag{2.4}$$

where $\mathscr{G}_f^{a^p,b^p}(n,k) = \Delta_g^{a^p,b^p}(n,k)$ with $g(t) = f(t^{\frac{1}{p}})$, namely

$$\begin{aligned} \mathscr{G}_f^{a^p,b^p}(n,k) =& f\left(\left(\frac{2^n-k+1}{2^n}a^p+\frac{k-1}{2^n}b^p\right)^{\frac{1}{p}}\right) + f\left(\left(\frac{2^n-k}{2^n}a^p+\frac{k}{2^n}b^p\right)^{\frac{1}{p}}\right) \\ &- 2f\left(\left(\frac{2^{n+1}-2k+1}{2^{n+1}}a^p+\frac{2k-1}{2^{n+1}}b^p\right)^{\frac{1}{p}}\right) \end{aligned} \tag{2.5}$$

and

$$\mathscr{P}^{a,b}(n,k) = \left(\frac{2^n-k+1}{2^n}a+\frac{k-1}{2^n}b\right)^p + \left(\frac{2^n-k}{2^n}a+\frac{k}{2^n}b\right)^p \\ -2\left(\frac{2^{n+1}-2k+1}{2^{n+1}}a+\frac{2k-1}{2^{n+1}}b\right)^p. \tag{2.6}$$

Proof. We consider the function $g(u) = f(u^{\frac{1}{p}})$ for $u \in [0,\infty)$ and denote

$$\varphi_N^{a^p,b^p}(\nu) = (1-\nu)g(a^p)+\nu g(b^p) - \sum_{n=0}^{N-1} r_n(\nu)\sum_{k=1}^{2^n}\Delta_g^{a^p,b^p}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)$$

for $\nu \in [0,1]$. Following the proof of [4, Lemma 2], we can show that

$$\varphi_N^{a^p,b^p}(\nu) = (k-2^N\nu)g\left(\frac{2^n-k+1}{2^n}a^p+\frac{k-1}{2^n}b^p\right) \\ +(2^N\nu-k+1)g\left(\frac{2^n-k}{2^n}a^p+\frac{k}{2^n}b^p\right)$$

for $\frac{k-1}{2^N} \leqslant \nu \leqslant \frac{k}{2^N}$ and $k = 1,2,\ldots,2^N$. This implies

$$\begin{aligned} &\varphi_N^{a^p,b^p}(\nu) \\ &\geqslant g\left((k-2^N\nu)\left(a^p+\frac{k-1}{2^n}(b^p-a^p)\right)+(2^N\nu-k+1)\left(a^p+\frac{k}{2^n}(b^p-a^p)\right)\right) \\ &= g((1-\nu)a^p+\nu b^p). \end{aligned}$$

Using the inequality (2.1) for convex function $((1-\nu)a+\nu b)^p$ on $\nu \in [0,1]$ we have

$$(1-\nu)a^b+\nu b^p \geqslant \sum_{n=0}^{N_1-1} r_n(\nu)\sum_{k=1}^{2^n}\mathscr{P}_{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)+((1-\nu)a+\nu b)^p.$$

Since g is an increasing function on $[0,\infty)$, we have

$$\begin{aligned} &g\left((1-\nu)a^b+\nu b^p\right) \\ &\geqslant g\left(\sum_{n=0}^{N_1-1} r_n(\nu)\sum_{k=1}^{2^n}\mathscr{P}_{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)+((1-\nu)a+\nu b)^p\right) \\ &\geqslant g\left(\sum_{n=0}^{N_1-1} r_n(\nu)\sum_{k=1}^{2^n}\mathscr{P}_{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)+g\left(((1-\nu)a+\nu b)^p\right) \\ &\geqslant \sum_{n=0}^{N_1-1} g\left(r_n(\nu)\sum_{k=1}^{2^n}\mathscr{P}_{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)+g\left(((1-\nu)a+\nu b)^p\right), \end{aligned} \tag{2.7}$$

where we have just used the super additive inequality of g (see [1, Problem II.5.12]).

This completes the proof. □

The following corollary provides a refinement of [21, Theorem 2.1].

COROLLARY 2.10. *Let N, N_1 be a nonnegative integer. Let f be a 2-radical convex function and $a, b \geqslant 0$. Then for any $0 \leqslant \nu \leqslant 1$, we have*

$$\begin{aligned} f((1-\nu)a+\nu b) &+ \sum_{n=0}^{N_1-1} f\left(\frac{1}{2^n}\sqrt{\frac{r_n(\nu)}{2}}|a-b|\right) \\ &+ \sum_{n=0}^{N-1} r_n(\nu) \sum_{k=1}^{2^n} \mathscr{G}_f^{a^2,b^2}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) \\ &\leqslant f((1-\nu)a+\nu b) + f\left(\sqrt{\nu(1-\nu)}|a-b|\right) \\ &+ \sum_{n=0}^{N-1} r_n(\nu) \sum_{k=1}^{2^n} \mathscr{G}_f^{a^2,b^2}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) \\ &\leqslant f([(1-\nu)a^2+\nu b^2]^{\frac{1}{2}}) + f\left(\sqrt{\nu(1-\nu)}|a-b|\right) \\ &+ \sum_{n=0}^{N-1} r_n(\nu) \sum_{k=1}^{2^n} \mathscr{G}_f^{a^2,b^2}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) \\ &\leqslant (1-\nu)f(a)+\nu f(b), \end{aligned} \tag{2.8}$$

where $\mathscr{G}_f^{a^2,b^2}(n,k)$ are defined as in (2.5) with $p=2$, that is

$$\begin{aligned} \mathscr{G}_f^{a^2,b^2}(n,k) =& f\left(\left(\frac{2^n-k+1}{2^n}a^2+\frac{k-1}{2^n}b^2\right)^{\frac{1}{2}}\right) + f\left(\left(\frac{2^n-k}{2^n}a^2+\frac{k}{2^n}b^2\right)^{\frac{1}{2}}\right) \\ &- 2f\left(\left(\frac{2^{n+1}-2k+1}{2^{n+1}}a^2+\frac{2k-1}{2^{n+1}}b^2\right)^{\frac{1}{2}}\right). \end{aligned} \tag{2.9}$$

Proof. When $p=2$, $\mathscr{P}^{a,b}(n,k)$ in (2.6) becomes

$$\mathscr{P}^{a,b}(n,k) = \frac{(b-a)^2}{2^{2n+1}}, \quad k=1,\dots,2^n.$$

Then for each fixed $\nu \in [0,1]$ and $p=2$,

$$\begin{aligned} &\lim_{N_1\to\infty}\left(\sum_{n=0}^{N_1-1} r_n(\nu) \sum_{k=1}^{2^n} \mathscr{P}^{a,b}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)\right) \\ &= (b-a)^2 \lim_{N_1\to\infty}\left(\sum_{n=0}^{N_1-1} \frac{r_n(\nu)}{2^{2n+1}} \sum_{k=1}^{2^n} \chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)\right) \\ &= (b-a)^2\nu(1-\nu). \end{aligned} \tag{2.10}$$

This together with the inequality (2.4) holds for all nonnegative integers N_1 implies (2.8). □

Applying the quadratic interpolation, we obtain the following refinement of Jensen type inequalities for twice differentiable radical convex functions.

THEOREM 2.11. *Let N, N_1 be two nonnegative integers and $p \geqslant 2$. Let f be a twice differentiable p-radical convex function satisfying that $0 < m \leqslant (f(t^{\frac{1}{p}}))'' \leqslant M < \infty$ for all $t > 0$ and $a, b \geqslant 0$. Then for any $0 \leqslant \nu \leqslant 1$, we have*

$$\begin{aligned} &f((1-\nu)a+\nu b)+f\left(\left(\sum_{n=0}^{N_1-1} r_n(\nu)\sum_{k=1}^{2^n}\mathscr{P}^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)^{\frac{1}{p}}\right) \\ &+\sum_{n=0}^{N-1}\tau_n(\nu)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^p,b^p}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \leqslant (1-\nu)f(a)+\nu f(b), \end{aligned} \tag{2.11}$$

where the quadratic functions τ_n are defined as in (2.2) with $\alpha = \frac{6m}{(4^N-1)M}$ and $\mathscr{G}_f^{a^p,b^p}(n,k)$, $\mathscr{P}^{a,b}(n,k)$ are defined in Theorem 2.9.

Proof. Let $g(u) = f(u^{\frac{1}{p}})$ for $u \in [0,\infty)$ and

$$\psi_N^{a^p,b^p}(\nu) = (1-\nu)g(a^p)+\nu g(b^p)-\sum_{n=0}^{N-1}\tau_n(\nu)\sum_{k=1}^{2^n}\Delta_g^{a^p,b^p}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)$$

for $\nu \in [0,1]$. We will show $\psi_N^{a^p,b^p}(\nu) \geqslant g((1-\nu)a^p+\nu b^p)$ for all $\nu \in [0,1]$. To see this, let us consider the function

$$h(\nu) := \psi_N^{a^p,b^p}(\nu) - g\left((1-\nu)a^p+\nu b^p\right), \quad \nu \in [0,1].$$

It is easy to see that $h\left(\frac{k-1}{2^N}\right) = h\left(\frac{k}{2^N}\right) = 0$ for $1 \leqslant k \leqslant 2^N$. Hence, it suffices to prove that h is concave on intervals $\left(\frac{k-1}{2^N},\frac{k}{2^N}\right)$ for $1 \leqslant k \leqslant 2^N$. Indeed, applying Jensen's inequality to the convex function $M\frac{t^2}{2} - f(t)$, one has

$$0 \leqslant \Delta_f^{a^p,b^p}(n,k) \leqslant 2^{-2n-2}(b^p-a^p)^2 M.$$

On the other hand, it is easy to see that the quadratic functions $\tau_n(\nu)$ are twice differentiable on intervals $\left(\frac{k-1}{2^N},\frac{k}{2^N}\right)$. Therefore, we obtain, for all $\nu \in \left(\frac{k-1}{2^N},\frac{k}{2^N}\right)$,

$$\begin{aligned} h''(\nu) &= \sum_{n=0}^{N-1} 2\alpha\,(r'_{N-1}(\nu))^2\sum_{k=1}^{2^n}\Delta_g^{a^p,b^p}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \\ &\qquad -(b^p-a^p)^2 g''((1-\nu)a^p+\nu b^p) \\ &\leqslant 2\alpha\sum_{n=0}^{N-1}2^{2N-2}\sum_{k=1}^{2^n}\Delta_g^{a^p,b^p}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \\ &\qquad -(b^p-a^p)^2 g''((1-\nu)a^p+\nu b^p) \\ &\leqslant \frac{\alpha(b^p-a^p)^2M}{8}\sum_{n=0}^{N-1}2^{2(N-n)} - (b^p-a^p)^2 g''((1-\nu)a^p+\nu b^p) \\ &= \frac{\alpha(b^p-a^p)^2M}{6}(4^N-1) - (b^p-a^p)^2 g''((1-\nu)a^p+\nu b^p) \\ &\leqslant \frac{\alpha(b^p-a^p)^2M}{6}(4^N-1) - (b^p-a^p)^2 m = 0, \end{aligned}$$

which proves that h is concave on $\left(\frac{k-1}{2^N}, \frac{k}{2^N}\right)$ for $1 \leqslant k \leqslant 2^N$. This fact leads to $h(v) \geqslant 0$ on intervals $\left(\frac{k-1}{2^N}, \frac{k}{2^N}\right)$, namely,

$$\psi_N^{a^p,b^p}(v) \geqslant g((1-v)a^p + vb^p)$$

for all $v \in [0,1]$. Together with (2.7) we know that (2.11) holds. □

Combining Theorem 2.11 and (2.10), we get easily the following result.

COROLLARY 2.12. *Let N be a nonnegative integer. Let f be a twice differentiable 2-radical convex function satisfying that $0 < m \leqslant (f(\sqrt{t}))'' \leqslant M < \infty$ for all $t > 0$ and $a,b \geqslant 0$. Then for any $0 \leqslant v \leqslant 1$, we have*

$$f((1-v)a+vb) + f\left(\sqrt{v(1-v)}|a-b|\right) + \sum_{n=0}^{N-1} \tau_n(v) \sum_{k=1}^{2^n} \mathscr{G}_f^{a^2,b^2}(n,k)\chi_{\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right)}(v) \leqslant (1-v)f(a) + vf(b),$$

where the quadratic functions τ_n are defined as in (2.2) with $\alpha = \frac{6m}{(4^N-1)M}$ and $\mathscr{G}_f^{a^2,b^2}(n,k)$ are defined in Corollary 2.10.

3. Some multiple-term refinements for upper bound of Hermite-Hadamard inequality

In this section, we present two new classes of multiple-term refinements for upper bound of the Hermite-Hadamard inequality. Our method is to utilize linear and quadratic interpolations for convex and radical convex functions. Firstly, for convex functions we have

THEOREM 3.1. *Let N be a nonnegative integer. If f is a convex function defined on $[a,b]$ then*

$$\frac{1}{b-a}\int_a^b f(t)dt + \sum_{n=0}^{N-1} \frac{1}{2^{n+2}} \sum_{k=1}^{2^n} \Delta_f^{a,b}(n,k) \leqslant \frac{f(a)+f(b)}{2}. \tag{3.1}$$

Furthermore, if f is a twice differentiable convex function defined on $[a,b]$ satisfying that $0 < m \leqslant f'' \leqslant M < \infty$, then

$$\frac{1}{b-a}\int_a^b f(t)dt + \sum_{n=0}^{N-1} \frac{1}{2^{n+2}} \left(1 + \frac{m}{(4^N-1)M}\right) \sum_{k=1}^{2^n} \Delta_f^{a,b}(n,k) \leqslant \frac{f(a)+f(b)}{2}. \tag{3.2}$$

Proof. First, since $g(\nu)=f((1-\nu)a+\nu b)$ in convex on $[0,1]$, we see that the inequality (2.1) implies the inequality

$$\begin{aligned}&f((1-\nu)a+\nu b)+\sum_{n=0}^{N-1}r_n(\nu)\sum_{k=1}^{2^n}\Delta_f^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\\&\leqslant(1-\nu)f(a)+\nu f(b).\end{aligned}\tag{3.3}$$

Integrating the inequality (3.3) from 0 to 1, we obtain

$$\begin{aligned}&\int_0^1 f((1-\nu)a+\nu b)d\nu+\int_0^1\left(\sum_{n=0}^{N-1}r_n(\nu)\sum_{k=1}^{2^n}\Delta_f^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)d\nu\\&\leqslant\int_0^1((1-\nu)f(a)+\nu f(b))\,d\nu=\frac{f(a)+f(b)}{2}.\end{aligned}$$

In view of

$$\begin{aligned}&\int_0^1\left(\sum_{n=0}^{N-1}r_n(\nu)\sum_{k=1}^{2^n}\Delta_f^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)d\nu\\&=\sum_{n=0}^{N-1}\int_0^1 r_n(\nu)\sum_{k=1}^{2^n}\Delta_f^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)d\nu\\&=\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\left(\Delta_f^{a,b}(n,k)\int_0^1 r_n(\nu)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)d\nu\right)\\&=\sum_{n=0}^{N-1}\sum_{k=1}^{2^n}\left(\Delta_f^{a,b}(n,k)\left[\int_{\frac{k-1}{2^n}}^{\frac{2k-1}{2^{n+1}}}(2^n\nu-k+1)d\nu+\int_{\frac{2k-1}{2^{n+1}}}^{\frac{k}{2^n}}(k-2^n\nu)d\nu\right]\right)\\&=\sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_f^{a,b}(n,k)\end{aligned}$$

we get the inequality (3.1).

The second inequality will follow on the same arguments. Since

$$\begin{aligned}&\int_0^1\tau_n(\nu)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)d\nu\\&=\int_0^1\left(r_n(\nu)+\frac{6m}{(4^N-1)M}\left(\frac{1}{2}r_{N-1}(\nu)-r_{N-1}^2(\nu)\right)\right)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)d\nu\\&=\frac{1}{2^{n+2}}\left(1+\frac{m}{(4^N-1)M}\right)\end{aligned}$$

we have

$$\begin{aligned}&\int_0^1\left(\sum_{n=0}^{N-1}\tau_n(\nu)\sum_{k=1}^{2^n}\Delta_f^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)d\nu\\&=\sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\left(1+\frac{m}{(4^N-1)M}\right)\sum_{k=1}^{2^n}\Delta_f^{a,b}(n,k).\end{aligned}\tag{3.4}$$

Applying the inequality (2.3) for the twice differentiable convex function $f((1-v)a+vb)$ and using (3.4) we deduce the inequality (3.2). □

REMARK 3.2. From (3.1) we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)dt &\leqslant \frac{1}{b-a}\int_a^b f(t)dt+\sum_{n=1}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_f^{a,b}(n,k)\\ &\leqslant \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right]\\ &\leqslant \frac{f(a)+f(b)}{2},\end{aligned}$$

which offers a refinement of the well known Bullen's inequality [8] and [27, Theorem 2.1]. It is worthy to note that the above inequality leads to the following inequality

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\\ &\leqslant \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt-\sum_{n=1}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_f^{a,b}(n,k)\\ &\leqslant \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt.\end{aligned}$$

This means that the residual in our inequality is smaller in comparison with the upper bound in the inequality (1.1).

Secondly, for radical convex functions we have the following result.

THEOREM 3.3. *Let N, N_1 be two nonnegative integers, $p \geqslant 2$ and $0 \leqslant a < b$. If f is a p-radical convex function, then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(t)dt+\int_0^1 f\left(\left(\sum_{n=0}^{N_1-1} r_n(v)\sum_{k=1}^{2^n}\mathscr{P}^{a,b}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v)\right)^{\frac{1}{p}}\right)dv\\ &\qquad+\sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\mathscr{G}_f^{a^b,b^p}(n,k)\\ &\leqslant \frac{f(a)+f(b)}{2},\end{aligned} \tag{3.5}$$

where $\mathscr{P}^{a,b}(n,k)$ and $\mathscr{G}_f^{a^b,b^p}(n,k)$ are defined in Theorem 2.9.

If f is a twice differentiable p-radical convex function satisfying that $0 < m \leqslant$

$(f(t^{\frac{1}{p}}))'' \leqslant M < \infty$ *for all* $t > 0$, *then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)dt + \int_0^1 f\left(\left(\sum_{n=0}^{N_1-1} r_n(v)\sum_{k=1}^{2^n}\mathscr{P}^{a,b}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\right)^{\frac{1}{p}}\right)dv \\ &\qquad + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\left(1+\frac{m}{(4^N-1)M}\right)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^b,b^p}(n,k) \\ &\leqslant \frac{f(a)+f(b)}{2}. \end{aligned} \tag{3.6}$$

Proof. For the first inequality, integrating the inequality (2.4) over the interval $[0,1]$, calculating the integral

$$\int_0^1\left(\sum_{n=0}^{N-1} r_n(v)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^b,b^p}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\right)dv = \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\mathscr{G}_f^{a^b,b^p}(n,k),$$

imply the first desired inequality.

For the second desired inequality, integrating the inequality (2.11) over the interval $[0,1]$ implies

$$\begin{aligned} &\int_0^1 (f((1-v)a+vb))dv \\ &\qquad + \int_0^1 f\left(\left(\sum_{n=0}^{N_1-1} r_n(v)\sum_{k=1}^{2^n}\mathscr{P}^{a,b}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\right)^{\frac{1}{p}}\right)dv \\ &\qquad + \int_0^1\left(\sum_{n=0}^{N-1}\tau_n(v)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^b,b^p}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\right)dv \\ &\leqslant \int_0^1((1-v)f(a)+vf(b))dv. \end{aligned}$$

Noting that

$$\begin{aligned} &\int_0^1\left(\sum_{n=0}^{N-1} r_n(v)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^b,b^p}(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\right)dv \\ &= \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\left(1+\frac{m}{(4^N-1)M}\right)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^b,b^p}(n,k) \end{aligned}$$

implies the desired inequality (3.6). □

COROLLARY 3.4. *Let* N *be a nonnegative integer and* $0 \leqslant a < b$. *If* f *is a* 2*-radical convex function, then*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_0^{\frac{b-a}{2}}\frac{4tf(t)}{\sqrt{(b-a)^2-4t^2}}dt + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\mathscr{G}_f^{a^2,b^2}(n,k) \\ &\leqslant \frac{f(a)+f(b)}{2}, \end{aligned}$$

where $\mathscr{G}_f^{a^2,b^2}(n,k)$ *are defined in Corollary* 2.10.

If f *is a twice differentiable* 2*-radical convex function satisfying that* $0 < m \leqslant (f(\sqrt{t}))'' \leqslant M < \infty$ *for* $t > 0$*, then*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\int_0^{\frac{b-a}{2}} \frac{4tf(t)}{\sqrt{(b-a)^2-4t^2}}dt \\ + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\left(1+\frac{m}{(4^N-1)M}\right)\sum_{k=1}^{2^n}\mathscr{G}_f^{a^2,b^2}(n,k) \\ \leqslant \frac{f(a)+f(b)}{2}.\end{aligned}$$

Proof. For the first inequality, noting that the quantity $f\left(\sqrt{\nu(1-\nu)}|a-b|\right)$ is symmetric about $\frac{1}{2}$, calculating the integral

$$\begin{aligned}\int_0^1 f\left(\sqrt{\nu(1-\nu)}|a-b|\right)d\nu &= 2\int_0^{\frac{1}{2}} f\left(\sqrt{\nu(1-\nu)}|a-b|\right)d\nu \\ &= \frac{1}{b-a}\int_0^{\frac{b-a}{2}}\frac{4tf(t)}{\sqrt{(b-a)^2-4t^2}}dt.\end{aligned}$$

This together with (2.10) and the inequality (3.5) holding for all nonnegative integers N_1 imply the first desired inequality. The second desired inequality can be treated similarly. □

4. Applications to norm inequalities

In this section, we use established conclusions to improve some operator norm, numerical radius and unitarily invariant norm inequalities. From the following discussion, we can see that the previous inequalities provide convenient tools for investigating refinements of norm inequalities.

4.1. Refinements of operator norm inequalities

In this subsection, relying on obtained inequalities in the previous section, we refine and generalize some well known norm and numerical radius inequalities of bounded linear operators on a complex Hilbert space.

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle , \rangle$. A real valued continuous function f on an interval J is said to be operator convex if

$$f((1-\nu)A+\nu B) \leqslant (1-\nu)f(A)+\nu f(B)$$

in the operator order, for all $\nu \in [0,1]$, and for all self adjoint operators $A,B \in B(H)$ whose spectra are contained in J.

For $A \in B(H)$, let $w(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of A, respectively. Recall that

$$w(A) = \sup\{|\langle Ax,x\rangle| : x \in H, \|x\| = 1\}$$

and

$$\|A\| = \sup\{\|Ax\| : x \in H, \|x\| = 1\}.$$

One of the important properties of w is that it is weakly unitarily invariant, that is $w(U^*AU) = w(A)$, for every for every unitary $U \in B(H)$.

It is also readily seen that an operator convex function is also convex. Therefore, such functions comply with the Hermite-Hadamard inequality (1.1). Moreover, Dragomir [6] proved that the following modified operator version of (1.1)

$$f\left(\frac{A+B}{2}\right) \leqslant \int_0^1 f((1-v)A+vB)\,dv \leqslant \frac{f(A)+f(B)}{2} \tag{4.1}$$

holds for $f : J \to \mathbb{R}$ is an operator convex function and A, B are two self adjoint operators with spectra in J. The above operator inequality can be reduced to the following norm inequality

$$\left\|f\left(\frac{A+B}{2}\right)\right\| \leqslant \left\|\int_0^1 f((1-v)A+vB)\,dv\right\| \leqslant \left\|\frac{f(A)+f(B)}{2}\right\| \tag{4.2}$$

when f is nonnegative. In [23, Theorem 2.1], it has been shown that for $A \in B(H)$,

$$f(w(A)) \leqslant \left\|\int_0^1 f((1-v)|A|+v|A^*|)\,dv\right\| \leqslant \left\|\frac{f(|A|)+f(|A^*|)}{2}\right\|, \tag{4.3}$$

where $f : [0,\infty) \to [0,\infty)$ is an increasing operator convex function, A^* is the adjoint operator of A, $|A| = (A^*A)^{\frac{1}{2}}$ and $|A^*| = (AA^*)^{\frac{1}{2}}$.

In the following, we will target the operator inequalities (4.1), (4.2), (4.3). We first make a refinement of the inequality (4.1). For the sake of simplicity, in the following we denote

$$\begin{aligned}\Delta_f^{A,B}(n,k) =& f\left(\frac{2^n-k+1}{2^n}A+\frac{k-1}{2^n}B\right)+f\left(\frac{2^n-k}{2^n}A+\frac{k}{2^n}B\right)\\ &-2f\left(\frac{2^{n+1}-2k+1}{2^{n+1}}A+\frac{2k-1}{2^{n+1}}B\right)\end{aligned}$$

for $f : J \to \mathbb{R}$ is an operator convex function on the interval J, A and B are self adjoint operators with spectra in J, $n = 0,1,2,\ldots$ and $1 \leqslant k \leqslant 2^n$. Notice that if A is a self adjoint operator and f, g are continuous real valued functions on the spectrum $Sp(A)$ satisfying that $f(t) \geqslant g(t)$ for all $t \in Sp(A)$, we then have an operator inequality $f(A) \geqslant g(A)$. Since f is an operator convex function then we see that $\Delta_f^{A,B}(n,k) \geqslant 0$ in operator order for any $1 \leqslant k \leqslant 2^n$.

THEOREM 4.1. *Let N be a nonnegative integer. Let $f : J \to \mathbb{R}$ be an operator convex function on the interval J. Then for any self adjoint operators A and B with spectra in J we have the inequality*

$$\begin{aligned} &\int_0^1 f((1-\nu)A+\nu B)d\nu \\ &\leqslant \int_0^1 f((1-\nu)A+\nu B)d\nu + \sum_{n=0}^{N-1} \frac{1}{2^{n+2}} \sum_{k=1}^{2^n} \Delta_f^{A,B}(n,k) \\ &\leqslant \frac{f(A)+f(B)}{2}. \end{aligned} \tag{4.4}$$

Proof. Consider now $x \in H$, $\|x\| = 1$ and two self adjoint operators A and B with spectra in J. Define the real valued function $\gamma_{x,A,B} : [0,1] \to \mathbb{R}$ given by

$$\gamma_{x,A,B}(\nu) = \langle f((1-\nu)A+\nu B)x, x\rangle.$$

Since f is operator convex, then $\gamma_{x,A,B}$ is a convex function on $[0,1]$. By the inequality (3.1), we have

$$\begin{aligned} &\int_0^1 \langle f((1-\nu)A+\nu B)x,x\rangle d\nu + \sum_{n=0}^{N-1} \frac{1}{2^{n+2}} \sum_{k=1}^{2^n} \left\langle \Delta_f^{A,B}(n,k)x, x\right\rangle \\ &\leqslant \left\langle \frac{f(A)+f(B)}{2}x, x\right\rangle. \end{aligned} \tag{4.5}$$

The continuity of the function f implies that

$$\int_0^1 \langle f((1-\nu)A+\nu B)x,x\rangle d\nu = \left\langle \int_0^1 f((1-\nu)A+\nu B)d\nu x, x\right\rangle \tag{4.6}$$

for any $x \in H$, $\|x\| = 1$ and any two self adjoint operators A and B with spectra in J. Then, we deduce from (4.5) and (4.6) the desired inequality (4.4). □

The operator inequality (4.4) can be reduced to the following norm inequality.

COROLLARY 4.2. *Let the assumptions of Theorem 4.1 hold. If f is nonnegative, then*

$$\begin{aligned} \left\| \int_0^1 f((1-\nu)A+\nu B)d\nu \right\| &\leqslant \left\| \frac{f(A)+f(B)}{2} - \sum_{n=0}^{N-1} \frac{1}{2^{n+2}} \sum_{k=1}^{2^n} \Delta_f^{A,B}(n,k) \right\| \\ &\leqslant \left\| \frac{f(A)+f(B)}{2} \right\|. \end{aligned}$$

We now show the convex version of Corollary 4.2. Notice that in [16] Moradi and Sababheh proved that

THEOREM 4.3. [16, Theorem 2.1] *Let $A,B \in B(H)$. If $f:[0,\infty)\to[0,\infty)$ is a convex function, then*

$$\begin{aligned} f\left(\left\langle \frac{|A|+|B|}{2}x,x\right\rangle\right) &\leqslant \int_0^1 f\left(\left\|((1-\nu)|A|+\nu|B|)^{\frac{1}{2}}x\right\|^2\right)d\nu \\ &\leqslant \frac{1}{2}\left\|f(|A|)+f(|B|)\right\| \end{aligned} \tag{4.7}$$

for any unit vector $x \in H$.

Using the arguments of the proof of Theorem 4.3 and applying Theorem 3.1, we obtain the following refinements of the inequality (4.7).

THEOREM 4.4. *Let $f:[0,\infty)\to[0,\infty)$ is a convex function. Let N be a nonnegative integer and $A,B \in B(H)$. Then*

$$\begin{aligned} & f\left(\left\langle \frac{|A|+|B|}{2}x,x\right\rangle\right) \\ &\leqslant \int_0^1 f\left(\left\|((1-\nu)|A|+\nu|B|)^{\frac{1}{2}}x\right\|^2\right)d\nu \\ &\leqslant \int_0^1 f\left(\left\|((1-\nu)|A|+\nu|B|)^{\frac{1}{2}}x\right\|^2\right)d\nu + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_f^{\langle Ax,x\rangle,\langle Bx,x\rangle}(n,k) \\ &\leqslant \frac{1}{2}\left\|f(|A|)+f(|B|)\right\| \end{aligned}$$

for any unit vector $x \in H$. Furthermore, if $f:[0,\infty)\to[0,\infty)$ is a twice differentiable convex function satisfying that $0<m\leqslant f''\leqslant M<\infty$, then

$$\begin{aligned} & f\left(\left\langle \frac{|A|+|B|}{2}x,x\right\rangle\right) \\ &\leqslant \int_0^1 f\left(\left\|((1-\nu)|A|+\nu|B|)^{\frac{1}{2}}x\right\|^2\right)d\nu \\ &\leqslant \int_0^1 f\left(\left\|((1-\nu)|A|+\nu|B|)^{\frac{1}{2}}x\right\|^2\right)d\nu \\ &\quad + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\left(1+\frac{m}{(4^N-1)M}\right)\sum_{k=1}^{2^n}\Delta_f^{\langle Ax,x\rangle,\langle Bx,x\rangle}(n,k) \\ &\leqslant \frac{1}{2}\left\|f(|A|)+f(|B|)\right\| \end{aligned}$$

for any unit vector $x \in H$.

Our next result will be extending the inequality (4.3).

THEOREM 4.5. *Let $f:[0,\infty)\to[0,\infty)$ be an increasing operator convex function. Let N be a nonnegative integer and $A\in B(H)$. Then*

$$\begin{aligned} f(w(A)) &\leqslant \left\| \int_0^1 f\left((1-\nu)|A|+\nu|A^*|\right)d\nu \right\| \\ &\leqslant \left\| \frac{f(|A|)+f(|A^*|)}{2} - \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_f^{|A|,|A^*|}(n,k) \right\| \\ &\leqslant \frac{1}{2}\left\|f(|A|)+f(|A^*|)\right\|. \end{aligned} \tag{4.8}$$

Proof. Applying the arguments for the proof of [23, Theorem 2.1], in view of the inequality (2.1) for the convex function $\gamma_{x,|A|,|A^*|}(\nu)=\langle f((1-\nu)|A|+\nu|A^*|)x,x\rangle$ on $\nu\in[0,1]$ with $x\in H$, $\|x\|=1$, one can verify (4.8). Hence, we skip to present the details. $\square$

Our last result in this subsection, we employ Theorem 2.2 and Theorem 2.4 to improve the following inequality

$$\begin{aligned} & w^p(A) \\ &\leqslant \left\| (1-\nu)|A|^p+\nu|A^*|^p - 2r_0(\nu)\left(\frac{|A|^p+|A^*|^p}{2} - \left(\frac{|A|^2+|A^*|^2}{2}\right)^{\frac{p}{2}}\right) \right\| \\ &\leqslant \left\|(1-\nu)|A|^p+\nu|A^*|^p\right\| \end{aligned}$$

for $A\in B(H)$ and $2\leqslant p\leqslant 4$ which was shown in [16, Theorem 2.5].

THEOREM 4.6. *Let $A\in B(H)$ and $f:[0,\infty)\to[0,\infty)$ be an increasing operator convex function. Let $0\leqslant\nu\leqslant 1$ and N be a nonnegative integer. For each nonnegative integer n, we denote by*

$$\tau_n(\nu)=r_n(\nu)+\frac{6m}{(4^N-1)M}\left(\frac{1}{2}r_{N-1}(\nu)-r_{N-1}^2(\nu)\right),$$

where $m=\min\{\||A|^2\|,\||A^*|^2\|\}$ and $M=\max\{\||A|^2\|,\||A^*|^2\|\}$. Then, we have

$$\begin{aligned} & f\left(w^2(A)\right) \\ &\leqslant \left\| f\left((1-\nu)|A|^2+\nu|A^*|^2 - \sum_{n=0}^{N-1}\tau_n(\nu)\sum_{k=1}^{2^n}\mathscr{H}^{|A|^2,|A^*|^2}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)\right\| \\ &\leqslant \left\| f\left((1-\nu)|A|^2+\nu|A^*|^2\right)\right\| \\ &\leqslant \left\| (1-\nu)f(|A|^2)+\nu f(|A^*|^2) - \sum_{n=0}^{N-1}r_n(\nu)\sum_{k=1}^{2^n}\Delta_f^{|A|^2,|A^*|^2}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right\| \\ &\leqslant \left\|(1-\nu)f(|A|^2)+\nu f(|A^*|^2)\right\|, \end{aligned} \tag{4.9}$$

where

$$\mathscr{H}^{|A|^2,|A^*|^2}(n,k) = |A|^{1-\frac{k-1}{2^n}}|A^*|^{\frac{k-1}{2^n}} + |A|^{1-\frac{k}{2^n}}|A^*|^{\frac{k}{2^n}} - 2|A|^{1-\frac{2k-1}{2^{n+1}}}|A^*|^{\frac{2k-1}{2^{n+1}}}.$$

In particular,

$$\begin{aligned} & w^p(A) \\ & \leqslant \left\| \left((1-\nu)|A|^2 + \nu|A^*|^2 - \sum_{n=0}^{N-1} \tau_n(\nu) \sum_{k=1}^{2^n} \mathscr{H}^{|A|^2,|A^*|^2}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \right)^{\frac{p}{2}} \right\| \\ & \leqslant \left\| \left((1-\nu)|A|^2 + \nu|A^*|^2 \right)^{\frac{p}{2}} \right\| \\ & \leqslant \left\| (1-\nu)|A|^p + \nu|A^*|^p - \sum_{n=0}^{N-1} r_n(\nu) \sum_{k=1}^{2^n} \mathscr{K}^{|A|^2,|A^*|^2}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu) \right\| \\ & \leqslant \|(1-\nu)|A|^p + \nu|A^*|^p\| \end{aligned} \tag{4.10}$$

for $2 \leqslant p \leqslant 4$, *where*

$$\begin{aligned} \mathscr{K}^{|A|^2,|A^*|^2}(n,k) = & \left(\frac{2^n-k+1}{2^n}|A|^2 + \frac{k-1}{2^n}|A^*|^2 \right)^{\frac{p}{2}} \\ & + \left(\frac{2^n-k}{2^n}|A|^2 + \frac{k}{2^n}|A^*|^2 \right)^{\frac{p}{2}} \\ & - 2\left(\frac{2^{n+1}-2k+1}{2^{n+1}}|A|^2 + \frac{2k-1}{2^{n+1}}|A^*|^2 \right)^{\frac{p}{2}}. \end{aligned}$$

Proof. For $0 \leqslant \nu \leqslant 1$ and any unit vector $x \in H$, by the mixed Schwarz inequality [11] we have

$$f\left(|\langle Ax,x\rangle|^2\right) \leqslant f\left(\langle |A|^{2(1-\nu)}x,x\rangle\langle |A^*|^{2\nu}x,x\rangle\right) \leqslant f\left(\langle |A|^2x,x\rangle^{1-\nu}\langle |A^*|^2x,x\rangle^{\nu}\right).$$

We consider the twice differentiable and convex function

$$f_1(\nu) = \langle |A|^2x,x\rangle^{1-\nu}\langle |A^*|^2x,x\rangle^{\nu}$$

define on $\nu \in [0,1]$. The second derivative

$$f_1''(\nu) = \left(\ln\frac{\langle |A^*|^2x,x\rangle}{\langle |A|^2x,x\rangle}\right)^2 \langle |A|^2x,x\rangle^{1-\nu}\langle |A^*|^2x,x\rangle^{\nu}$$

of this function satisfy the inequalities

$$m\left(\ln\frac{\langle |A^*|^2x,x\rangle}{\langle |A|^2x,x\rangle}\right)^2 \leqslant f_1''(\nu) \leqslant M\left(\ln\frac{\langle |A^*|^2x,x\rangle}{\langle |A|^2x,x\rangle}\right)^2.$$

Applying the inequality (2.3) to the convex function $f_1(\nu)$, we get

$$\begin{aligned}
&\langle |A|^2x,x\rangle^{1-\nu}\langle |A^*|^2x,x\rangle^{\nu}\\
&\leqslant (1-\nu)\langle |A|^2x,x\rangle+\nu\langle |A^*|^2x,x\rangle\\
&\qquad -\sum_{n=0}^{N-1}\tau_n(\nu)\sum_{k=1}^{2^n}\mathscr{H}^{\langle |A|^2x,x\rangle,\langle |A^*|^2x,x\rangle}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\\
&\leqslant (1-\nu)\langle |A|^2x,x\rangle+\nu\langle |A^*|^2x,x\rangle,
\end{aligned} \tag{4.11}$$

where $\mathscr{H}^{\langle |A|^2x,x\rangle,\langle |A^*|^2x,x\rangle}(n,k)=\Delta_{f_1}(n,k)$, namely

$$\begin{aligned}
&\mathscr{H}^{\langle |A|^2x,x\rangle,\langle |A^*|^2x,x\rangle}(n,k)\\
&=\langle |A|^2x,x\rangle^{1-\frac{k-1}{2^n}}\langle |A^*|^2x,x\rangle^{\frac{k-1}{2^n}}+\langle |A|^2x,x\rangle^{1-\frac{k}{2^n}}\langle |A^*|^2x,x\rangle^{\frac{k}{2^n}}\\
&\quad -2\langle |A|^2x,x\rangle^{1-\frac{2k-1}{2^{n+1}}}\langle |A^*|^2x,x\rangle^{\frac{2k-1}{2^{n+1}}}.
\end{aligned}$$

Since f is an increasing operator convex function, we have

$$\begin{aligned}
&f\left(|\langle Ax,x\rangle|^2\right)\leqslant f\left(\langle |A|^2x,x\rangle^{1-\nu}\langle |A^*|^2x,x\rangle^{\nu}\right)\\
&\leqslant f\Bigg((1-\nu)\langle |A|^2x,x\rangle+\nu\langle |A^*|^2x,x\rangle\\
&\qquad -\sum_{n=0}^{N-1}\tau_n(\nu)\sum_{k=1}^{2^n}\mathscr{H}^{\langle |A|^2x,x\rangle,\langle |A^*|^2x,x\rangle}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\Bigg)\\
&\leqslant f\left((1-\nu)\langle |A|^2x,x\rangle+\nu\langle |A^*|^2x,x\rangle\right).
\end{aligned}$$

Taking the supremum over $x\in H$ with $\|x\|=1$, we infer that

$$\begin{aligned}
&f\left(w^2(A)\right)\\
&\leqslant \left\|f\left((1-\nu)|A|^2+\nu|A^*|^2-\sum_{n=0}^{N-1}\tau_n(\nu)\sum_{k=1}^{2^n}\mathscr{H}^{|A|^2,|A^*|^2}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right)\right\|\\
&\leqslant \left\|f\left((1-\nu)|A|^2+\nu|A^*|^2\right)\right\|.
\end{aligned} \tag{4.12}$$

On the other hand, the inequality (2.1) implies

$$\begin{aligned}
&f\left((1-\nu)\langle |A|^2x,x\rangle+\nu\langle |A^*|^2x,x\rangle\right)\\
&\leqslant (1-\nu)f\left(\langle |A|^2x,x\rangle\right)+\nu f\left(\langle |A^*|^2x,x\rangle\right)\\
&\qquad -\sum_{n=0}^{N-1}r_n(\nu)\sum_{k=1}^{2^n}\Delta_f^{\langle |A|^2x,x\rangle,\langle |A^*|^2x,x\rangle}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\\
&\leqslant (1-\nu)f\left(\langle |A|^2x,x\rangle\right)+\nu f\left(\langle |A^*|^2x,x\rangle\right).
\end{aligned}$$

Then

$$\begin{aligned}
&\left\|f\left((1-\nu)|A|^2+\nu|A^*|^2\right)\right\| \\
&\leqslant \left\|(1-\nu)f(|A|^2)+\nu f(|A^*|^2)-\sum_{n=0}^{N-1} r_n(\nu)\sum_{k=1}^{2^n}\Delta_f^{|A|^2,|A^*|^2}(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(\nu)\right\| \quad (4.13)\\
&\leqslant \left\|(1-\nu)f(|A|^2)+\nu f(|A^*|^2)\right\|.
\end{aligned}$$

Combining (4.12) and (4.13) we get (4.9). For (4.10), let $f(\nu)=\nu^{\frac{p}{2}}$, $2\leqslant p\leqslant 4$ and apply (4.9). □

4.2. Refinements of numerical radius inequalities

This subsection focuses on some inequalities for the numerical radius norms. In what follows, $\mathbb{M}_n$ will stand for the algebra of all $n\times n$ complex matrices. Let $\mathbb{M}_n^{++}$ denote the set of positive definite matrices. Recall that the numerical radius $w(A)$ of the matrix $A\in\mathbb{M}_n$ is given by

$$w(A)=\sup\{|\langle Ax,x\rangle| : x\in\mathbb{C}^n, \|x\|=1\}.$$

It has been shown in [20, Theorem 2.2 and Proposition 2.8] that the functions

$$F_1(\nu)=w(A^\nu XA^{1-\nu}+A^{1-\nu}XA^\nu)$$

and

$$F_2(\nu)=w(A^\nu XA^{1-\nu}-A^{1-\nu}XA^\nu)$$

are convex on $\mathbb{R}$ and attain their minimums at $\nu=\frac{1}{2}$ for any $A\in\mathbb{M}_n^{++}$ and $X\in\mathbb{M}_n$.

Applying the inequality (3.1) to the functions $F_1(\nu)$ and $F_2(\nu)$, respectively on the interval $[\mu,1-\mu]$ when $0\leqslant\mu<\frac{1}{2}$ and on the interval $[1-\mu,\mu]$ when $\frac{1}{2}<\mu\leqslant 1$, we obtain

THEOREM 4.7. *Let N be a nonnegative integer. Let $A\in\mathbb{M}_n^{++}$ and $X\in\mathbb{M}_n$. Then, for any $\mu\in[0,1]$,*

$$\begin{aligned}
&2w(A^{\frac{1}{2}}XA^{\frac{1}{2}}) \\
&\leqslant \frac{1}{|1-2\mu|}\left|\int_\mu^{1-\mu} w(A^\nu XA^{1-\nu}+A^{1-\nu}XA^\nu)d\nu\right| \\
&\leqslant \frac{1}{|1-2\mu|}\left|\int_\mu^{1-\mu} w(A^\nu XA^{1-\nu}+A^{1-\nu}XA^\nu)d\nu\right| \\
&\quad+\sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_{F_1}^{\mu,1-\mu}(n,k) \\
&\leqslant w(A^\mu XA^{1-\mu}+A^{1-\mu}XA^\mu)
\end{aligned} \quad (4.14)$$

and

$$\begin{aligned}
&\frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} w(A^{v}XA^{1-v}-A^{1-v}XA^{v})dv\right| \\
&\leqslant \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} w(A^{v}XA^{1-v}-A^{1-v}XA^{v})dv\right| \\
&\qquad + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_{F_2}^{\mu,1-\mu}(n,k) \\
&\leqslant w(A^{\mu}XA^{1-\mu}-A^{1-\mu}XA^{\mu}).
\end{aligned} \tag{4.15}$$

It should be noted that in the inequalities (4.14) and (4.15), respectively

$$\lim_{\mu\to\frac{1}{2}}\frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} w(A^{v}XA^{1-v}+A^{1-v}XA^{v})dv\right| = 2w(A^{\frac{1}{2}}XA^{\frac{1}{2}})$$

and

$$\lim_{\mu\to\frac{1}{2}}\frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} w(A^{v}XA^{1-v}-A^{1-v}XA^{v})dv\right| = 0.$$

REMARK 4.8. The inequality (4.14) offers a refinement of [20, Corollary 2.6].

Using the inequality (3.1) to the functions $F_1(v)$ and $F_2(v)$, respectively on the interval $[\mu,\frac{1}{2}]$ when $0\leqslant\mu<\frac{1}{2}$ and on the interval $[\frac{1}{2},\mu]$ when $\frac{1}{2}<\mu\leqslant 1$, we have

THEOREM 4.9. *Let N be a nonnegative integer. Let $A\in\mathbb{M}_n^{++}$ and $X\in\mathbb{M}_n$. Then, for any $\mu\in[0,1]$,*

$$\begin{aligned}
&w(A^{\frac{2\mu+1}{4}}XA^{\frac{3-2\mu}{4}}+A^{\frac{3-2\mu}{4}}XA^{\frac{2\mu+1}{4}}) \\
&\leqslant \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} w(A^{v}XA^{1-v}+A^{1-v}XA^{v})dv\right| \\
&\leqslant \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} w(A^{v}XA^{1-v}+A^{1-v}XA^{v})dv\right| \\
&\qquad + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_{F_1}^{\mu,\frac{1}{2}}(n,k) \\
&\leqslant \frac{1}{2}w(A^{\mu}XA^{1-\mu}+A^{1-\mu}XA^{\mu})+w(A^{\frac{1}{2}}XA^{\frac{1}{2}})
\end{aligned} \tag{4.16}$$

and

$$
\begin{aligned}
& w(A^{\frac{2\mu+1}{4}}XA^{\frac{3-2\mu}{4}} - A^{\frac{3-2\mu}{4}}XA^{\frac{2\mu+1}{4}}) \\
& \leqslant \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} w(A^{\nu}XA^{1-\nu} - A^{1-\nu}XA^{\nu})d\nu\right| \\
& \leqslant \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} w(A^{\nu}XA^{1-\nu} - A^{1-\nu}XA^{\nu})d\nu\right| + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_{F_2}^{\mu,\frac{1}{2}}(n,k) \\
& \leqslant \frac{1}{2}w(A^{\mu}XA^{1-\mu} - A^{1-\mu}XA^{\mu}).
\end{aligned}
$$

REMARK 4.10. The inequality (4.16) offers a further refinement of [20, Corollary 2.6].

4.3. Refinements of unitarily invariant norm inequalities

It is well known that if A,B and X be operators on a complex separable Hilbert space H such that A and B are positive, then for $0 \leqslant \nu \leqslant 1$ and for every unitarily invariant norm $|||\cdot|||$ the following inequality due to Bhatia and Davis [2] holds

$$
2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \leqslant |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}||| \leqslant |||AX+XB|||. \tag{4.17}
$$

By using the classical Hermite-Hadamard inequality, Kittaneh [12] derived several refinements of this norm inequality. In this part, our main objective is to draw attention to refining the Heinz operator inequalities (4.17) via the Hermite-Hadamard type inequality (3.1).

We now consider the convex function $F(\nu) = |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||$ on $\nu \in [0,1]$. Applying the inequality (3.1) to the function $F(\nu)$ on the interval $[\mu, 1-\mu]$ when $0 \leqslant \mu < \frac{1}{2}$ and on the interval $[1-\mu, \mu]$ when $\frac{1}{2} < \mu \leqslant 1$, we obtain the following refinement of [12, Theorem 1].

THEOREM 4.11. *Let N be a nonnegative integer. Let A,B and X be operators such that A,B are positive. Then, for any $\mu \in [0,1]$ and any unitarily invariant norm $|||\cdot|||$,*

$$
\begin{aligned}
& \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||d\nu\right| \\
& \leqslant \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} |||A^{\nu}XB^{1-\nu} + A^{1-\nu}XB^{\nu}|||d\nu\right| \\
& \qquad + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_{F}^{\mu,1-\mu}(n,k) \\
& \leqslant |||A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}|||.
\end{aligned} \tag{4.18}
$$

It should be noted that in the inequality (4.18)

$$\lim_{\mu\to\frac{1}{2}} \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||d\nu\right| = 2|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||.$$

Using the inequality (3.1) to the function $F(\nu)$ on the interval $[\mu,\frac{1}{2}]$ when $0\leqslant\mu<\frac{1}{2}$ and on the interval $[\frac{1}{2},\mu]$ when $\frac{1}{2}<\mu\leqslant 1$, we have the following improvement of [12, Theorem 2].

THEOREM 4.12. *Let N be a nonnegative integer. Let A,B and X be operators such that A,B are positive. Then, for any $\mu\in[0,1]$ and any unitarily invariant norm $|||\cdot|||$,*

$$\begin{aligned}
&\frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||d\nu\right| \\
&\leqslant \frac{2}{|1-2\mu|}\left|\int_{\mu}^{\frac{1}{2}} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||d\nu\right| + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_F^{\mu,\frac{1}{2}}(n,k) \\
&\leqslant \frac{1}{2}|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||+|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||.
\end{aligned}$$

We consider the inequality (3.1) to the function $F(\nu)$ on the interval $[0,\mu]$ when $0\leqslant\mu\leqslant\frac{1}{2}$ and on the interval $[1,\mu]$ when $\frac{1}{2}\leqslant\mu\leqslant 1$ we get the following refinement of [12, Theorem 3].

THEOREM 4.13. *Let N be a nonnegative integer. Let A,B and X be operators such that A,B are positive. Then,*

(1) for $0\leqslant\mu\leqslant\frac{1}{2}$ and any unitarily invariant norm $|||\cdot|||$,

$$\begin{aligned}
&\frac{1}{\mu}\int_{0}^{\mu} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||d\nu \\
&\leqslant \frac{1}{\mu}\int_{0}^{\mu} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||d\nu + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_F^{0,\mu}(n,k) \\
&\leqslant \frac{1}{2}|||AX+XB|||+\frac{1}{2}|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||.
\end{aligned}$$

(2) for $\frac{1}{2}\leqslant\mu\leqslant 1$ and any unitarily invariant norm $|||\cdot|||$,

$$\begin{aligned}
&\frac{1}{|1-\mu|}\int_{\mu}^{1} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||d\nu \\
&\leqslant \frac{1}{|1-\mu|}\int_{\mu}^{1} |||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}|||d\nu + \sum_{n=0}^{N-1}\frac{1}{2^{n+2}}\sum_{k=1}^{2^n}\Delta_F^{\mu,1}(n,k) \\
&\leqslant \frac{1}{2}|||AX+XB|||+\frac{1}{2}|||A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}|||.
\end{aligned}$$

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Tran Dinh Phung
Department of Mathematics and Statistics
University of Finance – Marketing
778 Nguyen Kiem, Ho Chi Minh City, Vietnam
e-mail: td.phung@ufm.edu.vn

Nguyen Ngoc Hue
Department of Mathematics
Tay Nguyen University
567 Le Duan, Buon Ma Thuot Daklak, Vietnam
e-mail: nnhue@ttn.edu.vn

Duong Quoc Huy
Department of Mathematics
Tay Nguyen University
567 Le Duan, Buon Ma Thuot Daklak, Vietnam
e-mail: duongquochuy@ttn.edu.vn