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FURTHER INEQUALITIES FOR NORMAL MATRICES

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Abstract. In this paper, we study the properties of normal matrices and obtain some nonnegative function inequalities between normal matrices and their modulus. Some related works are also presented. Furthermore, we also give a novel approach to prove

$$\left\| f\left(\frac{1}{2}\begin{pmatrix} |A|+|B| & A^*+B^* \\ A+B & |A^*|+|B^*| \end{pmatrix}\right)\right\| \leqslant \|f(|A|)+f(|B|)\|$$

for normal matrices A, B.

1. Introduction

Let M_n be the set of complex matrices with order n. For $A \in M_n$, the conjugate transpose is denoted as A^*. We write the singular values of A as $s_i(A)$ $(1 \leqslant i \leqslant n)$ in decreasing order, which are defined as $s_i(A) = \lambda_i(|A|)$ ($\lambda_i(A)$ are the eigenvalues of A) and $|A| = (A^*A)^{\frac{1}{2}}$. The notation $A \geqslant 0$ indicates that $A = A^*$ with nonnegative eigenvalues and $A \oplus B$ is the block matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. A matrix $A \in M_n$ is contractive if $s_1(A) \leqslant 1$. For $A = (a_{ij})$, $B = (b_{ij})$ are two matrices with same size, the Schur product $A \circ B$ is the matrix $(a_{ij}b_{ij})$.

Majorization theory is a key tool for obtaining matrix inequalities. The basic definitions and concepts of Majorization could be found in [1]. For $A, B \in M_n$, let $s(A) = (s_1(A), s_2(A), \cdots, s_n(A))$ and $s(B) = (s_1(B), s_2(B), \cdots, s_n(B))$, we use $s(A) \prec_w s(B)$ to represent

$$\sum_{i=1}^{k} s_i(A) \leqslant \sum_{i=1}^{k} s_i(B)$$

for $1 \leqslant k \leqslant n$ and $s(A) \prec_{w\log} s(B)$ if

$$\prod_{i=1}^{k} s_i(A) \leqslant \prod_{i=1}^{k} s_i(B)$$

with $k \in \{1, 2, \cdots, n\}$. Let $A \in M_n$, a norm $\|\cdot\|$ on M_n is said unitarily invariant if $\|UAV\| = \|A\|$ for any unitary matrices U, V. It's known in [1] that $\|A\| \leqslant \|B\|$ if and

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only if $s(A) \prec_w s(B)$. Let $A \geqslant 0$, $B \geqslant 0$ and let $f(t)$ be a nonnegative concave function on $[0,\infty)$. Bourin and Uchiyama [7] obtained

$$\|f(A+B)\| \leqslant \|f(A)+f(B)\|. \tag{1}$$

A matrix A is normal if $AA^* = A^*A$. Bourin [4] extended inequality (1) to

$$\|f(|A+B|)\| \leqslant \|f(|A|)+f(|B|)\| \tag{2}$$

for normal matrices A, B. Inequality (2) contains the following inequality

$$s(A+B) \prec_w s(|A|+|B|) \tag{3}$$

by using Fan's dominance principle [1]. Some related works, elegant generalizations and refinements of inequality (3) has been given in [3], [5], [6], [8], [12]. We are most interested in the results presented in [8].

In [8], Zhang obtained

$$s\left(\sum_{i=1}^{m} A_i\right) \prec_{w\log} s\left(\sum_{i=1}^{m} |A_i|\right) \tag{4}$$

and

$$s\left(\circ_{i=1}^{m} A_i\right) \prec_{w\log} s\left(\circ_{i=1}^{m} |A_i|\right) \tag{5}$$

for normal matrices A_i $(i = 1,2,\cdots,m)$. Bourin and Lee in [6] provided more stronger results of inequalities (4) and (5).

Motivated by Bourin's work in [4]. Huang [11] proved that

$$\|f(|A+B|)\| \leqslant \left\| f\left(\frac{1}{2}\begin{pmatrix} |A|+|B| & A^*+B^* \\ A+B & |A^*|+|B^*| \end{pmatrix}\right)\right\| \leqslant \|f(|A|)+f(|B|)\|. \tag{6}$$

After reading [11], we find that the method used by Huang in [11] is similar to that of Zhang [13], so we believe that it is necessary to provide a new proof.

The purpose of this paper is to give a further version of Theorem 2.7, Theorem 2.13 and Theorem 2.18 in [8], some related works are also given. The left ride of inequality (6) is equal to the left ride of Theorem 10 in [13]. Therefore, in order to give some significant applications of inequality (6), we present a new method to get the right ride in inequality (6).

2. Main results

In the rest of this paper, we consider f is a nonnegative function on $[0,\infty)$.

LEMMA 1. [9] *Let* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \geqslant 0$ *and* $\begin{pmatrix} A & X^* \\ X & B \end{pmatrix} \geqslant 0$. *Then*

$$\prod_{j=1}^{k} s_j(X) \leqslant \prod_{j=1}^{k} s_j\left(A^{\frac{1}{2}}B^{\frac{1}{2}}\right).$$

THEOREM 1. *Let $A_i \in M_n$ be normal matrices with polar decomposition $A_i = U_i|A_i|$. Then*

$$s\left(\sum_{i=1}^{m} U_i f(|A_i|)\right) \prec_{w\log} s\left(\sum_{i=1}^{m} f(|A_i|)\right).$$

Proof. Since f is nonnegative, we get

$$\begin{pmatrix} f(|A_i|) & (U_i f(|A_i|))^* \\ U_i f(|A_i|) & f(|A_i^*|) \end{pmatrix} = 2W_i \begin{pmatrix} f(|A_i|) & 0 \\ 0 & 0 \end{pmatrix} W_i^* \geqslant 0 \tag{7}$$

for $W_i = \frac{1}{\sqrt{2}}\begin{pmatrix} I & I \\ U_i & -U_i \end{pmatrix}$.

Therefore,

$$\begin{pmatrix} \sum_{i=1}^{m} f(|A_i|) & \sum_{i=1}^{m} (U_i f(|A_i|))^* \\ \sum_{i=1}^{m} U_i f(|A_i|) & \sum_{i=1}^{m} f(|A_i|) \end{pmatrix} \geqslant 0$$

due to $|A_i| = |A_i^*|$.

Using Lemma 1 we obtain

$$s\left(\sum_{i=1}^{m} U_i f(|A_i|)\right) \prec_{w\log} s\left(\sum_{i=1}^{m} f(|A_i|)\right). \quad \square$$

As an application of Theorem 1, we have the following corollary:

COROLLARY 1. *Let $A_i \in M_n$ be normal matrices, for $i = 1, 2, \ldots, m$. Then*

$$\left\|\sum_{i=1}^{m} U_i f(|A_i|)\right\| \leqslant \left\|\sum_{i=1}^{m} f(|A_i|)\right\|.$$

Next, we establish a connection between Theorem 1 and Theorem 2.10 in [8].

REMARK 1. Let z_i be complex numbers with $|z_i| = 1$ and let $A_i \geqslant 0$ $(1 \leqslant i \leqslant m)$. Then $z_i A_i$ is normal and

$$z_i A_i = (z_i I) A_i,$$

where $z_i I$ is a unitary matrix.

It follows from Theorem 1 that

$$s\left(\sum_{i=1}^{m} z_i f(A_i)\right) \prec_{w\log} s\left(\sum_{i=1}^{m} f(A_i)\right). \tag{8}$$

Putting $f(x) = x$ in inequality (8), we get

$$s\left(\sum_{i=1}^{m} z_i A_i\right) \prec_{w\log} s\left(\sum_{i=1}^{m} |z_i| A_i\right) \tag{9}$$

for any positive integer m and $|z_i| = 1$.

For complex number z_i with $|z_i| \neq 0$, we set $z_i = l_i|z_i|$ with $l_i = \frac{z_i}{|z_i|}$. We replace z_i and A_i in Inequality (9) with l_i and $|z_i|A_i$, respectively.

Therefore, inequality (9) holds for any complex number z_i with $|z_i| \neq 0$.

If $z_i = 0$, we have

$$z_iA_i = |z_i|A_i = 0.$$

Thus, inequality (9) also holds for any z_i. In fact, inequality (9) is the Theorem 2.10 in [8].

In order to give our second main result, we list the following lemmas. The first lemma is an inequality for the Hadamard product of positive semidefinite matrices, which can be found on page 7 of [2].

LEMMA 2. *Let $A \geqslant 0$, $B \geqslant 0$. Then $A \circ B \geqslant 0$.*

The second lemma provides a characterization of the block elements for positive semidefinite 2×2 block matrices, see page 13 on [2] for more details.

LEMMA 3. *Let $A, B \in M_n$ with $A \geqslant 0, B \geqslant 0$. Then $\begin{bmatrix} A & X^* \\ X & B \end{bmatrix} \geqslant 0$ if and only if $X = A^{\frac{1}{2}}KB^{\frac{1}{2}}$ for some contraction K.*

The next lemma is due to Horn.

LEMMA 4. [10] *Let $A, B \in M_n$. Then $s(AB) \prec_{w\log} s(A)s(B)$.*

Our second theorem is an inequality related to Hadamard product and nonnegative functions.

THEOREM 2. *Let $A_i \in M_n$ be normal matrices with polar decomposition $A_i = U_i|A_i|$. Then*

$$s\left(\circ_{i=1}^m U_i f\left(|A_i|\right)\right) \prec_{w\log} s\left(\circ_{i=1}^m f\left(|A_i|\right)\right).$$

Proof. It follows form Theorem 1 and Lemma 2 that

$$\begin{pmatrix} \circ_{i=1}^m f\left(|A_i|\right) & \circ_{i=1}^m \left(U_i f\left(|A_i|\right)\right)^* \\ \circ_{i=1}^m U_i f\left(|A_i|\right) & \circ_{i=1}^m f\left(|A_i|\right) \end{pmatrix} \geqslant 0.$$

By Lemma 3, we have

$$\circ_{i=1}^m U_i f\left(|A_i|\right) = \left(\circ_{i=1}^m f\left(|A_i|\right)\right)^{\frac{1}{2}} K \left(\circ_{i=1}^m f\left(|A_i|\right)\right)^{\frac{1}{2}}$$

for some contraction K.

From Lemma 4, we can conclude that

$$\begin{aligned} s\left(\circ_{i=1}^{m} U_i f(|A_i|)\right) &\prec_{w\log} s\left(\left(\circ_{i=1}^{m} f(|A_i|)\right)^{\frac{1}{2}} K \left(\circ_{i=1}^{m} f(|A_i|)\right)^{\frac{1}{2}}\right) \\ &\prec_{w\log} s\left(\circ_{i=1}^{m} f(|A_i|)\right)^{\frac{1}{2}} s(K) s\left(\circ_{i=1}^{m} f(|A_i|)\right)^{\frac{1}{2}} \\ &\prec_{w\log} s\left(\circ_{i=1}^{m} f(|A_i|)\right)^{\frac{1}{2}} s\left(\circ_{i=1}^{m} f(|A_i|)\right)^{\frac{1}{2}} \\ &\prec_{w\log} s\left(\circ_{i=1}^{m} f(|A_i|)\right). \quad \square \end{aligned}$$

COROLLARY 2. *Let A_i $(i=1,2,\cdots,m)$ be normal matrices. Then*

$$s\left(\circ_{i=1}^{m} A_i\right) \prec_{w\log} s\left(\circ_{i=1}^{m} |A_i|\right). \tag{10}$$

Proof. Corollary 2 follows from Theorem 2 by letting $f(x)=x$. $\square$

THEOREM 3. *Let $A_i \in M_n$ $(1 \leqslant i \leqslant 4)$ be normal matrices with $A_i = U_i|A_i|$. Then*

$$\begin{aligned} & s\left(\begin{bmatrix} U_1 f(|A_1|) & U_2 f(|A_2|) \\ U_3 f(|A_3|) & U_4 f(|A_4|) \end{bmatrix}\right) \\ \prec_{w\log} & s\left(\begin{bmatrix} f(|A_1|)+f(|A_2|) & 0 \\ 0 & f(|A_3|)+f(|A_4|) \end{bmatrix}\right)^{\frac{1}{2}} \\ & \times \left(\begin{bmatrix} f(|A_1|)+f(|A_3|) & 0 \\ 0 & f(|A_2|)+f(|A_4|) \end{bmatrix}\right)^{\frac{1}{2}}. \end{aligned}$$

Proof. Let $U = \begin{bmatrix} U_1 & 0 \\ 0 & U_4 \end{bmatrix}$, $V = \begin{bmatrix} 0 & U_2 \\ U_3 & 0 \end{bmatrix}$. Then

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_4 \end{bmatrix} = U \begin{bmatrix} |A_1| & 0 \\ 0 & |A_4| \end{bmatrix}$$

and

$$\begin{bmatrix} 0 & A_2 \\ A_3 & 0 \end{bmatrix} = V \begin{bmatrix} |A_3| & 0 \\ 0 & |A_2| \end{bmatrix}.$$

It follows from inequality (7) and Lemma 3 that

$$\begin{bmatrix} U_1 f(|A_1|) & 0 \\ 0 & U_4 f(|A_4|) \end{bmatrix} = \begin{bmatrix} f(|A_1|) & 0 \\ 0 & f(|A_4|) \end{bmatrix}^{\frac{1}{2}} K \begin{bmatrix} f(|A_1|) & 0 \\ 0 & f(|A_4|) \end{bmatrix}^{\frac{1}{2}}$$

and

$$\begin{bmatrix} 0 & U_2 f(|A_2|) \\ U_3 f(|A_3|) & 0 \end{bmatrix} = \begin{bmatrix} f(|A_2|) & 0 \\ 0 & f(|A_3|) \end{bmatrix}^{\frac{1}{2}} L \begin{bmatrix} f(|A_3|) & 0 \\ 0 & f(|A_2|) \end{bmatrix}^{\frac{1}{2}}$$

for some contractive matrices K, L.

Let

$$B_1=\begin{bmatrix} f(|A_1|) & 0 \\ 0 & f(|A_4|) \end{bmatrix},$$
$$B_2=\begin{bmatrix} f(|A_1|) & 0 \\ 0 & f(|A_4|) \end{bmatrix},$$
$$B_3=\begin{bmatrix} f(|A_2|) & 0 \\ 0 & f(|A_3|) \end{bmatrix},$$
$$B_4=\begin{bmatrix} f(|A_3|) & 0 \\ 0 & f(|A_2|) \end{bmatrix}.$$

As a consequence of Lemma 4, we obtain

$$\begin{aligned}
& s\left(\begin{bmatrix} U_1 f(|A_1|) & U_2 f(|A_2|) \\ U_3 f(|A_3|) & U_4 f(|A_4|) \end{bmatrix} \oplus 0\right) \\
= & s\left(\begin{bmatrix} B_1^{\frac{1}{2}} & B_3^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix}\begin{bmatrix} B_2^{\frac{1}{2}} & 0 \\ B_4^{\frac{1}{2}} & 0 \end{bmatrix}\right) \\
\prec_{w\log} & s\left(\begin{bmatrix} B_1^{\frac{1}{2}} & B_3^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\right) s\left(\begin{bmatrix} K & 0 \\ 0 & L \end{bmatrix}\right) s\left(\begin{bmatrix} B_2^{\frac{1}{2}} & 0 \\ B_4^{\frac{1}{2}} & 0 \end{bmatrix}\right) \\
\prec_{w\log} & s\left(\begin{bmatrix} B_1^{\frac{1}{2}} & B_3^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\right) s\left(\begin{bmatrix} B_2^{\frac{1}{2}} & 0 \\ B_4^{\frac{1}{2}} & 0 \end{bmatrix}\right).
\end{aligned}$$

Observe that

$$s\left(\begin{bmatrix} B_1^{\frac{1}{2}} & B_3^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}\right)=s^{\frac{1}{2}}\left(\begin{bmatrix} f(|A_1|)+f(|A_2|) & 0 \\ 0 & f(|A_3|)+f(|A_4|) \end{bmatrix}\oplus 0\right)$$

and

$$s\left(\begin{bmatrix} B_2^{\frac{1}{2}} & 0 \\ B_4^{\frac{1}{2}} & 0 \end{bmatrix}\right)=s\left(\begin{bmatrix} f(|A_1|)+f(|A_3|) & 0 \\ 0 & f(|A_2|)+f(|A_4|) \end{bmatrix}\oplus 0\right).$$

Thus, we get our desired result. □

REMARK 2. Theorem 2. 13 in [8] is a special case of Theorem 3 by letting $f(x)=x$.

LEMMA 5. [3] *Let* $A \geqslant 0,\ B \geqslant 0$. *Then* $\left\|\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right\| \leqslant \|A+B\|$.

THEOREM 4. *Let* $f(t)$ *be concave on* $[0,\infty)$. *Then*

$$\begin{aligned}
&\left\| f\left(\frac{1}{2}\begin{pmatrix} |A|+|B| & A^*+B^* \\ A+B & |A^*|+|B^*| \end{pmatrix}\right) \oplus f\left(\frac{1}{2}\begin{pmatrix} |A|+|B| & A^*+B^* \\ A+B & |A^*|+|B^*| \end{pmatrix}\right)\right\| \\
\leqslant\ & \|[f(|A|)+f(|B|)] \oplus [f(|A^*|)+f(|B^*|)]\|
\end{aligned}$$

for $A, B \in M_n$.

Proof. Let $A = U|A|$ and $B = V|B|$ be the polar decomposition of A, B. First, let's consider the case when $f(0) = 0$.

$$\begin{aligned}
&\left\| f\left(\frac{1}{2}\begin{pmatrix} |A|+|B| & A^*+B^* \\ A+B & |A^*|+|B^*| \end{pmatrix}\right) \oplus f\left(\frac{1}{2}\begin{pmatrix} |A|+|B| & A^*+B^* \\ A+B & |A^*|+|B^*| \end{pmatrix}\right)\right\| \\
&= \left\| f\left(\frac{1}{2}\begin{pmatrix} |A|+|B| & A^*+B^* \\ A+B & |A^*|+|B^*| \end{pmatrix} \oplus \frac{1}{2}\begin{pmatrix} |A|+|B| & -A^*-B^* \\ -A-B & |A^*|+|B^*| \end{pmatrix}\right)\right\| \\
&\leqslant \left\| f\left(\left[\frac{1}{2}\begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix} \oplus \frac{1}{2}\begin{pmatrix} |A| & -A^* \\ -A & |A^*| \end{pmatrix}\right]\right.\right. \\
&\quad \left.\left. + f\left(\left[\frac{1}{2}\begin{pmatrix} |B| & B^* \\ B & |B^*| \end{pmatrix} \oplus \frac{1}{2}\begin{pmatrix} |B| & -B^* \\ -B & |B^*| \end{pmatrix}\right]\right)\right)\right\| \\
&\leqslant \left\| f\left(\frac{1}{2}\begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix}\right) + f\left(\frac{1}{2}\begin{pmatrix} |B| & B^* \\ B & |B^*| \end{pmatrix}\right)\right. \\
&\quad \left. + f\left(\frac{1}{2}\begin{pmatrix} |A| & -A^* \\ -A & |A^*| \end{pmatrix}\right) + f\left(\frac{1}{2}\begin{pmatrix} |B| & -B^* \\ -B & |B^*| \end{pmatrix}\right)\right\| \\
&= \left\| f\begin{pmatrix} |A| & 0 \\ 0 & |A^*| \end{pmatrix} + f\begin{pmatrix} |B| & 0 \\ 0 & |B^*| \end{pmatrix}\right\|.
\end{aligned}$$

The first inequality is obtained by inequality (1) and the second inequality is due to Lemma 5, the final step follows from

$$\begin{aligned}
&f\left(\frac{1}{2}\begin{pmatrix} |A| & A^* \\ A & |A^*| \end{pmatrix}\right) + f\left(\frac{1}{2}\begin{pmatrix} |A| & -A^* \\ -A & |A^*| \end{pmatrix}\right) \\
&= U_1 f\begin{pmatrix} |A| & 0 \\ 0 & 0 \end{pmatrix} U_1^* + \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} U_1 f\begin{pmatrix} |A| & 0 \\ 0 & 0 \end{pmatrix} U_1^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} f(|A|) & f(|A|)U^* \\ Uf(|A|) & Uf(|A|)U^* \end{pmatrix} + \frac{1}{2}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} f(|A|) & f(|A|)U^* \\ Uf(|A|) & Uf(|A|)U^* \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} f(|A|) & f(|A|)U^* \\ Uf(|A|) & f(|A^*|) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} f(|A|) & -f(|A|)U^* \\ -Uf(|A|) & f(|A^*|) \end{pmatrix} \\
&= f\begin{pmatrix} |A| & 0 \\ 0 & |A^*| \end{pmatrix},
\end{aligned}$$

where $U_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ U & U \end{bmatrix}$.

By following the same argument, we get

$$f\left(\frac{1}{2}\begin{pmatrix} |B| & B^* \\ B & |B^*| \end{pmatrix}\right) + f\left(\frac{1}{2}\begin{pmatrix} |B| & -B^* \\ -B & |B^*| \end{pmatrix}\right) = f\begin{pmatrix} |B| & 0 \\ 0 & |B^*| \end{pmatrix}.$$

For the general case, i.e., $f(0)>0$. We suppose that $g(x)=f(x)-f(0)$. Then $g(x)$ is a nonnegative and concave function on $[0,\infty)$ with $g(0)=0$. Thus,

$$\begin{aligned}&\left\|g\left(\frac{1}{2}\begin{pmatrix}|A|+|B| & A^*+B^*\\ A+B & |A^*|+|B^*|\end{pmatrix}\right)\oplus g\left(\frac{1}{2}\begin{pmatrix}|A|+|B| & A^*+B^*\\ A+B & |A^*|+|B^*|\end{pmatrix}\right)\right\|\\ &\leqslant \|[g(|A|)+g(|B|)]\oplus[g(|A^*|)+g(|B^*|)]\|.\end{aligned} \tag{11}$$

Let $M=\left(\frac{1}{2}\begin{pmatrix}|A|+|B| & A^*+B^*\\ A+B & |A^*|+|B^*|\end{pmatrix}\right)\oplus\left(\frac{1}{2}\begin{pmatrix}|A|+|B| & A^*+B^*\\ A+B & |A^*|+|B^*|\end{pmatrix}\right)$, $X=|A|\oplus|A^*|$, $Y=|B|\oplus|B^*|$. Then inequality (11) is equivalent to

$$\|g(M)\|\leqslant\|g(X)+g(Y)\|. \tag{12}$$

For $1\leqslant k\leqslant 2n$, we get

$$\begin{aligned}\sum_{j=1}^{k}s_j(f(M)) &= \sum_{j=1}^{k}s_j(g(M))+kf(0)\\ &\leqslant \sum_{j=1}^{k}s_j(g(X)+g(Y))+kf(0)\\ &\leqslant \sum_{j=1}^{k}s_j(f(X)+f(Y))\end{aligned}$$

from inequality (12).

For $2n\leqslant k\leqslant 4n$, we obtain

$$\begin{aligned}\sum_{j=1}^{k}s_j(f(M)) &= \sum_{j=1}^{k}s_j(g(M))+kf(0)\\ &\leqslant \sum_{j=1}^{2n}s_j(g(X)+g(Y))+kf(0)\\ &\leqslant \sum_{j=1}^{2n}s_j(g(X)+g(Y))+4nf(0)\\ &= \sum_{j=1}^{2n}s_j(f(X)+f(Y)).\end{aligned}$$

from inequality (12).

Thus, we derive

$$\|f(M)\|\leqslant\|f(X)+f(Y)\|$$

for $f(x)>0$. $\square$

REMARK 3. The following inequality

$$\left\|f\left(\frac{1}{2}\begin{pmatrix}|A|+|B| & A^*+B^*\\ A+B & |A^*|+|B^*|\end{pmatrix}\right)\right\|\leqslant\|f(|A|)+f(|B|)\|$$

is established if $|A|=|A^*|$ and $|B|=|B^*|$.

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