

MAPS PRESERVING THE LOCAL SPECTRAL SUBSPACE OF SKEW-PRODUCT OF OPERATORS

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(Communicated by M. Omladić)

Abstract. Let $B(H)$ be the algebra of all bounded linear operators on an infinite-dimensional complex Hilbert space H . For $T \in B(H)$ and $\lambda \in \mathbb{C}$, let $H_T(\{\lambda\})$ denotes the local spectral subspace of T associated with $\{\lambda\}$. We prove that if $\varphi : B(H) \rightarrow B(H)$ be an additive map such that its range contains all operators of rank at most two and satisfies

$$H_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = H_{TS^*}(\{\lambda\})$$

for all $T, S \in B(H)$ and $\lambda \in \mathbb{C}$, then there exist a unitary operator V in $B(H)$ and a nonzero scalar μ such that $\varphi(T) = \mu TV^*$ for all $T \in B(H)$. We also show if φ_1 and φ_2 be additive maps from $B(H)$ into $B(H)$ such that their ranges contain all operators of rank at most two and satisfies

$$H_{\varphi_1(T)^*\varphi_2(S)}(\{\lambda\}) = H_{T^*S}(\{\lambda\})$$

for all $T, S \in B(H)$ and $\lambda \in \mathbb{C}$. Then $\varphi_2(I)^*$ is invertible, and $\varphi_1(T) = (\varphi_2(I)^*)^{-1}T$ and $\varphi_2(T) = \varphi_2(I)T$ for all $T \in B(H)$.

1. Introduction

Throughout this paper, H and K are infinite-dimensional complex Hilbert spaces. As usual $B(H, K)$ denotes the space of all bounded linear operators from H into K . When $H = K$ we simply write $B(H)$ instead of $B(H, H)$, and its unit will be denoted by I . The inner product of H or K will be denoted by $\langle \cdot, \cdot \rangle$ if there is no confusion. For an operator $T \in B(H, K)$, let T^* denote as usual its adjoint. A preserver problem generally deals with characterizing those maps on some specific algebraic structures which preserve a particular subset, property or relation. This subject has a long history and its origins goes back well over a century to the so-called first linear preserver problem, due to Frobenius [11], that determines linear maps preserving the determinant of matrices. As we mentioned earlier, the main of this subject goal is to describe the general form of linear maps between two Banach algebras which preserve a certain property, or a certain

Mathematics subject classification (2020): 47A11, 47A15, 47B48.

Keywords and phrases: Local spectrum, local spectral subspace, nonlinear preservers, rank-one operators.

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class of elements, or a certain relation. One of the most famous related problems is Kaplansky's problem [17] asking whether every surjective unital invertibility preserving linear map between two semisimple Banach algebras is a Jordan homomorphism. His question was motivated by two classical results, the result of Marcus and Moyls [18] on linear maps preserving eigenvalues of matrices and the Gleason-Kahane-Zelazko theorem [15, 16] stating that every unital invertibility preserving linear functional on a unital complex Banach algebra is necessarily multiplicative. The later this result was obtained independently by Gleason in [15] and Kahane-Zelazko in [16], and was refined by Zelazko in [23]. In the non-commutative case, the best known result so far are due to Sourour [14]. He answered to the Kaplansky's question in the affirmative for bijective unital linear invertibility preserving maps acting on the algebra of all bounded operators on a Banach space. Note that when the maps are unital, then preserving invertibility is equivalent to preserving spectrum. These results opened the gate for many authors who investigate linear (or additive) maps preserving spectrum; see for instance [1, 12, 13] and the references therein. Along this line, Molnar [19] investigated maps preserving the spectrum of operator products without assuming linearity or additivity.

The local resolvent set, $\rho_T(x)$, of an operator $T \in B(H)$ at a point $x \in H$ is the union of all open subsets U of the complex plane \mathbb{C} for which there is an analytic function $f : U \rightarrow H$ such that $(\mu I - T)f(\mu) = x$ for all $\mu \in U$. The complement of local resolvent set is called the local spectrum of T at x , denoted by $\sigma_T(x)$, and is obviously a closed subset (possibly empty) of $\sigma(T)$, the spectrum of T . We recall that an operator $T \in B(H)$ is said to have the single-valued extension property (henceforth abbreviated to SVEP) if, for every open subset U of \mathbb{C} , there exists no nonzero analytic solution, $f : U \rightarrow H$, of the equation

$$(\mu I - T)f(\mu) = 0, \quad \forall \mu \in U.$$

Every operator $T \in B(H)$ for which the interior of its point spectrum, $\sigma_p(T)$, is empty enjoys this property.

For every subset $F \subseteq \mathbb{C}$ the local spectral subspace $H_T(F)$ is defined by

$$H_T(F) = \{x \in H : \sigma_T(x) \subseteq F\}.$$

Clearly, if $F_1 \subseteq F_2$ then $H_T(F_1) \subseteq H_T(F_2)$. For more information about these notions one may see the books [2, 20].

The study of linear and nonlinear local spectra preserver problems attracted the attention of a number of authors. Bourhim and Ransford were the first ones to consider this type of preserver problem, characterizing in [8] additive maps on $B(X)$, the algebra of all linear bounded operators on infinite-dimensional complex Banach space X , that preserves the local spectrum of operators at each vector of X . Their results motivated several authors to describe maps on matrices or operators that preserve local spectrum, local spectral radius, and local inner spectral radius; see, for instance, the survey articles [5, 22] and the references therein. Based on the results from the theory of linear preservers proved by Jafarian and Sourour [14], Dolinar et al. [9], characterised the form of maps preserving the lattice of sum of operators. They showed that the map (not necessarily linear) $\varphi : B(X) \rightarrow B(X)$ satisfies

$\text{Lat}(\varphi(T) + \varphi(S)) = \text{Lat}(T + S)$ for all $T, S \in B(X)$, if and only if there are a non zero scalar α and a map $\phi : B(X) \rightarrow \mathbb{F}$ such that $\varphi(T) = \alpha T + \phi(T)I$ for all $T \in B(X)$ (See [9, Theorem 1]), where \mathbb{F} is the complex field \mathbb{C} or the real field \mathbb{R} and $\text{Lat}(T)$ is denoted the lattice of T , that is, the set of all invariant subspaces of T . They proved also, in the same paper, that a not necessarily linear maps $\varphi : B(X) \rightarrow B(X)$ satisfies $\text{Lat}(\varphi(T)\varphi(S)) = \text{Lat}(TS)$ (resp. $\text{Lat}(\varphi(T)\varphi(S)\varphi(T)) = \text{Lat}(TST)$, resp. $\text{Lat}(\varphi(T)\varphi(S) + \varphi(S)\varphi(T)) = \text{Lat}(TS + ST)$) for all $T, S \in B(X)$, if and only if there is a map $\phi : B(X) \rightarrow \mathbb{F}$ such that $\varphi(T) \neq 0$ if $T \neq 0$ and $\varphi(T) = \phi(T)T$ for all $T \in B(X)$ (See [9, Theorem 2]).

For a Banach space X , it is well-known that $X_T(F)$, the local spectral subspace of T associated with a subset F of \mathbb{C} , is an element of $\text{Lat}(T)$, so one can replace the lattice preserving property by the local spectral subspace preserving property. In [10], the authors described additive maps on $B(X)$ that preserve the local spectral subspace of operators associated with any singleton. More precisely, they proved that the only additive map φ on $B(X)$ for which $X_{\varphi(T)}(\{\lambda\}) = X_T(\{\lambda\})$ for all $T \in B(X)$ and $\lambda \in \mathbb{C}$, is the identity. In [4], Benbouziane et al. characterized the forms of surjective weakly continuous maps φ from $B(X)$ into $B(X)$ which satisfy

$$X_{\varphi(T)-\varphi(S)}(\{\lambda\}) = X_{T-S}(\{\lambda\}), \quad (T, S \in B(X), \lambda \in \mathbb{C}).$$

Afterwards, in [3], the authors studied surjective maps that preserve the local spectral subspace of the sum of two operators associated with non-fixed singletons. In other word, they characterized surjective maps φ on $B(X)$ which satisfy

$$X_{\varphi(T)+\varphi(S)}(\{\lambda\}) = X_{T+S}(\{\lambda\}), \quad (T, S \in B(X), \lambda \in \mathbb{C}).$$

They also gave a characterization of maps on $B(X)$ that preserve the local spectral subspace of the difference of operators associated with non-fixed singletons. Furthermore, they investigated the product case as well as the triple product case. Namely, they described surjective maps φ on $B(X)$ satisfying

$$X_{\varphi(T)\varphi(S)}(\{\lambda\}) = X_{TS}(\{\lambda\}), \quad (T, S \in B(X), \lambda \in \mathbb{C}),$$

and also surjective maps φ on $B(X)$ satisfying

$$X_{\varphi(T)\varphi(S)\varphi(T)}(\{\lambda\}) = X_{TST}(\{\lambda\}) \quad (T, S \in B(X), \lambda \in \mathbb{C}).$$

Bourhim and Lee [6] investigated the form of all maps φ_1 and φ_2 on $B(X)$ such that, for every T and S in $B(X)$, the local spectra of TS and $\varphi_1(T)\varphi_2(S)$ are the same at a nonzero fixed vector x_0 . In this paper, We show that if $\varphi : B(H) \rightarrow B(H)$ is an additive map such that its range contains all operators of rank at most two and satisfies

$$H_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = H_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}),$$

then there exist a unitary operator V in $B(H)$ and a nonzero scalar μ such that $\varphi(T) = \mu TV^*$ for all $T \in B(H)$. We also investigate the form of all maps φ_1 and φ_2 on $B(H)$ such that, for every T and S in $B(H)$, the local spectral subspaces of T^*S and $\varphi_1(T)^*\varphi_2(S)$, associated with the singleton $\{\lambda\}$, coincide.

2. Preliminaries

The first lemma summarizes some known basic and properties of the local spectrum.

LEMMA 2.1. (See [2, 20].) *Let $T \in B(H)$. For every $x, y \in H$ and a scalar $\alpha \in \mathbb{C}$ the following statements hold.*

- (i) $\sigma_T(\alpha x) = \sigma_T(x)$ if $\alpha \neq 0$, and $\sigma_{\alpha T}(x) = \alpha \sigma_T(x)$.
- (ii) If $Tx = \lambda x$ for some $\lambda \in \mathbb{C}$, then $\sigma_T(x) \subseteq \{\lambda\}$. In particular, if $x \neq 0$ and T has SVEP, then $\sigma_T(x) = \{\lambda\}$.

In the next theorem we collect some of the basic properties of the subspaces $H_T(F)$.

LEMMA 2.2. (See [2, 20].) *Let $T \in B(H)$. For $F \subseteq \mathbb{C}$ the following statements hold.*

- (i) $H_T(F)$ is a T -hyperinvariant subspace of H .
- (ii) $(T - \lambda I)H_T(F) = H_T(F)$ for every $\lambda \in \mathbb{C} \setminus F$.
- (iii) If $x \in H$ satisfies $(T - \lambda I)x \in H_T(F)$, then $x \in H_T(F)$.
- (v) $\ker(T - \lambda I) \subseteq H_T(F)$.
- (iv) $H_{\alpha T}(\lambda) = H_T(\frac{\lambda}{\alpha})$ for every $\lambda \in \mathbb{C}$ and non-zero scalar α .

For a nonzero $h \in H$ and $T \in B(H)$, we use a useful notation defined by Bourhim and Mashreghi in [7]:

$$\sigma_T^*(h) := \begin{cases} \{0\} & \text{if } \sigma_T(h) = \{0\}, \\ \sigma_T(h) \setminus \{0\} & \text{if } \sigma_T(h) \neq \{0\}. \end{cases}$$

For two nonzero vectors x and y in H , let $x \otimes y$ stands for the operator of rank at most one defined by

$$(x \otimes y)z = \langle z, y \rangle x, \quad \forall z \in H.$$

Note that every rank one operator in $B(H)$ can be written in this form, and that every finite rank operator $T \in B(H)$ can be written as a finite sum of rank one operators; i.e., $T = \sum_{i=1}^n x_i \otimes y_i$ for some $x_i, y_i \in H$ and $i = 1, 2, \dots, n$. By $F(H)$ and $F_n(H)$, we mean the set of all finite rank operators in $B(H)$ and the set of all operators of rank at most n , n is a positive integer, respectively.

The following lemma is an elementary observation which describes the nonzero local spectrum of any rank one operator.

LEMMA 2.3. (See [7, Lemma 2.2].) *Let h_0 be a nonzero vector in H . For every vectors $x, y \in H$, we have*

$$\sigma_{x \otimes y}^*(h_0) := \begin{cases} \{0\} & \text{if } \langle h_0, y \rangle = 0, \\ \langle x, y \rangle & \text{if } \langle h_0, y \rangle \neq 0. \end{cases}$$

The following theorem, which may be of independent interest, gives a spectral characterization of rank one operators in term of local spectrum.

THEOREM 2.4. (See [7, Theorem 4.1].) *For a nonzero vector $h \in H$ and a nonzero operator $R \in B(H)$, the following statements are equivalent.*

- (a) R has rank one.
- (b) $\sigma_{RT}^*(h)$ contains at most one element for all $T \in B(H)$.
- (c) $\sigma_{RT}^*(h)$ contains at most one element for all $T \in F_2(H)$.

The following Lemma is a key tool for the proofs in the sequel.

LEMMA 2.5. (See [3, Lemma 1.6].) *Let h be a nonzero vector in H and $T, S \in B(H)$. If $H_T(\{\lambda\}) = H_S(\{\lambda\})$ for all $\lambda \in \mathbb{C}$. Then, $\sigma_T(h) = \{\mu\}$ if and only if $\sigma_S(h) = \{\mu\}$ for all $\mu \in \mathbb{C}$.*

Moreover, this theorem will be useful in the proofs of our main result.

THEOREM 2.6. (See [3, Theorem 2.1].) *Let $T, S \in B(H)$. The following statements are equivalent.*

- (1) $T = S$.
- (2) $H_{TR}(\{\lambda\}) = H_{SR}(\{\lambda\})$ for all $\lambda \in \mathbb{C}$ and $R \in F_1(H)$.

The next theorem describes additive maps on $B(H)$ that preserve the local spectral subspace of operators associated with any singleton set.

THEOREM 2.7. (See [10, Theorem 2.1].) *Let $\varphi : B(H) \rightarrow B(H)$ be an additive map such that $H_{\varphi(T)}(\{\lambda\}) = H_T(\{\lambda\})$ for all $T \in B(H)$ and $\lambda \in \mathbb{C}$. Then $\varphi(T) = T$ for all $T \in B(H)$.*

The following theorem will be useful in the sequel. We recall that if $h : \mathbb{C} \rightarrow \mathbb{C}$ is a ring homomorphism, then an additive map $A : H \rightarrow H$ satisfying $A(\alpha x) = h(\alpha)x$, ($x \in H, \alpha \in \mathbb{C}$) is called an h -quasilinear operator.

THEOREM 2.8. (See [21, Theorem 3.3].) *Let $\varphi : F(H) \rightarrow F(H)$ be a bijective additive map preserving rank one operators in both directions. Then there exist a ring automorphism $h : \mathbb{C} \rightarrow \mathbb{C}$, and either there are h -quasilinear bijective maps $A : H \rightarrow H$ and $B : H \rightarrow H$ such that*

$$\varphi(x \otimes y) = Ax \otimes By, \quad x, y \in H,$$

or there are h -quasilinear bijective maps $C : H \rightarrow H$ and $D : H \rightarrow H$ such that

$$\varphi(x \otimes y) = Cy \otimes Dx, \quad x, y \in H.$$

Note that, if in Theorem 2.8 the map φ is linear, then h is the identity map on \mathbb{C} and so the maps A, B, C and D are linear.

3. Main results

The following theorem is the first main result of this paper, which characterizes those maps preserving the local spectral subspace of skew-product operators.

THEOREM 3.1. *Let $\varphi : B(H) \rightarrow B(H)$ be an additive map such that its range contains $F_2(H)$. If*

$$H_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = H_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}), \tag{1}$$

then there exist a unitary operator V in $B(H)$ and a nonzero scalar μ such that $\varphi(T) = \mu TV^$ for all $T \in B(H)$.*

Proof. The proof breaks down into several claims.

Claim 1. φ is injective.

If $\varphi(T) = \varphi(S)$ for some $T, S \in B(H)$, we get that

$$H_{TR^*}(\{\lambda\}) = H_{\varphi(T)\varphi(R)^*}(\{\lambda\}) = H_{\varphi(S)\varphi(R)^*}(\{\lambda\}) = H_{SR^*}(\{\lambda\})$$

for all $R \in F_1(X)$ and $\lambda \in \mathbb{C}$. By Theorem 2.6, we see that $T = S$ and hence φ is injective.

Claim 2. φ preserves rank one operators in both directions.

Let $R = x \otimes y$ be a rank one operator where $x, y \in H$. Note that, $\varphi(R) \neq 0$, since $\varphi(0) = 0$ and φ is injective. Let $T \in B(H)$ be an arbitrary operator. Since $RT^*x = \langle x, Ty \rangle x$ and RT^* has the SVEP, then $\sigma_{RT^*}(x) = \{\langle x, Ty \rangle\}$. We have

$$x \in H_{RT^*}(\{\langle x, Ty \rangle\}) = H_{\varphi(R)\varphi(T)^*}(\{\langle x, Ty \rangle\}).$$

As the range of φ contains $F_2(H)$, using Lemma 2.5, $\sigma_{\varphi(R)\varphi(S)^*}^*(x)$ contains at most one element for all operators $S \in F_2(H)$. By Theorem 2.4, we see that $\varphi(R)$ has rank one. The converse holds in a similar way and thus φ preserves the rank one operators in both directions.

Claim 3. φ is linear.

We show that φ is homogeneous. Let R be an arbitrary rank-one operator. By the previous claim, there exists a rank one operator S in $B(H)$ such that $\varphi(S) = R$. For every $\alpha, \lambda \in \mathbb{C}$ with $\alpha \neq 0$ and $T \in B(H)$, we have

$$\begin{aligned} H_{\alpha\varphi(T)R^*}(\{\lambda\}) &= H_{\alpha\varphi(T)\varphi(S)^*}(\{\lambda\}) = H_{\varphi(T)\varphi(S)^*} \left(\left\{ \frac{\lambda}{\alpha} \right\} \right) \\ &= H_{TS^*} \left(\left\{ \frac{\lambda}{\alpha} \right\} \right) = H_{(\alpha T)S^*}(\{\lambda\}) \\ &= H_{\varphi(\alpha T)\varphi(S)^*}(\{\lambda\}) = H_{\varphi(\alpha T)R^*}(\{\lambda\}). \end{aligned}$$

By Theorem 2.6, we see that $\varphi(\alpha T) = \alpha\varphi(T)$. Since φ is assumed to be additive, the map φ is, in fact, linear.

Claim 4. There are bijective linear mappings $A : H \rightarrow H$ and $B : H \rightarrow H$ such that $\varphi(x \otimes y) = Ax \otimes By$ for all $x, y \in H$.

By the previous claim φ is a bijective linear map from $F(H)$ onto $F(H)$ and preserves rank one operators in both directions, thus by Theorem 2.8, either there are bijective linear mappings $A : H \rightarrow H$ and $B : H \rightarrow H$ such that

$$\varphi(x \otimes y) = Ax \otimes By, \quad x, y \in H, \tag{2}$$

or there are bijective linear mappings $C : H \rightarrow H$ and $D : H \rightarrow H$ such that

$$\varphi(x \otimes y) = Cy \otimes Dx, \quad x, y \in H. \tag{3}$$

Assume that φ takes the form (3). Let y_1 be a nonzero vector in H . Choose a nonzero vector v such that $\langle y_1, v \rangle = 0$. Set $x = C^{-1}y_1$, since x and v are nonzero vectors in H , there exists a $y \in H$ such that $\langle x, y \rangle = 1$ and $\langle v, y \rangle \neq 0$, since $x \otimes y$ is idempotent, we have

$$\begin{aligned} H_{x \otimes y}(\{\lambda\}) &= H_{(x \otimes y)(x \otimes y)}(\{\lambda\}) \\ &= H_{(x \otimes y)(y \otimes x)^*}(\{\lambda\}) \\ &= H_{(Cy \otimes Dx)(Cx \otimes Dy)^*}(\{\lambda\}) \\ &= H_{(Cy \otimes Dx)(Dy \otimes Cx)}(\{\lambda\}) \\ &= H_{\langle Dy, Dx \rangle \langle Cy \otimes Cx \rangle}(\{\lambda\}) \\ &= H_{\langle Dy, Dx \rangle \langle Cy \otimes y_1 \rangle}(\{\lambda\}). \end{aligned}$$

On the other hand, since $\langle y_1, v \rangle = 0$, we have $\sigma_{C y \otimes y_1}^*(v) = \{0\}$ and consequently $\langle Dy, Dx \rangle \sigma_{C y \otimes y_1}(v) = \{0\}$. This implies that

$$v \in H_{\langle Dy, Dx \rangle \langle Cy \otimes y_1 \rangle}(\{0\}) = H_{x \otimes y}(\{0\}).$$

Using Lemma 2.5, $\sigma_{x \otimes y}(v) = \{0\}$. But lemma 2.3 implies that

$$\sigma_{x \otimes y}^*(v) \neq \{0\}.$$

This contradiction shows that φ only takes the form (2).

Claim 5. A and B are bounded unitary operators multiplied by positive scalars α and β such that $\alpha\beta = 1$.

Let x, y be nonzero vectors in H , since $\sigma_{(x \otimes y)(y \otimes x)}(x) = \{\langle y, y \rangle \langle x, x \rangle\}$, by the previous claim, we have

$$\begin{aligned} H_{(x \otimes y)(y \otimes x)}(\{\|y\|^2 \|x\|^2\}) &= H_{(x \otimes y)(x \otimes y)^*}(\{\|y\|^2 \|x\|^2\}) \\ &= H_{(Ax \otimes By)(Ax \otimes By)^*}(\{\|y\|^2 \|x\|^2\}) \\ &= H_{(Ax \otimes By)(By \otimes Ax)}(\{\|y\|^2 \|x\|^2\}) \\ &= H_{\langle By, By \rangle \langle Ax \otimes Ax \rangle}(\{\|y\|^2 \|x\|^2\}). \end{aligned}$$

By the Lemma 2.5, we see that

$$\begin{aligned} \{\|y\|^2\|x\|^2\} &= \sigma_{(x\otimes y)(y\otimes x)}(x) \\ &= \sigma_{\langle (By,By)(Ax\otimes Ax) \rangle}(x) = \{\|By\|^2\|Ax\|^2\}. \end{aligned} \tag{4}$$

Now, let y_0 be a fixed unit vector in H and let $\alpha = \frac{1}{\|By_0\|}$. By (4), we have

$$\|Ax\|^2 = \alpha^2\|x\|^2$$

for all $x \in H$. Hence, $U = \frac{1}{\alpha}A$ is an isometry and thus it is a unitary operator in $B(H)$, because A is bijective. Similarly, fix a unit vector $x_0 \in H$ and take $\beta = \frac{1}{\|Ax_0\|}$, and note that $V = \frac{1}{\beta}B$ is a unitary operator in $B(H)$. Finally, by (4), we see that $\alpha\beta = 1$.

Claim 6. A^* and I are linearly dependent.

Assume, by the way of contradiction, that there exists a nonzero vector $x \in H$ such that Ax and x are linearly independent. Let $u \in H$ be a vector such that $\langle x, u \rangle = 1$ and $\langle A^*x, u \rangle = 0$. Since $\sigma_{x\otimes u}(x) = \{1\}$, then

$$\begin{aligned} x \in H_{x\otimes u}(\{1\}) &= X_{(x\otimes u)(x\otimes u)}(\{1\}) \\ &= H_{(x\otimes u)(u\otimes x)^*}(\{1\}) \\ &= H_{\varphi_{(x\otimes u)\varphi(u\otimes x)^*}}(\{1\}) \\ &= H_{(Ax\otimes Bu)(Au\otimes Bx)^*}(\{1\}) \\ &= H_{(Ax\otimes Bu)(Bx\otimes Au)}(\{1\}) \\ &= H_{\langle Bx, Bu \rangle (Ax\otimes Au)}(\{1\}), \end{aligned}$$

using Lemma 2.5, we have

$$\{1\} = \sigma_{x\otimes u}(x) = \sigma_{\langle Bx, Bu \rangle (Ax\otimes Au)}(x) = \{0\}.$$

This contradiction shows that there is a nonzero scalar $\gamma \in \mathbb{C}$ such that $A^* = \gamma I$.

Claim 7. $\varphi(T) = \mu TV^*$ for all $T \in B(H)$, where V is unitary operators and μ is a nonzero scalar.

By claim 5 we shall assume that $A = U$ and $B = V$ for some unitary operators $U, V \in B(H)$. Using the previous claim and (1), for every rank one operator $R \in B(H)$ and every operator $T \in B(H)$ we have

$$\begin{aligned} H_{\varphi(T)\varphi(R)^*}(\{\lambda\}) &= H_{TR^*}(\{\lambda\}) \\ &= H_{UTR^*U^*}(\{\lambda\}) \\ &= H_{UTV^*VR^*U^*}(\{\lambda\}) \\ &= H_{UTV^*(URV^*)^*}(\{\lambda\}) \\ &= H_{UTV^*\varphi(R)^*}(\{\lambda\}). \end{aligned}$$

Since φ preserves rank one operators in both directions, Theorem 2.6 shows that $\varphi(T) = UTV^*$ for all $T \in B(H)$. Claim 6 tells us that for ever $T \in B(H)$ we have $\varphi(T) = \mu TV^*$ for some $\mu \in \mathbb{C}$. \square

From this result, it is easy to deduce a generalization for the case of two different Hilbert spaces H, K .

COROLLARY 3.2. *Suppose $U \in B(H, K)$ be a unitary operator. Let φ be an additive map from $B(H)$ into $B(K)$ such that its range contains $F_2(K)$. If*

$$K_{\varphi(T)\varphi(S)^*}(\{\lambda\}) = UH_{TS^*}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}).$$

Then there exist a unitary operator $V : H \rightarrow K$ and a nonzero scalar μ such that $\varphi(T) = \mu UTV^$ for all $T \in B(H)$.*

Proof. We consider the map $\psi : B(H) \rightarrow B(H)$ defined by $\psi(T) = U^*\varphi(T)U$ for all $T \in B(H)$. We have,

$$\begin{aligned} H_{\psi(T)\psi(S)^*}(\{\lambda\}) &= H_{U^*\varphi(T)UU^*\varphi(S)^*U}(\{\lambda\}) \\ &= H_{U^*\varphi(T)\varphi(S)^*U}(\{\lambda\}) \\ &= U^*K_{\varphi(T)\varphi(S)^*}(\{\lambda\}) \\ &= H_{TS^*}(\{\lambda\}) \end{aligned}$$

for every $T, S \in B(H)$ and $\lambda \in \mathbb{C}$. So by Theorem 3.1, there exist a unitary operator $P : H \rightarrow H$ and $\mu \in \mathbb{C}$ such that $\psi(T) = \mu TP^*$ for all $T \in B(H)$. Therefore $\varphi(T) = \mu UTV^*$ for all $T \in B(H)$, where $V = UP$. \square

In the next theorem, we investigate the form of all maps φ_1 and φ_2 on $B(H)$ such that, for every T and S in $B(H)$, the local spectral subspaces of T^*S and $\varphi_1(T)^*\varphi_2(S)$, associated with the singleton $\{\lambda\}$, coincide.

THEOREM 3.3. *Let φ_1 and φ_2 be additive maps from $B(H)$ into $B(H)$ which satisfy*

$$H_{\varphi_1(T)^*\varphi_2(S)}(\{\lambda\}) = H_{T^*S}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}). \tag{5}$$

If the range of φ_1 and φ_2 contain $F_2(H)$, then $\varphi_2(I)^$ is invertible, and $\varphi_1(T) = (\varphi_2(I)^*)^{-1}T$ and $\varphi_2(T) = \varphi_2(I)T$ for all $T \in B(H)$.*

Proof. The proof is rather long and we break it into several claims.

Claim 1. φ_1 is a one to one map preserving rank one operators in both directions.

Similar to the proof of Theorem 3.1, we can shows that φ_1 is a one to one map preserving rank one operators in both directions.

Claim 2. φ_1 is linear.

We show that φ_1 is homogeneous. Let R be an arbitrary rank-one operator, as the range of φ_2 contains $F_2(H)$, so $\varphi_2(S) = R$ for some operator $S \in B(H)$. For every $\alpha, \lambda \in \mathbb{C}$ with $\alpha \neq 0$ and $T \in B(H)$, we have

$$\begin{aligned} H_{\varphi_1(\alpha T)^*R}(\{\lambda\}) &= H_{\varphi_1(\alpha T)^*\varphi_2(S)}(\{\lambda\}) = H_{\overline{\alpha T}^*S}(\{\lambda\}) \\ &= H_{T^*S}\left(\left\{\frac{\lambda}{\alpha}\right\}\right) = H_{\varphi_1(T)^*\varphi_2(S)}\left(\left\{\frac{\lambda}{\alpha}\right\}\right) \\ &= H_{\overline{\alpha}\varphi_1(T)^*\varphi_2(S)}(\{\lambda\}) = H_{\overline{\alpha}\varphi_1(T)^*R}(\{\lambda\}). \end{aligned}$$

By Theorem 2.6, we see that $\varphi_1(\alpha T)^* = \overline{\alpha}\varphi_1(T)^*$, and so $\varphi_1(\alpha T) = \alpha\varphi_1(T)$ for all $T \in B(H)$. Since φ_1 is assumed to be additive, the map φ_1 is, in fact, linear.

Claim 3. There are bijective linear mappings $A : H \rightarrow H$ and $B : H \rightarrow H$ such that $\varphi_1(x \otimes y) = Ax \otimes By$ for all $x, y \in H$.

By the claim 1, $\varphi_1 : F(H) \rightarrow F(H)$ is a bijective linear map which preserves rank one operators in both directions. Thus by Theorem 2.8, φ_1 has one of the following forms.

- (1) There exist bijective linear maps $A : H \rightarrow H$ and $B : H \rightarrow H$ such that

$$\varphi_1(x \otimes y) = Ax \otimes By, \quad x, y \in H. \tag{6}$$

- (2) There exist bijective linear maps $C : H \rightarrow H$ and $D : H \rightarrow H$ such that

$$\varphi_1(x \otimes y) = Cy \otimes Dx, \quad x, y \in H. \tag{7}$$

Assume that φ_1 takes the form (7). Let y be a nonzero vector in H , choose a nonzero vector $v \in H$ such that $\langle \varphi_2(I)y, v \rangle = 0$. Set $u = C^{-1}v$, since u and y are nonzero vectors in H , there exists a vector $x \in H$ such that $\langle y, x \rangle \neq 0$ and $\langle u, x \rangle \neq 0$. Since $\sigma_{(Dx \otimes v)\varphi_2(I)}^*(y) = \{0\}$, we have

$$y \in H_{(Dx \otimes v)\varphi_2(I)}(\{0\}) = H_{(Dx \otimes Cu)\varphi_2(I)}(\{0\}) = H_{(Cu \otimes Dx)^*\varphi_2(I)}(\{0\}) = H_{(x \otimes u)^*}(\{0\}).$$

Using Lemma 2.5, $\sigma_{u \otimes x}(y) = \{0\}$. But lemma 2.3 implies that

$$\sigma_{u \otimes x}^*(y) = \{\langle u, x \rangle\} \neq \{0\}.$$

This contradiction shows that φ_1 only takes the form (6).

Claim 4. For every $x, y \in H$, $\langle y, x \rangle = \langle By, \varphi_2(I)^*(Ax) \rangle$.

Assume that x and y are arbitrary vectors in H . We have, $\sigma_{y \otimes x}(y) = \{\langle y, x \rangle\}$, so the previous claim and (5) imply that

$$\begin{aligned} y \in H_{y \otimes x}(\{\langle y, x \rangle\}) &= H_{(x \otimes y)^*}(\{\langle y, x \rangle\}) = H_{\varphi_1(x \otimes y)^*\varphi_2(I)}(\{\langle y, x \rangle\}) \\ &= H_{(Ax \otimes By)^*\varphi_2(I)}(\{\langle y, x \rangle\}). \end{aligned}$$

Assume first that $\langle y, x \rangle \neq 0$, using lemma 2.5,

$$\{0\} \neq \{\langle y, x \rangle\} = \sigma_{y \otimes x}(y) = \sigma_{(By \otimes Ax)\varphi_2(I)}(y),$$

which means that $\langle y, \varphi_2(I)^*(Ax) \rangle \neq 0$. Then Lemma 2.3 implies that

$$\{\langle y, x \rangle\} = \sigma_{y \otimes x}(y) = \sigma_{(By \otimes Ax)\varphi_2(I)}(y) = \{\langle By, \varphi_2(I)^*(Ax) \rangle\}.$$

Now, if $\langle y, x \rangle = 0$, we choose a vector $u \in H$ such that $\langle y, u \rangle \neq 0$. By application of what has been shown previously to both u and $y + u$, we have $\langle y, u \rangle = \langle By, \varphi_2(I)^*(Au) \rangle$ and $\langle y, x + u \rangle = \langle By, \varphi_2(I)^*(A(u + x)) \rangle$. So

$$\begin{aligned} \langle y, x \rangle + \langle y, u \rangle &= \langle y, x + u \rangle \\ &= \langle By, \varphi_2(I)^*(A(x + u)) \rangle \\ &= \langle By, \varphi_2(I)^*(Ax) \rangle + \langle By, \varphi_2(I)^*(Au) \rangle \\ &= \langle By, \varphi_2(I)^*(Ax) \rangle + \langle y, u \rangle. \end{aligned}$$

This shows that $\langle y, x \rangle = \langle By, \varphi_2(I)^*(Ax) \rangle$ in this case too.

Claim 5. $\varphi_2(I)^*$ is invertible.

It is clear that $\varphi_2(I)^*$ is injective, if not, there is a nonzero vector $y \in H$ such that $\varphi_2(I)^*y = 0$. Take $x = A^{-1}y$, and let $u \in H$ be a vector such that $\langle u, x \rangle = 1$. By the previous claim, we have $1 = \langle u, x \rangle = \langle Bu, \varphi_2(I)^*(Ax) \rangle = \langle Bu, \varphi_2(I)^*y \rangle = 0$. This contradiction tells us that $\varphi_2(I)^*$ is injective. Now, we show that B is continuous and $A^*\varphi_2(I) = B^{-1}$. Assume that $(x_n)_n$ is a sequence in H such that $\lim_{n \rightarrow \infty} x_n = x \in H$ and $\lim_{n \rightarrow \infty} Bx_n = y \in H$. Then, for every $u \in H$, we have

$$\begin{aligned} \langle y, \varphi_2(I)^*(Au) \rangle &= \lim_{n \rightarrow \infty} \langle Bx_n, \varphi_2(I)^*(Au) \rangle \\ &= \lim_{n \rightarrow \infty} \langle x_n, u \rangle = \langle x, u \rangle = \langle Bx, \varphi_2(I)^*(Au) \rangle. \end{aligned}$$

Since A is bijective and $u \in H$ is an arbitrary vector, the closed graph theorem shows that B is continuous. Moreover, we have $\langle y, x \rangle = \langle By, \varphi_2(I)^*(Ax) \rangle = \langle y, B^*\varphi_2(I)^*(Ax) \rangle$ for all $x, y \in H$, and thus $I = B^*\varphi_2(I)^*A$. It follows that $\varphi_2(I)^*$ is invertible.

Claim 6. φ_1 and φ_2 have the desired forms.

We define the map $\psi_1 : B(H) \rightarrow B(H)$ by $\psi_1(T) = \varphi_1(T^*)^*\varphi_2(I)$ for all $T \in B(H)$. We have,

$$H_{\psi_1(T)}(\{\lambda\}) = H_{\varphi_1(T^*)^*\varphi_2(I)}(\{\lambda\}) = H_T(\{\lambda\})$$

for all $T \in B(H)$ and $\lambda \in \mathbb{C}$. So by Theorem 2.7, $\psi_1(T) = T$ for all $T \in B(H)$, and so $\varphi_1(T^*)^* = T\varphi_2(I)^{-1}$ for all $T \in B(H)$. Therefore $\varphi_1(T) = (\varphi_2(I)^*)^{-1}T$ for all $T \in B(H)$. Once again, we consider the map $\psi_2 : B(H) \rightarrow B(H)$ defined by $\psi_2(T) = \varphi_1(I)^*\varphi_2(T)$ for all $T \in B(H)$. We see that for all $T \in B(H)$ and $\lambda \in \mathbb{C}$,

$$H_{\psi_2(T)}(\{\lambda\}) = H_T(\{\lambda\}),$$

by Theorem 2.7, $\psi_2(T) = T$ for all $T \in B(H)$. Hence $\varphi_2(T) = (\varphi_1(I)^*)^{-1}T$, and so $\varphi_2(T) = \varphi_2(I)T$ for all $T \in B(H)$. \square

Theorem 3.3 leads directly to the following corollary.

COROLLARY 3.4. *Suppose $U \in B(H, K)$ be a unitary operator. Let φ_1 and φ_2 be two additive map from $B(H)$ into $B(K)$ which satisfy*

$$K_{\varphi_1(T)^*\varphi_2(S)}(\{\lambda\}) = UH_{T^*S}(\{\lambda\}), \quad (T, S \in B(H), \lambda \in \mathbb{C}).$$

If the range of φ_1 and φ_2 contain $F_2(K)$, then there exists a bijective linear map $V : K \rightarrow H$ such that $\varphi_1(T) = (V^)^{-1}TU^*$ and $\varphi_2(T) = VTU^*$ for all $T \in B(H)$.*

Proof. We consider the maps $\psi_1 : B(H) \rightarrow B(H)$ defined by $\psi_1(T) = U^*\varphi_1(T)U$ and $\psi_2 : B(H) \rightarrow B(H)$ defined by $\psi_2(T) = U^*\varphi_2(T)U$ for all $T \in B(H)$. We have,

$$\begin{aligned} H_{\psi_1(T)^*\psi_2(S)}(\{\lambda\}) &= H_{U^*\varphi_1(T)^*UU^*\varphi_2(S)U}(\{\lambda\}) \\ &= H_{U^*\varphi_1(T)^*\varphi_2(S)U}(\{\lambda\}) \\ &= U^{-1}K_{\varphi_1(T)^*\varphi_2(S)}(\{\lambda\}) = H_{T^*S}(\{\lambda\}) \end{aligned}$$

for every $T, S \in B(H)$ and $\lambda \in \mathbb{C}$. So by Theorem 3.3, $\psi_1(T) = (\psi_2(I)^*)^{-1}T$ and $\psi_2(T) = \psi_2(I)T$ for all $T \in B(H)$. Therefore $\varphi_1(T) = (V^*)^{-1}TU^*$ and $\varphi_2(T) = VTU^*$ for all $T \in B(H)$, where $V = \varphi_2(I)U$. \square

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(Received October 6, 2023)

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