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ON APPROXIMATE AND ACTUAL REDUCIBILITY OF MATRIX GROUPS

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Abstract. We introduce the notions of ε-approximate fixed point and weak ε-approximate fixed point. We show that for a group of unitary matrices even the existence of a nontrivial weak ε-approximate fixed point for sufficiently small ε gives an actual nontrivial common eigenvector. We give estimates for ε in terms of the size n of matrices and prove that the dependence is polynomial. Moreover, we show that the common eigenvector is polynomially close to the starting weak approximate fixed point.

1. Introduction

Let $\mathcal{G}$ be a group of complex matrices. An approximate fixed point $\xi \in \mathbb{C}^n$ is defined to be a nonzero vector for which the norm $\|G\xi - \xi\|$ is uniformly small for all $G \in \mathcal{G}$ in an appropriate sense. For a general group, given $\varepsilon > 0$, one demands that $\|G\xi - \xi\| \leqslant \|G\| \|\xi\| \varepsilon$, for all $G \in \mathcal{G}$. (In this paper $\|\cdot\|$ will always mean Euclidean norm $\|\cdot\|_2$ unless stated otherwise.) Since we will be concerned with groups of unitary matrices, we might as well assume $\|G\| = \|\xi\| = 1$ and simplify our definitions. The above definition then is equivalent to the requirement that the inner product $\langle G\xi, \xi\rangle$ be uniformly close to 1. An obviously weaker condition is to require that the modulus $|\langle G\xi, \xi\rangle|$ be uniformly close to 1. That kind of ξ will then be called a weak approximate fixed point. The problem of determining groups for which the existence of weak approximate fixed points in the above sense implies the existence of a nontrivial invariant subspace of $\mathbb{C}^n$ for $\mathcal{G}$ (i.e., reducibility of $\mathcal{G}$) doesn't seem to be easy. In the present paper we impose additional hypotheses on $\mathcal{G}$ to get affirmative results. Motivation behind this problem and some related results are discussed in the next section entitled Preliminaries.

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2. Preliminaries

Let an integer $n > 0$, a real number $\varepsilon > 0$ and a group $\mathcal{G} \subseteq M_n(\mathbb{C})$ be given (all the groups in this paper will consist of invertible complex matrices of order n so that they may be seen as subgroups of the general linear group).

We call a point $\xi \in \mathbb{C}^n$ of norm one *a fixed point of* $\mathcal{G}$ if $G\xi = \xi$ for each $G \in \mathcal{G}$, and we call it an ε*-approximate fixed point* of $\mathcal{G}$ if $\|G\xi - \xi\| \leqslant \varepsilon$ for all $G \in \mathcal{G}$. When ε is understood from the context, we call ξ shortly an *approximate fixed point* of the group $\mathcal{G}$. Recall that the existence of an approximate fixed point is closely related to Kazhdan property T introduced in a seminal paper of Kazhdan [8]; we refer to [1] for a historical background and a wealth of information on the topic. In particular (see [1, Proposition 1.1.5]), if a compact unitary subgroup has an $\varepsilon < \sqrt{2}$ approximate fixed point ξ, then it has a fixed point and we have the estimate $\|\xi - P\xi\| \leqslant 1$ where P is the projection to the span of all fixed points (see [1, Proposition 1.1.9]). Clearly, $P\xi$ is a common eigenvector of $\mathcal{G}$, but note that in general its norm is less than one, so it does not qualify as a fixed point according to our definition above. Occasionally, e.g., in Proposition 1, we will be satisfied with the existence of a nonzero vector ξ such that $G\xi = \xi$ for every $G \in \mathcal{G}$; the vector with this property will be called *a fixed vector of* $\mathcal{G}$. *Thus, a fixed point is a norm-one fixed vector.*

We call $\xi \in \mathbb{C}^n$ of norm one a *weak* ε*-approximate fixed point* of $\mathcal{G}$ if $0 < \varepsilon < 1$ and

$$|\langle G\xi, \xi\rangle| \geqslant 1 - \varepsilon \tag{1}$$

for all $G \in \mathcal{G}$. Again, when ε is understood from the context, we call ξ a *weak approximate fixed point* of the group $\mathcal{G}$. The condition $\|\xi\| = 1$ will be tacitly assumed for any fixed point ξ in this paper in connection with either of these definitions, approximate or not, in any of the claims to follow.

The main result of our paper is that for $0 < \varepsilon < \frac{1}{3600n^{11}}$, any unitary subgroup $\mathcal{G}$ of $M_n(\mathbb{C})$ with a weak ε-approximate fixed point ξ has a common eigenvector η that is within $3600n^{11}\varepsilon$ of ξ (see Theorem 22). The existence of an $\varepsilon > 0$ for which existence of weak ε-approximate fixed points implies reducibility for unitary subgroups of $M_n(\mathbb{C})$ follows from a significantly more general result from [9]. There the authors study continuous multi-variable functions

$$f\colon M_n(\mathbb{C}) \times \ldots \times M_n(\mathbb{C}) \to [0, \infty),$$

that are reducing for unitary groups in the sense that any unitary group $\mathcal{G}$ on which f is identically 0 (that is, $f(\mathcal{G}, \ldots, \mathcal{G}) = \{0\}$) is reducible. They show that for any fixed n, and for any such function f, there is an $\varepsilon > 0$ such that any unitary group $\mathcal{G}$ on which f is bounded by ε (that is, $|f(\mathcal{G}, \ldots, \mathcal{G})| < \varepsilon$) is also reducible. The main ingredient of establishing this is applying the theory of collections of nonempty compact subsets of compact metric spaces equipped with the Hausdorff topology to the metric space of complex matrices with the usual topology. The just mentioned result easily implies that, for any n, there is an $\varepsilon > 0$ such that any unitary subgroup $\mathcal{G}$ of $M_n(\mathbb{C})$ with a weak ε-approximate fixed point ξ has a common eigenvector η. However, the techniques of [9] do not yield any estimate on the size of ε (they do not even guarantee that ε will depend on $\frac{1}{n}$ polynomially), nor do they guarantee that the common eigenvector η will be polynomially close to the starting weak approximate fixed point ξ.

The paper is organized as follows. In Section 3 we study finite groups generated by their commutators. We show, in particular, that if every element of the group is a product of a commutator and a scalar, the existence of a weak ε-approximate fixed point for sufficiently small ε implies reducibility (Theorem 6). In Section 4 we study monomial groups and prove in Theorem 13 that a monomial group of unitary matrices having a weak ε-approximate fixed point has a common eigenvector provided that ε is of the order of magnitude n^{-11}. (Recall that all our eigenvectors, common, approximate or otherwise, will be of norm 1.) Some important results on decompositions of unitary groups are presented in Section 5, and in Section 6 we combine the results of the previous sections to give an analogous result for a general finite group of unitaries. In Section 7 we present the desired result for connected compact groups using a well-known result on connected compact Lie groups. Then we use the fact that the connected component of the identity of a compact group has finite index to extend our finite-group results to arbitrary compact groups of unitaries via a simple device. Finally, in Section 8 we present the main result (Theorem 22) mentioned above.

We will make frequent use of the properties of the elements of the group as matrices and will consequently prefer to see our groups as subsets of $M_n(\mathbb{C})$. We will also use the fact that these matrices act as operators on the underlying space $\mathbb{C}^n$ which will be equipped with the usual Hilbert space norm, i.e. inner-product norm. As a matter of fact, most of our groups will consist of unitary matrices and this general assumption will make our theory work.

3. Commutator groups and fixed points

Let us start with a well-known fact for which we give a simple proof for the sake of completeness.

PROPOSITION 1. *Assume that a compact group $\mathcal{G} \subseteq M_n(\mathbb{C})$ of unitaries has an ε-approximate fixed point ξ $(0 < \varepsilon < 1)$. Then $\mathcal{G}$ has a fixed vector $\eta \neq 0$ such that $\|\xi - \eta\| \leqslant \varepsilon$. Moreover, $\eta' := \eta/\|\eta\|$ is a fixed point of $\mathcal{G}$ and $\|\xi - \eta'\| \leqslant 2\varepsilon$.*

Proof. Let μ be the Haar measure on $\mathcal{G}$ and define

$$\eta = \int_{G \in \mathcal{G}} G\xi \, d\mu.$$

Using standard arguments and the fact that the Haar measure is left invariant, we see that $G\eta = \eta$ for all $G \in \mathcal{G}$, so that η is a fixed vector, provided we show it is nonzero. Use the fact that μ is a positive measure with $\mu(\mathcal{G}) = 1$ to get

$$\|\eta - \xi\| = \sup_{\|\vartheta\| \leqslant 1} |\langle \eta - \xi, \vartheta \rangle| = \sup_{\|\vartheta\| \leqslant 1} \left| \int_{G \in \mathcal{G}} \langle G\xi - \xi, \vartheta \rangle \, d\mu \right| \leqslant \varepsilon.$$

It follows that $1-\varepsilon \leqslant \|\eta\| \leqslant 1+\varepsilon$, so $\eta' := \eta/\|\eta\|$ is a fixed point of $\mathcal{G}$ and $\|\eta' - \xi\| \leqslant |1-\frac{1}{\|\eta\|}| \cdot \|\eta\| + \|\eta - \xi\| \leqslant 2\varepsilon$, as desired. □

Observe that the assumption "of unitaries" is not necessary in the result above. This is because the existence of the Haar measure is assured solely by the assumption that $\mathcal{G}$ be compact. As a matter of fact we may only assume it is bounded. We can close a bounded group to make it compact, and using the Haar measure we can find a similarity after which all members of the group are unitary operators; we will omit the proof since all this is well known. However, all these assertions are true only up to simultaneous similarity, while one of our main points is to determine the ε as a polynomial function of n. And this point can be studied only if the groups are assumed unitary upfront.

We will now show that a weak approximate fixed point may always be seen as an approximate eigenvector for all elements of the group. So, let $\mathcal{G} \subseteq M_n(\mathbb{C})$ be a compact group of unitaries and let $\xi \in \mathbb{C}^n$ of norm one be a *weak approximate fixed point* of $\mathcal{G}$. We want to show that this vector is an approximate eigenvector of every element of the group with the corresponding eigenvalue depending on the element. Introduce for every $G \in \mathcal{G}$ the scalar

$$\lambda_G = \frac{\langle G\xi, \xi\rangle}{|\langle G\xi, \xi\rangle|}. \tag{2}$$

Observe that $|\lambda_G| = 1$ and $\lambda_{G^*} = \overline{\lambda}_G$ since $\langle G^*\xi, \xi\rangle = \langle \xi, G\xi\rangle = \overline{\langle G\xi, \xi\rangle}$.

PROPOSITION 2. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. If $\xi \in \mathbb{C}^n$ is a weak ε-approximate fixed point of a compact group $\mathcal{G} \subseteq M_n(\mathbb{C})$ of unitaries, then*

$$\|G\xi - \lambda_G\xi\| \leqslant \sqrt{2\varepsilon} \quad \text{for any} \quad G \in \mathcal{G}.$$

Proof. A straightforward computation reveals

$$\begin{aligned}\|G\xi - \lambda_G\xi\|^2 &= \langle G\xi - \lambda_G\xi, G\xi - \lambda_G\xi\rangle \\ &= \langle G\xi, G\xi\rangle - 2\mathbb{R}\mathrm{e}(\langle G\xi, \lambda_G\xi\rangle) + \langle \lambda_G\xi, \lambda_G\xi\rangle \\ &= 2 - 2|\langle G\xi, \xi\rangle| \leqslant 2\varepsilon. \quad \square\end{aligned}$$

THEOREM 3. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. If $\xi \in \mathbb{C}^n$ is a weak ε-approximate fixed point of a group of unitaries $\mathcal{G} \subseteq M_n(\mathbb{C})$, and $\mathcal{H}$ is a subset of commutators in $\mathcal{G}$, then ξ is also an ε'-approximate fixed point of $\mathcal{H}$ for $\varepsilon' = 4\sqrt{2\varepsilon}$.*

Recall that an element of the form $[G,H] = GHG^{-1}H^{-1}$ is called a *commutator* for any $G, H \in \mathcal{G}$; since the group under consideration consists of unitary matrices we may also compute the commutator as $[G,H] = GHG^*H^*$. Of course, not every group of matrices (even if finite) has the property that all of its elements are commutators, the smallest nonabelian counterexample is the group S_3, seen as a group of permutation matrices in $M_3(\mathbb{C})$. The smallest counterexample of an abstract perfect group (a group that equals its derived group, i.e. the group generated by its commutators) in which there are elements that are not commutators, is the group $G = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_5$ of size 960 (cf. [5]).

Observe that Theorem 3 is a simple corollary of the following lemma.

LEMMA 4. *Let $G \in \mathcal{G}$ be a commutator and $\xi \in \mathbb{C}^n$ be a weak ε-approximate fixed point of $\mathcal{G}$. Then*

$$\|G\xi - \xi\| \leqslant 4\sqrt{2\varepsilon}.$$

Proof. Write a commutator $G \in \mathcal{G}$ as $G = [A,B] = ABA^*B^*$ for some $A,B \in \mathcal{G}$. We estimate $\|G\xi - \xi\|$ using Proposition 2:

$$\begin{aligned} \|G\xi - \xi\| &\leqslant \|ABA^*(B^* - \overline{\lambda}_B I_n)\xi\| + \|ABA^*\overline{\lambda}_B\xi - \xi\| \\ &\leqslant \sqrt{2\varepsilon} + \|AB\overline{\lambda}_B(A^* - \overline{\lambda}_A I_n)\xi\| + \|AB\overline{\lambda}_B\overline{\lambda}_A\xi - \xi\| \\ &\leqslant 2\sqrt{2\varepsilon} + \|A\overline{\lambda}_B\overline{\lambda}_A(B - \lambda_B I_n)\xi\| + \|A\overline{\lambda}_A\xi - \xi\| \leqslant 4\sqrt{2\varepsilon}. \quad \square \end{aligned}$$

The following corollary is a consequence of Theorem 3 and Proposition 1. The notation is borrowed from Theorem 3, in particular, $\mathcal{H} \subseteq \mathcal{G}$ is the set of all commutators in a group $\mathcal{G}$.

COROLLARY 5. *Given $n \in \mathbb{N}$, let $0 < \varepsilon < \dfrac{1}{32}$. If a compact group of unitaries $\mathcal{G} \subseteq M_n(\mathbb{C})$ has a weak ε-approximate fixed point ξ, then $\mathcal{H}$ has a fixed vector $\eta \neq 0$ satisfying $\|\eta - \xi\| \leqslant 4\sqrt{2\varepsilon}$.*

THEOREM 6. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given, and assume that $n \geqslant 2$ and that $\varepsilon < \dfrac{1}{32n^2}$. Let $\mathcal{G} \subseteq M_n(\mathbb{C})$ be a subgroup of unitary matrices such that $\mathcal{G} \subseteq \mathbb{C}\mathcal{H}$.*

If $\mathcal{G}$ has a weak ε-approximate fixed point $\xi \in \mathbb{C}^n$, then $\mathcal{G}$ has a common eigenvector η, such that $\|\xi - \eta\| \leqslant 4\sqrt{2\varepsilon} < \dfrac{1}{n}$.

Proof. Assume with no loss of generality that $\mathcal{G} = \overline{\mathbb{T} \cdot \mathcal{G}}$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and "bar" denotes the closure in the Euclidean topology. Now note that it is sufficient to prove the result for $\mathcal{G}_0 = \{G \in \mathcal{G} \mid \det(G) = 1\}$. This is because $\mathcal{G}_0$ also satisfies the assumptions of the theorem and because every element of $\mathcal{G}$ is a $\mathbb{T}$-multiple of an element of $\mathcal{G}_0$.

Choose a $G \in \mathcal{G}_0$ and write

$$G = \lambda[A,B] \tag{3}$$

for some $A,B \in \mathcal{G}$ and a scalar λ. Note that $\lambda I = G[A,B]^{-1} \in \mathcal{G}_0$ and hence $\lambda^n = 1$.

By Lemma 4 we have that

$$\|[A,B]\xi - \xi\| \leqslant 4\sqrt{2\varepsilon}$$

implying that

$$\|G\xi - \lambda\xi\| = \|\lambda[A,B]\xi - \lambda\xi\| \leqslant 4\sqrt{2\varepsilon}. \tag{4}$$

Since the adjacent n-th roots of unity are at the distance

$$d_n = \left|e^{2\pi i/n} - 1\right| \geqslant \frac{4}{n},$$

and

$$4\sqrt{2\varepsilon} < 4\sqrt{2\frac{1}{32n^2}} = \frac{1}{n} < \frac{1}{2}d_n,$$

we conclude that λ defined by (3) is unique. Consequently, we may define a map ρ going from $\mathcal{G}_0$ to $\mathcal{G}_0$, defined by $\rho\colon G \mapsto \lambda I$. We will show that this map is a group homomorphism.

Indeed, choose $G = \lambda[A,B]$ for some $A,B \in \mathcal{G}_0$ and a scalar $\lambda = \rho(G)$ and choose $\widetilde{G} = \widetilde{\lambda}\,[\widetilde{A},\widetilde{B}]$ for some $\widetilde{A},\widetilde{B} \in \mathcal{G}_0$ and a scalar $\widetilde{\lambda} = \rho(\widetilde{G})$. Then,

$$\begin{aligned}\|G\widetilde{G}\xi - \lambda\widetilde{\lambda}\xi\| &= \|(\lambda[A,B]\widetilde{\lambda}[\widetilde{A},\widetilde{B}]\xi - \lambda\widetilde{\lambda}\xi)\| = \|\lambda\widetilde{\lambda}([A,B][\widetilde{A},\widetilde{B}]\xi - \xi)\| \\ &\leqslant \|[A,B]([\widetilde{A},\widetilde{B}]\xi - \xi)\| + \|[A,B]\xi - \xi\| \leqslant 8\sqrt{2\varepsilon} < \frac{2}{n}\end{aligned}$$

Hence

$$\begin{aligned}\left|\rho(G\widetilde{G}) - \rho(G)\rho(\widetilde{G})\right| &= \left\|\left(\rho(G\widetilde{G}) - \rho(G)\rho(\widetilde{G})\right)\xi\right\| \\ &\leqslant \left\|\rho(G\widetilde{G})\xi - (G\widetilde{G})\xi\right\| + \left\|(G\widetilde{G})\xi - \rho(G)\rho(\widetilde{G})\xi\right\| \\ &< \frac{1}{n} + \frac{2}{n} < d_n.\end{aligned}$$

Thus $\rho(G)\rho(\widetilde{G}) = \lambda\widetilde{\lambda} = \rho(G\widetilde{G})$. As in the proof of Proposition 1 let μ be the Haar measure on $\mathcal{G}_0$ and introduce

$$\eta = \int_{G\in\mathcal{G}_0} \overline{\rho(G)}G\xi\,d\mu,$$

so that

$$H\eta = \int_{G\in\mathcal{G}_0} \overline{\rho(G)}HG\xi\,d\mu = \rho(H)\int_{G\in\mathcal{G}_0} \overline{\rho(HG)}HG\xi\,d\mu = \rho(H)\eta$$

for all $H \in \mathcal{G}_0$, implying that η, if nonzero, is a common eigenvector of all elements of the group. Also

$$\|\eta - \xi\| = \sup_{\|\vartheta\|\leqslant 1} |\langle \eta - \xi | \vartheta\rangle| = \sup_{\|\vartheta\|\leqslant 1}\left|\int_{G\in\mathcal{G}_0} \overline{\rho(G)}\langle G\xi - \rho(G)\xi | \vartheta\rangle\,d\mu\right| \leqslant 4\sqrt{2\varepsilon}$$

as desired. □

4. Monomial groups

In the following proposition we need a version of the well-known Hardy-Littlewood-Pólya theorem [6], given as Theorem 368 on p. 261 of [7] (the inequality is often called the "Rearrangement Inequality"). Let $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$ and $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_n \geqslant 0$ be two n-tuples of reals and let π be a permutation of n indices. Then,

$$\sum_{i=1}^{n} x_i y_{\pi(i)} \leqslant \sum_{i=1}^{n} x_i y_i. \tag{5}$$

In what follows we will denote by $|\xi|$ the vector whose entries consist of the absolute values of the corresponding entries of ξ. Similarly, $|G|$ will denote the matrix whose entries are equal to the absolute values of the corresponding entries of $G \in M_n(\mathbb{C})$. For a group $\mathcal{G} \subseteq M_n(\mathbb{C})$ we will denote by $|\mathcal{G}|$ the set of all $|G|$ for $G \in \mathcal{G}$. Recall that a unitary group $\mathcal{G}$ is *monomial* whenever $|\mathcal{G}|$ consists of permutation matrices, which implies that it is a permutation group. Furthermore, recall that a group of matrices is called *indecomposable* if it has no nontrivial invariant subspace spanned by a subset of the standard basis vectors.

In the following proposition we will need the vector

$$\eta = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{6}$$

PROPOSITION 7. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given, and let η be as defined by (6). If an indecomposable monomial unitary group $\mathcal{G} \subseteq M_n(\mathbb{C})$ has a weak ε-approximate fixed point ξ, then:*

(a) *η is a fixed point of $|\mathcal{G}|$ and $\| \eta - |\xi| \| \leqslant \varepsilon_1$, where $\varepsilon_1 = n\sqrt{n\varepsilon}$;*

(b) *if $\varepsilon_1 < \dfrac{1}{3}$ and $\xi = |\xi|$, then η is a weak ε_2-approximate fixed point of $\mathcal{G}$, where $\varepsilon_2 = 3\varepsilon_1$;*

(c) *if $\varepsilon_1 < \dfrac{1}{3}$ then there is a weak ε_2-approximate fixed point ζ of $\mathcal{G}$, where $\varepsilon_2 = 3\varepsilon_1$, and $|\zeta| = \eta$.*

Proof. Assume with no loss that $\xi = |\xi|$ (otherwise use an appropriate unitary diagonal similarity on the group). Then, assume with no loss that $\xi_1 \geqslant \xi_2 \geqslant \cdots \geqslant \xi_n \geqslant 0$ (otherwise use an appropriate permutational similarity which is necessarily unitary). Now, any $G \in \mathcal{G}$ is determined by a permutation π of n indices and a choice of n complex numbers $\gamma_1, \gamma_2, \ldots, \gamma_n$ of modulus one, so that for any $x \in \mathbb{C}^n$ we have $(Gx)_i = \gamma_i x_{\pi(i)}$. By the definition of a weak approximate fixed point and (6) we get

$$1 - \varepsilon \leqslant |\langle G\xi, \xi \rangle| \leqslant \left| \sum_{i=1}^{n} \gamma_i \xi_{\pi(i)} \xi_i \right| \leqslant 1.$$

The fact that $\mathcal{G}$ is indecomposable implies that the permutation group corresponding to $|\mathcal{G}|$ acts transitively on the standard basis vectors. So, for any $k, 1 \leqslant k < n$, there exists a permutation π corresponding to a certain $|G| \in |\mathcal{G}|$ such that $\pi(k) = k+1$, implying

$$1-\varepsilon \leqslant \xi_k \xi_{k+1} + \sum_{\substack{i=1 \\ i\neq k}}^{n} \xi_i \xi_{\pi(i)} \leqslant 2\xi_k \xi_{k+1} + \sum_{\substack{i=1 \\ i\neq k,k+1}}^{n} \xi_i^2 .$$

In the first sum above we use (5) with x_i's equal to ξ_i's after omitting ξ_k and y_i's equal to ξ_i's after omitting ξ_{k+1} in order to get the estimate. Recall that $\sum_{i=1}^{n} \xi_i^2 = 1$ to get, after rearranging,

$$\xi_k^2 + \xi_{k+1}^2 - 2\xi_k \xi_{k+1} \leqslant \varepsilon,$$

so that $0 \leqslant \xi_k - \xi_{k+1} \leqslant \sqrt{\varepsilon}$. It follows that $0 \leqslant \xi_1 - \xi_n = \sum_{k=1}^{n-1}(\xi_k - \xi_{k+1}) \leqslant (n-1)\sqrt{\varepsilon}$, implying finally

$$\max_{1\leqslant i,j\leqslant n} |\xi_i - \xi_j| \leqslant (n-1)\sqrt{\varepsilon}. \tag{7}$$

We now want to show that the vector ξ is close to a fixed point of the group $|\mathcal{G}|$. Actually, we may choose for that purpose the vector η defined by (6). The simple observation $n\xi_1^2 \geqslant \sum_{i=1}^{n} \xi_i^2 = 1 \geqslant n\xi_n^2$ yields $\xi_1 \geqslant \eta_i \geqslant \xi_n$ for all $i, 1 \leqslant i \leqslant n$, so that (7) implies $|\eta_i - \xi_i| \leqslant \max\limits_{1\leqslant j\leqslant n} |\xi_i - \xi_j| \leqslant (n-1)\sqrt{\varepsilon}$, for all $i, 1 \leqslant i \leqslant n$, and consequently

$$\|\eta - \xi\| \leqslant (n-1)\sqrt{n\varepsilon} < n\sqrt{n\varepsilon} = \varepsilon_1$$

yielding **(a)**. This expression also implies that

$$\Big| |\langle G\xi, \eta\rangle| - |\langle G\xi, \xi\rangle| \Big| \leqslant |\langle G\xi, \eta - \xi\rangle| \leqslant \varepsilon_1$$

and similarly $\Big| |\langle G\eta, \eta\rangle| - |\langle G\xi, \eta\rangle| \Big| \leqslant \varepsilon_1$. So,

$$\Big| |\langle G\xi, \xi\rangle| - |\langle G\eta, \eta\rangle| \Big| \leqslant 2\varepsilon_1,$$

and the fact that $\varepsilon \leqslant \varepsilon_1$ now implies that

$$|\langle G\eta, \eta\rangle| \geqslant 1 - \varepsilon_2,$$

where $\varepsilon_2 = 3\varepsilon_1$, thus showing **(b)**. (Note that $\xi = |\xi|$ was assumed "with no loss" at the beginning of this proof.) **(c)** The general case is now obtained using a unitary diagonal similarity on the group. If this similarity is given by a diagonal U, then $\zeta = U\eta$ yields the desired result. □

LEMMA 8. *Let* $\varphi \in [-\pi, \pi]$. *Then*

$$\frac{1}{2}|\varphi| \leqslant \frac{2}{\pi}|\varphi| \leqslant |e^{\varphi i} - 1| \leqslant |\varphi|. \tag{8}$$

Proof. Clearly, it suffices to give the proof only for $\varphi \geqslant 0$ since $|e^{-\varphi i}-1| = |e^{\varphi i}-1|$. Observe that the function $f(\varphi) = 2\sin\left(\frac{\varphi}{2}\right)$ is convex upwards on $[0,\pi]$ since its second derivative is nonpositive on this interval. This implies that the graph of the curve lies above the line segment connecting any two points on the graph and, in particular, the endpoints of this interval. So, $\frac{2}{\pi}\varphi \leqslant f(\varphi) = 2\sin\left(\frac{\varphi}{2}\right)$ since $\varphi \geqslant 0$, and the second of the three inequalities in (8) follows. Now, the first one is obvious and the third follows from the fact that a line segment is shorter than the arc connecting the same points. $\square$

LEMMA 9. *Let $\varphi_j \in [-\pi,\pi]$ for $j = 1,2,\ldots,k$. If*

$$\left|1+\sum_{j=1}^{k} e^{\varphi_j i}\right| \geqslant (k+1)(1-\varepsilon); \qquad (0<\varepsilon<1) \tag{9}$$

then

$$\sum_{j=1}^{k} |\varphi_j| < \pi\sqrt{k}(k+1)\sqrt{\frac{\varepsilon}{2}}. \tag{10}$$

Proof. Let $e^{\varphi_j i} = x_j + iy_j$, $x_j, y_j \in \mathbb{R}$ for $j = 1,2,\ldots,k$. We first want to show that (9) implies

$$\frac{2}{\pi^2}\sum_{j=1}^{k} |\varphi_j|^2 \leqslant k - \sum_{j=1}^{k} x_j. \tag{11}$$

To this end observe that for any $j \in \{1,2,\ldots,k\}$ and $x'_j = 1 - x_j$ we have $2x'_j = x'^2_j + y_j^2 = |e^{\varphi_j i}-1|^2 \geqslant \frac{4}{\pi^2}|\varphi_j|^2$ by Lemma 8. We sum these estimates and divide by 2 to get (11).

Clearly

$$\begin{aligned}\left|1+\sum_{j=1}^{k} e^{\varphi_j i}\right|^2 &= \left(1+\sum_{j=1}^{k} x_j\right)^2 + \left(\sum_{j=1}^{k} y_j\right)^2 \\ &= \left(1+2\sum_{j=1}^{k} x_j\right) + \sum_{j=1}^{k}\sum_{l=1}^{k}(x_j x_l + y_j y_l).\end{aligned}$$

Next, observe that $x_j x_l + y_j y_l = \mathrm{Re}\left(e^{\varphi_j i}\overline{e^{\varphi_l i}}\right) \leqslant 1$ for all $j,l = 1,2,\ldots,k$, so that

$$\left|1+\sum_{j=1}^{k} e^{\varphi_j i}\right|^2 - (k+1)^2 \leqslant 2\left(\sum_{j=1}^{k} x_j - k\right).$$

Combine this estimate with (9) and (11) to get

$$\frac{4}{\pi^2}\sum_{j=1}^{k} |\varphi_j|^2 \leqslant (k+1)^2 2\varepsilon, \tag{12}$$

where we have also used the simple estimate $1-(1-\varepsilon)^2 \leqslant 2\varepsilon$. So, finally

$$\sum_{j=1}^{k}|\varphi_j| \leqslant \left(k\sum_{j=1}^{k}|\varphi_j|^2\right)^{1/2} < \pi\sqrt{k}(k+1)\sqrt{\frac{\varepsilon}{2}}$$

and the desired Inequality (10) follows. □

In the following proposition we need the standard notations $\|\xi\|_1 = \sum_{i=1}^{n}|\xi_i|$ and $\|\xi\|_2^2 = \sum_{i=1}^{n}|\xi_i|^2$ for any $\xi \in \mathbb{C}^n$.

PROPOSITION 10. *Let* $0 < \varepsilon < \dfrac{2}{n^3}$ *and let* $g_1,\ldots,g_n \in \mathbb{T}$ *be such that*

$$g_1g_2\cdots g_n = 1, \quad \textit{and}$$

$$|g_1+\cdots+g_n| \geqslant n(1-\varepsilon).$$

Then there exists $\alpha \in \mathbb{T}$ *such that* $\alpha^n = 1$, *and*

$$\|(g_1,\ldots,g_n)-\alpha(1,\ldots,1)\|_1 < \pi n\sqrt{2n\varepsilon} \quad \text{and}$$
$$\|(g_1,\ldots,g_n)-\alpha(1,\ldots,1)\|_2 < \pi n\sqrt{2\varepsilon}.$$

Proof. Write $k = n-1$, and let $\varphi_j \in (-\pi,\pi]$ be such that

$$\frac{g_{j+1}}{g_1} = e^{\varphi_j i}$$

for $j = 1,\ldots,k$. Moreover, let $\alpha \in \mathbb{T}$ and $\varphi \in (-\pi/n,\pi/n]$ be such that $\alpha^n = 1$ and $g_1 = \alpha e^{\varphi i}$. By Lemma 9 we have that

$$\sum_{j=1}^{k}|\varphi_j| < \pi\sqrt{k}(k+1)\sqrt{\frac{\varepsilon}{2}} < \pi\sqrt{\frac{n^3\varepsilon}{2}} < \pi. \tag{13}$$

From $g_1\cdots g_n = 1$ we then get that $n\varphi+\varphi_1+\cdots+\varphi_k = 0$ (since the inequalities in (13) and the fact that $n|\varphi| \leqslant \pi$ give $-2\pi < n\varphi+\varphi_1+\cdots+\varphi_k < 2\pi$). So, by (13) again, we get

$$n|\varphi| \leqslant \sum_{j=1}^{k}|\varphi_j| < \pi\sqrt{\frac{n^3\varepsilon}{2}}. \tag{14}$$

Hence we have

$$\|(g_1,\ldots,g_n)-\alpha(1,1,\ldots,1)\|_1 \leqslant |\varphi| + \sum_{j=1}^{k}|\varphi+\varphi_j| \leqslant n|\varphi| + \sum_{j=1}^{k}|\varphi_j| < \pi\sqrt{2n^3\varepsilon},$$

where we have first used (8), and then (13) and (14).

The inequality involving the 2-norm is verified as follows:

$$\begin{aligned}\|(g_1,\ldots,g_n)-\alpha(1,1,\ldots,1)\|_2^2 &\leqslant |\varphi|^2+\sum_{j=1}^{k}|\varphi+\varphi_j|^2\\ &\leqslant |\varphi|^2+2\sum_{j=1}^{k}(|\varphi|^2+|\varphi_j|^2)\\ &=(2n-1)|\varphi|^2+2\sum_{j=1}^{k}|\varphi_j|^2<2\pi^2n^2\varepsilon.\end{aligned}$$

using (12) (recalling that $k+1=n$) and (14). □

THEOREM 11. *Let $n\in\mathbb{N}$ and $\varepsilon>0$ be given. If $n\geqslant 2$ and $\varepsilon<\frac{1}{3600n^{11}}$ then every indecomposable monomial unitary group $\mathcal{G}\subseteq M_n(\mathbb{C})$ with a weak ε-approximate fixed point ξ has a common eigenvector ζ such that $\|\xi-\zeta\|<\frac{1}{n}$.*

Proof. Assume first with no loss that $\xi=|\xi|$. We then use Proposition 7**(b)** to see that η defined by (6) is a weak $\widetilde{\varepsilon}$-approximate fixed point for

$$\widetilde{\varepsilon}=3n\sqrt{n\varepsilon}<\frac{3}{60}\sqrt{\frac{n^3}{n^{11}}}=\frac{1}{20n^4}.$$

We assume, with no loss, that $\mathcal{G}$ is compact and $\mathbb{T}$-homogeneous, i.e., that $\mathcal{G}=\overline{\mathbb{T}\mathcal{G}}$.

Let $\mathcal{G}_0$ be the subgroup of $\mathcal{G}$ consisting of those elements of $\mathcal{G}$ whose weights multiply to 1, i.e., $\mathcal{G}_0$ is the kernel of the group homomorphism $G\mapsto\det(G)\operatorname{sign}(|G|)$ (here we abuse the notation and use $\operatorname{sign}(|G|)\in\{\pm1\}$ to denote the parity of the permutation associated to $|G|$). One can think of this homomorphism as assigning to every matrix of $\mathcal{G}$ the product of its non-zero entries. Note that $\mathcal{G}_0$ is large in the sense that every element of $\mathcal{G}$ is a scalar multiple of an element of $\mathcal{G}_0$.

Next, denote by Ω_n the group of all complex n-th roots of unity. We want to prove the existence and uniqueness of a mapping $\alpha\colon\mathcal{G}_0\to\Omega_n$ such that for all $G\in\mathcal{G}_0$ we have that

$$\|G\eta-\alpha(G)\eta\|<\varepsilon'=\pi n\sqrt{2\widetilde{\varepsilon}}.$$

Indeed, for any such G there exist $g_1,\ldots,g_n\in\mathbb{T}$ such that $g_1g_2\cdots g_n=1$, and $G=\operatorname{Diag}(g_1,\ldots,g_n)|G|$ by the construction above. By Proposition 7**(b)** we have

$$|\langle G\eta,\eta\rangle|\geqslant 1-\widetilde{\varepsilon},$$

which implies that $|g_1+\ldots+g_n|\geqslant n(1-\widetilde{\varepsilon})$. By Proposition 10 there exists an $\alpha(G)\in\Omega_n$ such that

$$\|G\eta-\alpha(G)\eta\|<\varepsilon'<\frac{\pi}{\sqrt{10}n}<\frac{1}{n}.$$

Now, if d_n is the distance between the adjacent n-th roots of unity, then, by Lemma 8, $d_n \geqslant 4/n$. Then α is unique: Indeed, if there were a $\beta(G) \in \Omega_n$ with the same property, we would get

$$|\alpha(G)-\beta(G)| = \|\alpha(G)\eta-\beta(G)\eta\| \leqslant \|G\eta-\beta(G)\eta\|+\|G\eta-\alpha(G)\eta\| \\ < 2\varepsilon' < d_n$$

yielding $\alpha(G)=\beta(G)$. But this is true for every G.

Next we want to show that α is a group homomorphism. Clearly

$$\|GH\eta-\alpha(G)\alpha(H)\eta\| \leqslant \|G(H\eta-\alpha(H)\eta)\|+\|\alpha(H)(G\eta-\alpha(G)\eta)\| < 2\varepsilon'$$

and since also $\|GH\eta-\alpha(GH)\eta\| \leqslant \varepsilon'$, we get

$$|\alpha(GH)-\alpha(G)\alpha(H)| < 3\varepsilon' < d_n$$

so that $\alpha(GH)=\alpha(G)\alpha(H)$ as desired.

Let $\mathcal{G}_1$ be the kernel of α and note that η is an ε'-approximate fixed point for $\mathcal{G}_1$. Hence by Proposition 1 there is a nonzero fixed vector ζ for $\mathcal{G}_1$ with $\|\zeta-\eta\| < \varepsilon'$. By considerations above, respectively by Proposition 7**(a)**, we have

$$\|\zeta-\eta\| \leqslant \varepsilon' = \pi\sqrt{6}\sqrt[4]{n^7\varepsilon} \leqslant \frac{\pi}{n\sqrt{10}}, \text{ respectively } \|\eta-\xi\| \leqslant n\sqrt{n\varepsilon} \leqslant \frac{1}{60n^4}.$$

Since α is Ω_n-homogeneous we have that every element of $\mathcal{G}_0$ and hence also every element of $\mathcal{G}$ is a scalar multiple of an element of $\mathcal{G}_1$. Hence ζ is a common eigenvector for $\mathcal{G}$. The above two estimates combined give the desired inequality $\|\xi-\zeta\| < \frac{1}{n}$. □

5. Block decompositions

Let $\mathcal{G}$ be a group of unitary $n\times n$ matrices with a weak ε-approximate point ξ. Assume that $\mathcal{G}$ is either block diagonal or block monomial with respect to orthogonal decomposition (for the definition of a block-monomial group we refer the reader to the first paragraph of [11, Section 2.2])

$$\mathcal{V} = \mathcal{V}_1 \overset{\perp}{\oplus} \mathcal{V}_2 \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} \mathcal{V}_k.$$

Here, the spaces $\mathcal{V}_i$ may be of different dimensions. We do not assume that $\mathcal{G}$ is irreducible. In particular, if $\mathcal{G}$ is block-diagonal, it can only be irreducible if $k=1$. For $i=1,\ldots,k$ we write $n_i=\dim(\mathcal{V}_i)$, and we let ξ_i be the i-th component of ξ with respect to this decomposition. Furthermore, introduce

$$\mathcal{G}_{ii} = \{G|_{\mathcal{V}_i} \,|\, G\in\mathcal{G}, G(\mathcal{V}_i)\subseteq\mathcal{V}_i\} \subseteq M_{n_i}(\mathbb{C});$$

we view the groups $\mathcal{G}_{ii}$ either as diagonal blocks of those members of $\mathcal{G}$ for which $\mathcal{V}_i$ is invariant or as groups of linear mappings from $\mathcal{V}_i$ to itself. Clearly, for some $i\in\{1,\ldots,k\}$ we have that $\xi_i\neq 0$, since the sum of the squares of their norms equals 1.

The following lemma is useful in both situations described above.

LEMMA 12. **(a)** *If $\xi_i \neq 0$ for some $i \in \{1,\dots,k\}$ and $\varepsilon_i = \dfrac{\varepsilon}{\|\xi_i\|^2}$, then $\widetilde{\xi}_i = \dfrac{\xi_i}{\|\xi_i\|}$ is a weak ε_i-approximate fixed point for $\mathcal{G}_{ii}$.*

(b) *Let $a_1,\dots,a_k$ be non-negative real numbers such that $a_1+\dots+a_k = 1$. Then there exists an $i \in \{1,\dots,k\}$ such that $a_i \geqslant (n_i/n)$.*

(c) *There always exists an $i \in \{1,\dots,k\}$ such that $\widetilde{\xi}_i = \dfrac{\xi_i}{\|\xi_i\|} \neq 0$ is a weak $\widetilde{\varepsilon}$-approximate fixed point for $\mathcal{G}_{ii}$, where $\widetilde{\varepsilon} = \dfrac{n}{n_i}\varepsilon$.*

Proof. **(a)** Let $G \in \mathcal{G}$ be such that $G(\mathcal{V}_i) \subseteq \mathcal{V}_i$ and let $H = G|_{\mathcal{V}_i}$. For $j \neq i$, let $\pi_j \neq i$ be the unique index such that $G(\mathcal{V}_j) \subseteq \mathcal{V}_{\pi_j}$. Then

$$\begin{aligned}(1-\varepsilon) \leqslant |\langle G\xi,\xi\rangle| &\leqslant \left|\langle H\xi_i,\xi_i\rangle + \sum_{j\neq i}\langle G\xi_j,\xi_{\pi_j}\rangle\right| \\ &\leqslant |\langle H\xi_i,\xi_i\rangle| + \sum_{j\neq i}\|\xi_j\|\cdot\|\xi_{\pi_j}\| \\ &\leqslant |\langle H\xi_i,\xi_i\rangle| + \sum_{j\neq i}\|\xi_j\|^2 \\ &= |\langle H\xi_i,\xi_i\rangle| + 1 - \|\xi_i\|^2\end{aligned}$$

(we used the Rearrangement Inequality mentioned at the beginning of Section 3 to go from the second to the third line). Hence $|\langle H\xi_i,\xi_i\rangle| \geqslant \|\xi_i\|^2 - \varepsilon$ and therefore

$$|\langle H\widetilde{\xi}_i,\widetilde{\xi}_i\rangle| \geqslant 1 - \frac{\varepsilon}{\|\xi_i\|^2} = 1-\varepsilon_i.$$

(b) Suppose, if possible, that for every i we have $a_i < n_i/n$. Then $a_1+\dots+a_n < (n_1/n)+\dots+(n_k/n) = 1$. A contradiction.

(c) Immediate consequence of **(a)** and **(b)**. $\square$

THEOREM 13. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. If $n \geqslant 2$ and $\varepsilon < \dfrac{1}{3600n^{11}}$, then every monomial unitary group $\mathcal{G} \subseteq M_n(\mathbb{C})$ with a weak ε-approximate fixed point ξ has a common eigenvector ζ.*

Proof. If $\mathcal{G} \subseteq M_n(\mathbb{C})$ is indecomposable, we are done by Theorem 11. Otherwise, decompose the space and the group into indecomposable components, and then also the corresponding weak ε-approximate fixed point ξ. By Lemma 12**(c)** we choose an $i \in \{1,\dots,k\}$ such that $\widetilde{\xi}_i = \dfrac{\xi_i}{\|\xi_i\|} \neq 0$ is a weak $\widetilde{\varepsilon}$-approximate fixed point for $\mathcal{G}_{ii}$, where $\widetilde{\varepsilon} = \dfrac{n}{n_i}\varepsilon$, and n_i is the dimension of the underlying space of the group $\mathcal{G}_{ii}$. We conclude that

$$\widetilde{\varepsilon} = \frac{n}{n_i}\varepsilon \leqslant \frac{n}{3600n_in^{11}} = \frac{1}{3600n_in^{10}} \leqslant \frac{1}{3600n_i^{11}},$$

so that $\mathcal{G}_{ii}$ has a common eigenvector by Theorem 11, denoted by ζ. Let $\widehat{\zeta}$ be the vector of the starting space whose i-th component equals ζ and the rest of whose components are zero. Clearly, this is a common eigenvector for $\mathcal{G}$. □

REMARK. We will need the following observation about decompositions. Assume a decomposition of a finite-dimensional vector space $\mathcal{V} = \mathbb{C}^n$ into a direct sum

$$\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \cdots \oplus \mathcal{V}_r.$$

Let a unitary group $\mathcal{N}$ be block diagonal with respect to this decomposition. Then we may assume, with no loss of generality, that this sum is orthogonal. Indeed, choose any basis respecting this decomposition, and then orthogonalize this basis using the Gram-Schmidt process. Clearly $\mathcal{N}$ is still upper-triangular with respect to the associated "new", now orthogonal, decomposition

$$\mathcal{V} = \mathcal{V}_1' \overset{\perp}{\oplus} \mathcal{V}_2' \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} \mathcal{V}_r';$$

since $\mathcal{N}$ is unitary we conclude that $\mathcal{N}$ must actually be block-diagonal with respect to this orthogonal decomposition as well.

6. General finite groups

In the proof of the following theorem we need Clifford's theorem. A version of it that is close to our point of view is given in [11, Theorem 2.3].

THEOREM 14. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. If $n \geqslant 2$ and $\varepsilon < \dfrac{1}{3600n^{11}}$, then every finite unitary group $\mathcal{G} \subseteq M_n(\mathbb{C})$ with a weak ε-approximate fixed point is reducible.*

Proof. We will prove this by contradiction. Assume the contrary, so that there exist $n \geqslant 2$, $\varepsilon > 0$ with $\varepsilon < \dfrac{1}{3600n^{11}}$, and an irreducible finite unitary group $\mathcal{G} \subseteq M_n(\mathbb{C})$ with a weak ε-approximate fixed point ξ. Choose the smallest possible n for which such a group exists. From among all such groups choose one of the smallest possible order, and call it $\mathcal{G}$.

We want to show that every proper normal subgroup $\mathcal{N}$ of $\mathcal{G}$ consists of scalars only. Since $\mathcal{N}$ is strictly smaller than $\mathcal{G}$ and has ξ as a weak ε-approximate fixed point, it is reducible by assumption, so we can use [11, Theorem 2.3]. Then $\mathcal{V} = \mathbb{C}^n$ decomposes into a direct sum

$$\mathcal{V} = \mathcal{V}_1 \overset{\perp}{\oplus} \mathcal{V}_2 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} \mathcal{V}_r; \qquad r \geqslant 1,$$

of invariant subspaces for $\mathcal{N}$, maximal with the property (when translated from the group-representations language) that for any $i = 1, \ldots, r$ and for any $\mathcal{N}$-irreducible subspaces $\mathcal{U}_i, \mathcal{U}_i' \subseteq \mathcal{V}_i$ the representations $g \in \mathcal{N} \to g|_{\mathcal{U}_i}$ and $g \in \mathcal{N} \to g|_{\mathcal{U}_i'}$ are isomorphic. By the cited theorem the spaces $\mathcal{V}_i$ are of equal dimension, necessarily $\widetilde{n} = \frac{n}{r}$. Furthermore, since $\mathcal{N}$ is unitary, we can assume that the decomposition above is orthogonal by the remark of Section 5. We decompose ξ according to this decomposition, denoting the components by ξ_i, and choose an $i \in \{1, \ldots, k\}$ by Lemma 12**(c)** such that $\widetilde{\xi} = \frac{\xi_i}{\|\xi_i\|} \neq 0$ is a weak $\tilde{\varepsilon}$-approximate fixed point for $\widetilde{\mathcal{G}} = \mathcal{G}_{ii}$, where $\tilde{\varepsilon} = \frac{n}{n_i}\varepsilon = r\varepsilon$. Since

$$\widetilde{\varepsilon} = r\varepsilon < \frac{n}{3600\widetilde{n}n^{11}} = \frac{1}{3600\widetilde{n}n^{10}} \leqslant \frac{1}{3600\widetilde{n}^{11}},$$

we have found a group $\widetilde{\mathcal{G}}$ that satisfies the assumptions of the theorem with $\widetilde{n}$ and $\widetilde{\varepsilon}$ instead of n and ε. The possibility $n > \widetilde{n}$ implies, by [11, Theorem 2.3 and Proposition 2.2], that $\widetilde{\mathcal{G}}$ is irreducible. This contradicts the starting assumption of this proof, thus proving that $r = 1$.

The decomposition of $\mathcal{V}$ into irreducible invariant subspaces of $\mathcal{N}$

$$\mathcal{V} = \mathcal{U}_1 \overset{\perp}{\oplus} \mathcal{U}_2 \overset{\perp}{\oplus} \ldots \overset{\perp}{\oplus} \mathcal{U}_s,$$

may be assumed orthogonal with no loss as above. Observe that the representations $g \in \mathcal{N} \mapsto g|_{\mathcal{U}_i}$ are isomorphic. Since they act on a strictly smaller dimensional space than $\mathcal{G}$ and satisfy the assumptions of the theorem, these blocks are reducible by the starting assumption of this proof, so that $n/s = 1$. Consequently, $\dim(\mathcal{U}_i) = 1$ implying that $\mathcal{N}$ is scalar.

We know now that every proper normal subgroup $\mathcal{N}$ of $\mathcal{G}$ consists of scalars only. If for such a maximal $\mathcal{N}$, the group $\mathcal{G}/\mathcal{N}$ is commutative, then by Suprunenko's theorem [11, Proposition 3.1] (cf. also [14, Theorem 24, p. 60]) $\mathcal{G}$ is unitarily monomializable, and we come to a contradiction with the irreducibility of $\mathcal{G}$ using Theorem 13. It remains to consider the case that $\mathcal{G}/\mathcal{N}$ is a noncommutative simple group. The longstanding Ore conjecture which states that every element of every finite (non-abelian) simple group is a commutator, was proved to be true in [10]. We apply this result to this quotient group to see that every element $G \in \mathcal{G}$ is of the form $G = \lambda[A,B]$ for some $A, B \in \mathcal{G}$ and a scalar λ. This brings us to a contradiction with the irreducibility of $\mathcal{G}$ by Theorem 6, and we are done. □

7. Connected groups of unitary matrices

LEMMA 15. *If $\mathcal{G}$ is a connected compact group of unitary matrices, then every element of its derived subgroup $\mathcal{G}'$, i.e. the closed subgroup generated by its commutators, is a commutator of two elements of $\mathcal{G}$.*

Proof. Since $\mathcal{G}'$ is a semisimple (cf. [15, Corollary 20.5.5]) connected compact Lie group, the claim is an immediate consequence of §4, No. 5, Proposition 10, Corollary of [4]. □

PROPOSITION 16. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. If $n \geqslant 2$ and $\varepsilon < \frac{1}{32}$, then every connected compact unitary group $\mathcal{G} \subseteq M_n(\mathbb{C})$ with a weak ε-approximate fixed point is reducible.*

Proof. Since commutative groups are always reducible, we may assume with no loss that $\mathcal{G}$ is non-commutative. By Lemma 15 every element of $\mathcal{G}'$ is a commutator of two elements of $\mathcal{G}$. So, by Corollary 5, $\mathcal{G}' \subseteq \mathcal{H}$ has a nonzero fixed vector, say ξ. Denote by $\mathcal{A}$ the linear span of $\mathcal{G}$, which is the same as the algebra generated by $\mathcal{G}$. Since $\mathcal{G}'$ is a normal subgroup of $\mathcal{G}$, we have that for any three elements $G,H,A \in \mathcal{G}$ the element $A^*H^*G^*HGA$ belongs to $\mathcal{G}'$, so that it fixes the vector ξ. It follows that

$$GHA\xi = HGA\xi. \tag{15}$$

This relation is then true for all $A \in \mathcal{A}$. Note that

$$\mathcal{V} := \{A\xi \,|\, A \in \mathcal{A}\}$$

is a non-trivial subspace of $\mathbb{C}^n$, invariant for $\mathcal{G}$. If we prove that it is a proper subspace, we are done. Assume the contrary. Then, any two elements $G,H \in \mathcal{G}$ commute on $\mathcal{V} = \mathbb{C}^n$, and $\mathcal{G}$ is a commutative group, contradicting the above. □

COROLLARY 17. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. If $n \geqslant 2$ and $\varepsilon < \frac{1}{32n}$ then every connected compact unitary group $\mathcal{G} \subseteq M_n(\mathbb{C})$ with a weak ε-approximate fixed point has a common eigenvector.*

Proof. By Proposition 16 the group is reducible and we are able to decompose the space, the group, and the corresponding weak ε-approximate fixed point ξ as in Section 5. By Lemma 12**(c)** there always exists an $i \in \{1,\ldots,k\}$ such that $\widetilde{\xi}_i = \frac{\xi_i}{\|\xi_i\|} \neq 0$ is a weak $\widetilde{\varepsilon}$-approximate fixed point for $\mathcal{G}_{ii}$, where $\widetilde{\varepsilon} = \frac{n}{n_i}\varepsilon \leqslant n\frac{1}{32n} = \frac{1}{32}$. So, by Proposition 16, such a $\mathcal{G}_{ii}$ acts on a one-dimensional subspace, whose unique (modulo a scalar of absolute value 1) orthonormal basis vector we denote by ζ. It is clear that this is also a common eigenvector for $\mathcal{G}$. □

LEMMA 18. *Let a compact unitary group $\mathcal{G} \subseteq M_n(\mathbb{C})$ be given such that $\mathcal{G} = \overline{\mathbb{T}\mathcal{G}}$ and $\mathcal{G} = \mathcal{G}_0\mathcal{N}$, where $\mathcal{N}$ is a normal connected subgroup and $\mathcal{G}_0$ is a finite subgroup of $\mathcal{G}$. If $n \geqslant 2$, $\varepsilon < \frac{1}{3600n^{11}}$, and $\mathcal{G}$ has a weak ε-approximate fixed point ξ, then it is reducible.*

Proof. Towards a contradiction we assume that $\mathcal{G}$ is irreducible. By Clifford's theorem $\mathcal{V} = \mathbb{C}^n$ decomposes into a direct sum

$$\mathcal{V} = \mathcal{V}_1 \overset{\perp}{\oplus} \mathcal{V}_2 \overset{\perp}{\oplus} \cdots \overset{\perp}{\oplus} \mathcal{V}_r,$$

of invariant subspaces for $\mathcal{N}$, maximal such that for all $i = 1, \ldots, r$ the representations $g \in \mathcal{N} \mapsto g|_{\mathcal{U}_i}$, which restrict to $\mathcal{N}$-irreducible subspaces of $\mathcal{V}_i$, are isomorphic. Observe that this definition makes each of the spaces $\mathcal{V}_i$ unique, and since they could have been assumed orthogonal to each other by the Remark of Section 5, this decomposition is indeed orthogonal. Also, the spaces $\mathcal{V}_i$ are of equal dimension, necessarily $\widetilde{n} = \frac{n}{r}$, the group $\mathcal{G}$ is block-monomial with respect to this decomposition, and all the groups $\mathcal{G}_{ii}$, viewed as groups of linear mappings from $\mathcal{V}_i$ to itself, are irreducible.

We also decompose ξ, denoting its components by ξ_i, and choose an $i \in \{1, \ldots, r\}$ by Lemma 12**(c)** such that $\widetilde{\xi}_i = \frac{\xi_i}{\|\xi_i\|} \neq 0$ is a weak $\widetilde{\varepsilon}_i$-approximate fixed point for $\mathcal{G}_{ii}$ (hence also for $\mathcal{N}|_{\mathcal{V}_i}$), where $\widetilde{\varepsilon}_i = \frac{n}{n_i}\varepsilon = r\varepsilon < \frac{1}{3600(n_i)^{11}}$. Thus, $\mathcal{N}|_{\mathcal{V}_i}$ has a common eigenvector by Corollary 17, so that the dimension of an $\mathcal{N}$-irreducible subspace in $\mathcal{V}_i$ (and consequently everywhere) is one. It follows that $\mathcal{N}|_{\mathcal{V}_i}$ consists of scalar multiples of the identity.

Assume $\dim(\mathcal{V}_i) \geqslant 2$. Note that the group $\mathcal{H}_0 = \{G|_{\mathcal{V}_i} \mid G \in \mathcal{G}_0, G(\mathcal{V}_i) \subseteq \mathcal{V}_i\}$ is finite and that $\mathcal{G}_{ii} = \mathbb{T}\mathcal{H}_0$. Since $\mathcal{G}_{ii}$ is irreducible, we have that $\mathcal{H}_0$ is also irreducible. But this contradicts Theorem 14 (since $\mathcal{H}_0$ has a weak $\widetilde{\varepsilon}_i$-approximate fixed point and $\widetilde{\varepsilon}_i < \frac{1}{3600(n_i)^{11}}$ for $n_i = \dim(\mathcal{V}_i)$).

If $\dim(\mathcal{V}_i) = 1$, then $\mathcal{G}$ is a monomial unitary group having a weak ε-approximate fixed point and so, by Theorem 13, cannot be irreducible. □

THEOREM 19. *Let $n \in \mathbb{N}$ and $\varepsilon > 0$ be given. If $n \geqslant 2$ and $\varepsilon < \frac{1}{3600n^{11}}$, then every compact unitary group $\mathcal{G} \subseteq M_n(\mathbb{C})$ with a weak ε-approximate fixed point is reducible.*

Proof. We assume with no loss of generality that $\mathcal{G} = \overline{\mathbb{T}\mathcal{G}}$ and denote by $\mathcal{N}$ the connected component of the identity. It is well known that there is a finite group $\mathcal{G}_0$ such that $\mathcal{G} = \mathcal{G}_0\mathcal{N}$ [2, Lemma 3], and hence we are done by Lemma 18. □

8. Common eigenvectors

In this section we present the main results of the paper. Let $\mathcal{G}$ be a group of unitary $n \times n$ matrices with a weak ε-approximate point ξ, with $\|\xi\| = 1$ and $0 < \varepsilon < 1$ as assumed throughout the paper. The following theorem follows easily from Theorem 19 and Lemma 12.

THEOREM 20. *Let $n \geqslant 2$ and $0 < \varepsilon < \frac{1}{3600n^{11}}$. If a group $\mathcal{G}$ of $n \times n$ unitary matrices has a weak ε-approximate fixed point, then $\mathcal{G}$ has a common eigenvector.*

Proof. Assume, using a unitary similarity if necessary, that $\mathcal{G}$ is block diagonal with irreducible blocks. For the corresponding decomposition we recall the notation of

Section 5. By Lemma 12(**c**) we choose an $i \in \{1,\dots,k\}$ such that $\widetilde{\xi}_i = \frac{\xi_i}{\|\xi_i\|} \neq 0$ is a weak $\widetilde{\varepsilon}$-approximate fixed point for $\mathcal{G}_{ii}$, where

$$\widetilde{\varepsilon} = \frac{n}{n_i}\varepsilon < \frac{1}{3600 n_i^{11}}.$$

In case $n_i \geqslant 2$, this would yield that $\mathcal{G}_{ii}$ is reducible by Theorem 19 which is impossible. Hence $n_i = 1$ and therefore ξ_i is a common eigenvector for $\mathcal{G}$. $\square$

LEMMA 21. *Let $x \geqslant y \geqslant 0$, $\alpha \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$, and let $\xi = e^{\alpha i}$. Then $|x+\xi y| \leqslant x$.*

Proof. We can suppose $y > 0$ otherwise there is nothing to do. With no loss assume that $\alpha \in \left[\pi, \frac{4\pi}{3}\right]$ (otherwise replace ξ by $\overline{\xi}$). Consider the triangle in $\mathbb{C}$ with vertices 0, x and $-\xi y$. The lengths of its sides are x, y, and $z = |x+\xi y|$. Since the angle opposite z is $\alpha - \pi \leqslant \frac{\pi}{3}$, we have that z cannot be the longest side of the triangle. $\square$

THEOREM 22. *Let $n \geqslant 2$, $0 < \varepsilon < \frac{1}{3600 n^{11}}$, and let a group of $n \times n$ unitary matrices $\mathcal{G}$ have a weak ε-approximate fixed point ξ. Then there exists a common eigenvector η (possibly $\|\eta\| \neq 1$) such that $\|\xi - \eta\|^2 < 3600 n^{11}\varepsilon$.*

Proof. With no loss we assume that $\mathcal{G} = \overline{\mathbb{T}\mathcal{G}}$. By Theorem 20 $\mathcal{G}$ has a common eigenvector. Let r be the greatest number of linearly independent eigenvectors of $\mathcal{G}$. With no loss of generality we assume that $\mathcal{G}$ is block diagonal with $k = r+1$ blocks, where the first r blocks are 1×1 and the k-th block has no common eigenvector (or that the k-th block is of size 0×0). For this decomposition recall the notation of Section 5. We first prove that $\|\xi_k\|^2 \leqslant 3600(n-r)^{11}\varepsilon$. If $\xi_k = 0$, then this is obvious. If $\xi_k \neq 0$, then, by Lemma 12(a), $\tilde{\xi}_k = \frac{\xi_k}{\|\xi_k\|}$ is a weak ε_k-approximate fixed point for $\mathcal{G}_{kk}$ with $\varepsilon_k = \frac{\varepsilon}{\|\xi_k\|^2}$. Hence $\varepsilon_k \geqslant \frac{1}{3600(n-r)^{11}}$ by Theorem 20 and therefore $\|\xi_k\|^2 \leqslant 3600(n-r)^{11}\varepsilon$.

For $a_i = \|\xi_i\|^2$, $i \in \{1,2,\dots,r\}$, assume with no loss that $a_1 \geqslant a_2 \geqslant \dots \geqslant a_r$. Let $G \in \mathcal{G}$, and suppose, if possible, that for some $i \in \{2,\dots,r\}$ we have that $G_{ii} \neq G_{11}$. Let m be an integer such that an argument α of $(G_{ii}\overline{G_{11}})^m$ is between $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ and let $H = (\overline{G_{11}}G)^m \in \overline{\mathbb{T}\mathcal{G}} = \mathcal{G}$ so that $H_{11} = 1$ and an argument of H_{ii} is between

$\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. Using Lemma 21 we then get

$$\begin{aligned}1-\varepsilon &\leqslant |\langle H\xi,\xi\rangle| \\ &\leqslant |\langle H_{11}\xi_1,\xi_1\rangle+\langle H_{ii}\xi_i,\xi_i\rangle|+\sum_{j\notin\{1,i\}}\|\xi_j\|^2 \\ &= |a_1+H_{ii}a_i|+(1-a_1-a_i) \\ &\leqslant a_1+(1-a_1-a_i)=1-a_i,\end{aligned}$$

and therefore $a_i \leqslant \varepsilon$. If also $a_1 \leqslant \varepsilon$ we define $s=1$ and $\eta=\xi_1$ otherwise we let s be the largest integer such that $a_s>\varepsilon$ and let $\eta=\xi_1+\ldots+\xi_s$. Now note that η is a common eigenvector for $\mathcal{G}$. Indeed, by the above, we have that for every $G\in\mathcal{G}$, we have $G_{11}=\ldots=G_{ss}$, so $\xi_1,\ldots,\xi_s$ are eigenvectors of G corresponding to the same eigenvalue. Now note that

$$\begin{aligned}\|\xi-\eta\|^2 &= \sum_{j=s+1}^{k}\|\xi_j\|^2 \\ &\leqslant (r-1)\varepsilon+3600(n-r)^{11}\varepsilon \\ &< 3600n^{11}\varepsilon. \quad \square\end{aligned}$$

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