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DISTANCE LAPLACIAN EIGENVALUE DISTRIBUTION OF A GRAPH WITH GIVEN SOME PARAMETERS

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Abstract. For a connected graph G of order n and an interval I, denote by $m_{\mathscr{D}\mathscr{L}(G)}I$ the number of distance Laplacian eigenvalues of G in I. In this paper, applying two different methods, we prove that $m_{\mathscr{D}\mathscr{L}(G)}[n,n+1) \leqslant \kappa(G)$, where $\kappa(G)$ is the vertex connectivity of G. Moreover, it is shown that this upper bound is sharp. Finally, based on the dominating induced matching of a graph G, we give the distance Laplacian eigenvalue distribution of the graph G.

1. Introduction

All graphs considered in this paper are undirected, finite and simple. Let $G = \big(V(G),E(G)\big)$ be a graph with vertex set $V(G)$, edge set $E(G)$ and $|V(G)| = n$, $|E(G)| = m$. For $v \in V(G)$, the *neighbour set* of vertex v is defined as $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the number $d(v) = d_G(v) = |N_G(v)|$ is the *degree* of vertex v in graph G. If every vertex in $V(G)$ has the same degree r, then G is called r*-regular*. The *adjacency matrix* of G, denoted by $A(G)$, is a $(0,1)$-square matrix of order n, whose (i,j)-th entry is 1, if $v_iv_j \in E(G)$ and 0, otherwise. For the vertex degrees diagonal matrix $Deg(G)$, the real symmetric matrix $L(G) = Deg(G) - A(G)$ is said to be the *Laplacian matrix* of G, respectively. The *complement* of G, denoted by $\overline{G}$, is the simple graph with the vertex set $V(G)$ such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G.

The *union* $G \cup H$ of two graphs G and H is the graph with the vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H)$. Given two vertex disjoint graphs G and H, the *join* of G and H is the graph $G \vee H$ such that $V(G \vee H) = V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$. As usual, we denote by P_n the *path* of order n, by C_n the *cycle* with n vertices, by K_n the *complete graph* of order n. In particular, the *complete bipartite graph* with part sizes p and q denoted by $K_{p,q}$ and the *star* of order n is denoted by $K_{1,n-1}$. For a graph G, an *independent set* $S(G)$ of G is a subset of vertices of G if no two of its vertices are adjacent, the *vertex connectivity* $\kappa(G)$ is the minimum number of vertices whose removal gives rise to a disconnected

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or trivial graph. A *clique* is a complete subgraph of a given graph G. The cardinality of the maximum clique is called the *clique number* of G and is denoted by $\omega(G)$.

A *matching* of G is a set of mutually nonadjacent edges of G. An *induced matching* (IM) is a matching having no two edges joined by an edge. In other words, M is an induced matching of G if the subgraph of G induced by $V(M)$ is 1-regular. A maximum induced matching is an induced matching of maximum cardinality. A subset $S \subset E(G)$ is a *dominating edge set* if every edge $e \in E(G) \setminus S$ shares an endpoint with some edge $e' \in S$. A *dominating induced matching* (DIM) of G is an induced matching that dominates every edge of G, which is also a dominating edge set. All DIMs of a graph G have the same size which is the size of a maximum IM. Observe that if X is a DIM of G, then there is a partition of $V(G)$, which divide $V(G)$ into two disjoint subsets $V(X)$ and $V(Y)$, where $V(Y)$ is an independent set. Noting that every edge of P_4 is an IM. But only the internal edge is a DIM. Clearly, not every graph has DIM. For example, a connected graph having a DIM is illustrated in Fig. 1(a), and a connected graph having no a DIM can be illustrated in Fig. 1(b).

Now, we consider a connected graph G of order n with a DIM $X \subset E(G)$ such that $|X| = s$, where $V(G) = V_1(G) \cup V_2(G)$, $V_1(G) = V(X)$ and $V_2(G) = V(G) \setminus V_1(G)$ is an independent set. Obviously, the property of having a DIM does not change whether we add edges linking the vertices of $V_1(G)$ with the vertices of $V_2(G)$. The extremal graph G', obtained from G adding $\big(2s(n-2s)+2s\big) - |E(G)|$ edges (which is the maximum as possible) between $V_1(G)$ and $V_2(G)$, that is, such that $E(G') = X \cup \{e_1e_2 : e_1 \in V_1(G), e_2 \in V_2(G)\}$. Particularly, if $G \cong G'$, then the DIM is called a *complete dominating induced matching* of G, say a CDIM.

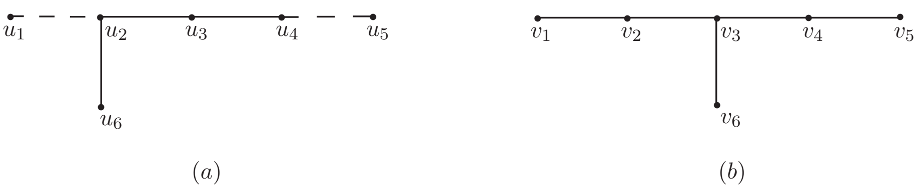

Figure 1: *(a) A connected graph with a DIM* $\{u_1u_2, u_4u_5\}$*. (b) A connected graph with no DIM.*

For a graph G, let $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $v_i, v_j \in V(G)$, the *distance* between v_i and v_j, denoted by $d_G(v_i, v_j)$, is the length of a shortest path from v_i to v_j in G. In particular, $d_G(v_i, v_i) = 0$ for any vertex $v_i \in V(G)$. The *vertex transmission* $Tr_G(v_i)$ of a vertex v_i is defined as the sum of the distances from v_i to all other vertices in G, that is, $Tr_G(v_i) = \sum_{v_j \in V(G)} d_G(v_i, v_j)$.

The *distance matrix* of G is denoted by $\mathscr{D}(G)$ and is defined as $\mathscr{D}(G) = \big(d_G(v_i, v_j)\big)_{v_i, v_j \in V(G)}$. Obviously, the transmission $Tr_G(v_i)$ of a vertex v_i is the sum of all coordinates of the row vector of $\mathscr{D}(G)$ indexed by v_i. Let $Tr(G) = diag\big(Tr_G(v_1), Tr_G(v_2), \ldots, Tr_G(v_n)\big)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [1] introduced the *distance Laplacian matrix* of a connected graph as $\mathscr{D}\mathscr{L}(G) = Tr(G) - \mathscr{D}(G)$. Note that $\mathscr{D}\mathscr{L}(G)$ is a positive semidefinite matrix.

We denote by $\Theta(G,x)$ the *characteristic polynomial* of $\mathscr{D}\mathscr{L}(G)$, that is $\Theta(G,x) = \det\big(xI_n - \mathscr{D}\mathscr{L}(G)\big)$. For a square matrix N, the collection of its eigenvalues together with their multiplicities is called the *spectrum* of N. Let $\{\partial_1^{\mathscr{L}}(G), \partial_2^{\mathscr{L}}(G), \ldots, \partial_n^{\mathscr{L}}(G)\}$ denote the spectrum of $\mathscr{D}\mathscr{L}(G)$, we call it the *distance Laplacian spectrum* of the graph G and we assume that the distance Laplacian eigenvalues are labeled such that $\partial_1^{\mathscr{L}}(G) \geqslant \partial_2^{\mathscr{L}}(G) \geqslant \cdots \geqslant \partial_n^{\mathscr{L}}(G) = 0$. In particular, let $\partial_1^{\mathscr{L}}(G) > \partial_2^{\mathscr{L}}(G) > \cdots > \partial_k^{\mathscr{L}}(G)$ be all distinct eigenvalues of $\mathscr{D}\mathscr{L}(G)$ with multiplicity $m_1, m_2, \ldots, m_k$. Then the spectrum of $\mathscr{D}\mathscr{L}(G)$ are denoted by

$$\begin{pmatrix} \partial_1^{\mathscr{L}}(G) & \partial_2^{\mathscr{L}}(G) & \cdots & \partial_k^{\mathscr{L}}(G) \\ m_1 & m_2 & \cdots & m_k \end{pmatrix}.$$

Given a real interval I, $m_{\mathscr{D}\mathscr{L}(G)}I$ denotes the number of distance Laplacian eigenvalues of G in I.

In recent years, the distribution of Laplacian eigenvalues of a graph G in relation to various graph parameters of G has been studied extensively. Similarly, distance Laplacian spectrum of graphs have attracted a lot of attention, see [1, 2, 3, 4, 5, 6, 7, 8, 11]. Aouchiche and Hansen [1] gave the distance Laplacian characteristic polynomials of some special graphs, and proved that the distance Laplacian eigenvalues do not decrease on deletion of edges. In [2], the authors investigated some particular distance Laplacian eigenvalues. Among other results, they showed that the complete graph is the unique graph with only two distinct distance Laplacian eigenvalues. Pirzada and Saleem Khan [9] gave the distribution of the distance Laplacian eigenvalues of G in terms of the chromatic number $\chi(G)$. In fact, we find that there are few results on the distribution of distance Laplacian eigenvalues.

Therefore, the main purpose of this article is to understand how the eigenvalues of the matrix $\mathscr{D}\mathscr{L}(G)$ are distributed and how this distribution is related to classical parameters of graphs. The paper is organized as follows. In Section 2, we give a list of some previously known results. In Section 3, we adapt two methods to study the relationship between vertex connectivity and the distribution of distance Laplacian eigenvalues. In Section 4, we obtain the distribution of distance Laplacian eigenvalues for a connected graph G in terms of the size of DIM.

2. Preliminaries

Here we mention some preliminary results that will be needed for proving our main results in the next two sections.

LEMMA 2.1. [2] *For a connected graph G, we have $\partial_n^{\mathscr{L}}(G) = 0$ with multiplicity 1.*

LEMMA 2.2. [1] *Let G be a connected graph on n vertices. Then $\partial_{n-1}^{\mathscr{L}}(G) \geqslant n$ with equality holding if and only if $\overline{G}$ is disconnected. Furthermore, the multiplicity of n as an eigenvalue of $\mathscr{D}\mathscr{L}(G)$ is one less than the number of components of $\overline{G}$.*

The following lemma is about how the distance Laplacian eigenvalues change under the deletion of an edge.

LEMMA 2.3. [1] *Let G be a connected graph on n vertices and $m \geqslant n$ edges. Consider the connected graph G^* obtained from G by the deletion of an edge. Let $\partial_1^{\mathscr{L}}(G) \geqslant \partial_2^{\mathscr{L}}(G) \geqslant \cdots \geqslant \partial_n^{\mathscr{L}}(G)$ and $\partial_1^{\mathscr{L}}(G^*) \geqslant \partial_2^{\mathscr{L}}(G^*) \geqslant \cdots \geqslant \partial_n^{\mathscr{L}}(G^*)$ be the distance Laplacian eigenvalues of G and G^*, respectively. Then $\partial_i^{\mathscr{L}}(G^*) \geqslant \partial_i^{\mathscr{L}}(G)$ for all $i = 1, \ldots, n$.*

Next, we give the distance Laplacian characteristic polynomial of the join of two graphs.

LEMMA 2.4. [10] *Let G_1 and G_2 be graphs of order n_1 and n_2, respectively. Let $\lambda_1(G_1) \geqslant \lambda_2(G_1) \geqslant \cdots \geqslant \lambda_{n_1}(G_1) = 0$ and $\lambda_1(G_2) \geqslant \lambda_2(G_2) \geqslant \cdots \geqslant \lambda_{n_2}(G_2) = 0$ be the Laplacian eigenvalues of G_1 and G_2. Then the distance Laplacian characteristic polynomial $\Theta(x)$ of $G_1 \vee G_2$ is given by*

$$\Theta(G_1 \vee G_2, x) = x(x - n_1 - n_2)\prod_{i=1}(x - 2n_1 - n_2 + \lambda_i(G_1))\prod_{j=1}(x - n_1 - 2n_2 + \lambda_j(G_2)),$$

where $\lambda_i(G_1)$ $(1 \leqslant i \leqslant n_1 - 1)$ and $\lambda_j(G_2)$ $(j = 1, 2, \ldots, n_2 - 1)$ are the non-zero Laplacian eigenvalues of G_1 and G_2 respectively.

For some graphs, a local regularity is enough to know some eigenvalue of a graph. The following lemma gives the distance Laplacian eigenvalues of a graph if it contains a clique whose all vertices share the same neighborhood.

LEMMA 2.5. [2] *Let G be a graph on n vertices. If $K = \{v_1, v_2, \ldots, v_k\}$ is a clique of G such that $N(v_i) - K = N(v_j) - K$ for all $i, j \in \{1, 2, \ldots, k\}$. Then $\partial = Tr_G(v_i) = Tr_G(v_j)$ for all $i, j \in \{1, 2, \ldots, k\}$ and $\partial + 1$ is an eigenvalue of $\mathscr{DL}(G)$ with multiplicity at least $k - 1$.*

3. Distribution of distance Laplacian eigenvalues and vertex-connectivity

In this section, we devote to investigate the distribution of distance Laplacian eigenvalues of a graph G in the interval $[n, n+1)$ with respect to the vertex-connectivity $\kappa(G)$. First, we apply vector method to give their relationships.

THEOREM 3.1. *Let G be a connected graph with n vertices having vertex-connectivity $\kappa(G)$. Then*

$$m_{\mathscr{DL}(G)}[n, n+1) \leqslant \kappa(G). \tag{1}$$

Proof I. Assume that $S \subset V(G)$ is a vertex cut of G with $|S| = \kappa(G)$ such that the subgraph induced by $V(G) \setminus S$ is disconnected. Then let $G - S = U_1 \cup U_2 \cup \cdots \cup U_t$, where U_i is the connected components of $G - S$, for $i = 1, 2, \ldots, t$.

Let G' be a graph obtained from G by adding edges between all nonadjacent vertices within each U_i, and then adding edges between vertices of U_i and vertices of S such that the vertices of U_i and S are all adjacent. Thus we have

$$G' - S = U_1' \cup U_2' \cup \cdots \cup U_t',$$

where each U_i' is complete subgraph for $i = 1, 2, \ldots, t$.

By using Lemmas 2.1 and 2.2, we know that $\partial_n^{\mathscr{L}}(G) = 0$ and $\partial_{n-1}^{\mathscr{L}}(G) \geqslant n$ for any connected graph. In addition, from Lemma 2.3, we get that deletion of an edge from a graph does not decrease the corresponding distance Laplacian eigenvalues. Therefore, we can obtain

$$m_{\mathscr{D}\mathscr{L}(G)}[n, n+1) \leqslant m_{\mathscr{D}\mathscr{L}(G')}[n, n+1).$$

To complete the proof of (1), we only need to show that the graph G' satisfies the following inequality

$$m_{\mathscr{D}\mathscr{L}(G')}[n, n+1) \leqslant |S|.$$

Suppose that the number of vertices in each U_i' is n_i, where $i = 1, 2, \ldots, t$. Then

$$\sum_{i=1}^{t} n_i + |S| = |V(G')| = n.$$

Now, we label the vertices in U_i' as $V(U_i') = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n_i}\}$. Based on the structure of G', we can acquire that the distance Laplacian matrix of G' is

$$\left(\begin{array}{cccc:c}
(2n-n_1-|S|)I_{n_1\times n_1} - J_{n_1\times n_1} & -2J_{n_1\times n_2} & \cdots & -2J_{n_1\times n_t} & -J_{n_1\times |S|} \\
-2J_{n_2\times n_1} & (2n-n_2-|S|)I_{n_2\times n_2} - J_{n_2\times n_2} & \cdots & -2J_{n_2\times n_t} & -J_{n_2\times |S|} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-2J_{n_t\times n_1} & -2J_{n_t\times n_2} & \cdots & (2n-n_t-|S|)I_{n_t\times n_t} - J_{n_t\times n_t} & -J_{n_t\times |S|} \\
\hdashline
-J_{|S|\times n_1} & -J_{|S|\times n_2} & \cdots & -J_{|S|\times n_t} & R_{|S|\times |S|}
\end{array}\right),$$

where $I_{i\times i}$ is the identity matrix of $i \times i$, $J_{p\times q}$ is the matrix with each entry 1 and $R_{|S|\times|S|}$ is a matrix with order $|S| \times |S|$.

Let $\theta_{i,j}$ $(i = 1, 2, \ldots, t;\ j = 2, 3, \ldots, n_i)$ be a column vector in R^n with respect to U_i' such that

$$(\theta_{i,j})_v = \begin{cases} 1, & \text{if } v = u_{i,1}; \\ -1, & \text{if } v = u_{i,j}; \\ 0, & \text{otherwise}, \end{cases}$$

where $(\theta_{i,j})_v$ denotes the entry of the vector $\theta_{i,j}$ indexed by v. Then we can easily find that all $\theta_{i,j}$ $(i = 1, 2, \ldots, t; j = 2, 3, \ldots, n_i)$ are linearly independent eigenvectors of $\mathscr{D}\mathscr{L}(G')$ corresponding to eigenvalue $2n - n_i - |S|$.

Moreover, for $i = 2, 3, \ldots, t$, let η_i be the vector in R^n with respect to a pair of (U_1', U_i') such that

$$(\eta_i)_v = \begin{cases} 1, & \text{if } v \in V(U_1'); \\ -\frac{n_1}{n_i}, & \text{if } v \in V(U_i'); \\ 0, & \text{otherwise.} \end{cases}$$

By a simple calculation, we get $\eta_2,\eta_3,\ldots,\eta_t$ are linearly independent eigenvectors of $\mathscr{D}\mathscr{L}(G')$ corresponding to eigenvalue $2n-|S|$.

Next, suppose that W is the set of all the eigenvectors we just mentioned,

$$W=\{\theta_{i,j}:1\leqslant i\leqslant t,2\leqslant j\leqslant n_i\}\bigcup\{\eta_i:2\leqslant i\leqslant t\}.$$

Apparently, all of the eigenvectors in W are also linearly independent and

$$|W|=(\sum_{i=1}^{t}n_i-t)+(t-1)=n-|S|-1. \tag{2}$$

It is observed that $0<|S|<n_i+|S|\leqslant n-1$, then we have

$$n+1\leqslant 2n-n_i-|S|<2n-|S|<2n. \tag{3}$$

Combining (2) and (3), we attain

$$m_{\mathscr{D}\mathscr{L}(G')}[n+1,2n)\geqslant n-|S|-1.$$

On the other hand, according to Lemmas 2.1 and 2.2, we can also obtain

$$m_{\mathscr{D}\mathscr{L}(G')}[0,n+1)\leqslant |S|+1,$$

which implies that the number of distance Laplacian eigenvalues of G' over the interval $[n,n+1)$ is at most $|S|$. This completes the proof. □

In what follows, we will use another method to prove the Theorem 3.1. Noting that the Laplacian spectrum of the complete graph K_n is $\{n^{n-1},0\}$.

Proof II. The assumption here is the same as above, $S\subset V(G)$ is still a set of vertices that $|S|=\kappa(G)$, the subgraph induced by $V(G)\setminus S$ is disconnected, $G-S=U_1\cup U_2\cup\cdots\cup U_t$, where U_i is a component of $G-S$, for $i=1,2,\ldots,t$ and $|U_i|=n_i$. However, the structure of G' is different from the above, and G' is obtained by the following way.

For each U_i $(i=1,2,\ldots,t)$, we add edges between non-adjacent vertices of U_i and join all vertices of U_i to vertices of S and add edges between all non-adjacent vertices of S. The resulting graph is G', that is to say, $G'=K_{\kappa(G)}\vee(K_{n_1}\cup K_{n_2}\cup\cdots\cup K_{n_t})$. By Lemma 2.4, we observe that the distance Laplacian characteristic polynomial of G' is

$$\begin{aligned}\Theta(G',x)&=x(x-n)^{\kappa(G)}(x-2n+\kappa(G))^{t-1}(x-2n+\kappa(G)+n_1)^{n_1-1}\\&\quad\cdots(x-2n+\kappa(G)+n_t)^{n_t-1}.\end{aligned}$$

Therefore, the distance Laplacian spectrum of G' is

$$\begin{pmatrix}2n-\kappa(G) & 2n-\kappa(G)-n_1 & 2n-\kappa(G)-n_2 & \cdots\cdots & 2n-\kappa(G)-n_t & n & 0\\ t-1 & n_1-1 & n_2-1 & \cdots\cdots & n_t-1 & \kappa(G) & 1\end{pmatrix}.$$

Since $n_i+\kappa(G)\leqslant n-1$, we have $2n-\kappa(G)-n_i\geqslant n+1$ and $m_{\mathscr{D}\mathscr{L}(G')}[n,n+1)=\kappa(G)$. Consequently, according to Lemma 2.3, $m_{\mathscr{D}\mathscr{L}(G)}[n,n+1)\leqslant\kappa(G)$. This completes the proof. □

REMARK 1. Among the above two proof methods, we can find that the Proof II is relatively simple, but it applies the technique of characteristic polynomial method of graph operations. Hence, this method can only be used to prove when the extremal graph is special and has a fixed structure. However, Proof I is more universal. Since the vector method mentioned in it is widely used in the proof of other problems, this approach is enlightening and consistent. Meanwhile, it is also a very effective tool in the theory of graph spectra.

In fact, by the proof II of Theorem 3.1, it is not difficult to verify that if G is an spanning subgraph of $K_{\kappa(G)} \vee (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t})$, then $m_{\mathscr{DL}(G)}[n, n+1) \leqslant \kappa(G)$ is always true. Now, we give some examples to show that for many graphs G, the bounds of $m_{\mathscr{DL}(G)}[n, n+1)$ are sharp.

EXAMPLE 1. The distance Laplacian spectrum of the star $K_{1,n-1}$ is

$$\begin{pmatrix} 2n-1 & n & 0 \\ n-2 & 1 & 1 \end{pmatrix}.$$

It is not difficult to find that $m_{\mathscr{DL}(K_{1,n-1})}[n, n+1) = 1 = \kappa(K_{1,n-1})$.

EXAMPLE 2. The complete split graph $CS(n,s)$, is a graph on n vertices consisting of a clique on $n-s$ vertices and an independent set on the remaining s vertices in which each vertex of the clique is adjacent to each vertex of the independent set. That is to say, $CS(n,s) = K_{n-s} \vee \overline{K_s}$. Then by Lemma 2.4, we have

$$\Theta(CS(n,s), x) = x(x-n)^{n-s}(x-n-s)^{s-1},$$

and its distance Laplacian spectrum is

$$\begin{pmatrix} n+s & n & 0 \\ s-1 & n-s & 1 \end{pmatrix}.$$

This shows that $m_{\mathscr{DL}(CS(n,s))}[n, n+1) = n-s = \kappa(CS(n,s))$.

EXAMPLE 3. Let $PA_{n,p}$ denote the pineapple graph, obtained from a clique K_{n-p} by attaching $p > 0$ pending edges to a vertex from the clique. Then we have the distance Laplacian spectrum of $PA_{n,p}$ as

$$\begin{pmatrix} 2n-1 & n+p & n & 0 \\ p & n-p-2 & 1 & 1 \end{pmatrix}.$$

Hence, there is one distance Laplacian eigenvalue which fall in the interval $[n, n+1)$ and equal to $\kappa(PA_{n,p})$.

4. Distribution of distance Laplacian eigenvalues with a DIM

In this section, we mainly focus our attention on the distribution of distance Laplacian eigenvalues of connected graphs with DIM. So the main results of this section are as follows.

THEOREM 4.1. *Let G be a connected graph of order n with a DIM $M(G) \subset E(G)$ such that $|M(G)| = s$. Then*

$$\begin{cases} m_{\mathscr{DL}(G)}[n, n+2s) \leqslant s+1, & \text{if } n \geqslant 4s, \\ m_{\mathscr{DL}(G)}[n, n+2s) \leqslant n-s, & \text{if } 2s \leqslant n < 4s. \end{cases} \tag{4}$$

Proof. Assume that G has a DIM $M(G) \subset E(G)$ such that $|M(G)| = s$. Thus $V(G)$ can be partitioned into two disjoint vertex subsets $V(M)$ and $V(S)$, where $V(S)$ is an independent set. It is known that all DIMs have the same size which is the size of a maximum IM, so the size of the DIM does not change whether we add edges linking the vertices of $V(M)$ with the vertices of $V(S)$. By adding $s(2(n-2s)+1) - E(G)$ edges between $V(M)$ and $V(S)$, we obtain G', and G' contains a CDIM.

Therefore, $G' = sK_2 \vee (n-2s)K_1$. According to Lemma 2.4, we acquire the distance Laplacian spectrum of G' is

$$\begin{pmatrix} 2(n-s) & n+2s & n+2s-2 & n & 0 \\ n-2s-1 & s-1 & s & 1 & 1 \end{pmatrix}.$$

Using Lemma 2.3, we have $m_{\mathscr{DL}(G)}[n, n+2s) \leqslant m_{\mathscr{DL}(G')}[n, n+2s)$. In order to complete the proof of (4), we just need to prove that

$$\begin{cases} m_{\mathscr{DL}(G')}[n, n+2s) \leqslant s+1, & \text{if } n \geqslant 4s, \\ m_{\mathscr{DL}(G')}[n, n+2s) \leqslant n-s, & \text{if } 2s \leqslant n < 4s. \end{cases}$$

According to the distance Laplace spectrum of G', we consider the following three cases.

Case 1. If $2(n-s) \geqslant n+2s$, that is $n \geqslant 4s$, the distance Laplacian spectrum of G' is given as follows

$$\begin{pmatrix} 2(n-s) & n+2s & n+2s-2 & n & 0 \\ n-2s-1 & s-1 & s & 1 & 1 \end{pmatrix}.$$

We can easily obtain that $m_{\mathscr{DL}(G')}[n, n+2s) = s+1$.

Case 2. If $n+2s-2 \leqslant 2(n-s) < n+2s$, that is $4s-2 \leqslant n < 4s$, the distance Laplace spectrum of G' can also be written as

$$\begin{pmatrix} n+2s & 2(n-s) & n+2s-2 & n & 0 \\ s-1 & n-2s-1 & s & 1 & 1 \end{pmatrix}.$$

It is easy to verify that $m_{\mathscr{DL}(G')}[n, n+2s) = n-s$.

Case 3. If $n \leqslant 2(n-s) < n+2s-2$, that is $2s \leqslant n < 4s-2$, the distance Laplace spectrum of G' is given by

$$\begin{pmatrix} n+2s & n+2s-2 & 2(n-s) & n & 0 \\ s-1 & s & n-2s-1 & 1 & 1 \end{pmatrix}.$$

In this case, we see that $m_{\mathscr{DL}(G')}[n, n+2s) = n-s$.

From the above analysis, it can be seen that if $2s \leqslant n < 4s$, $m_{\mathscr{DL}(G')}[n, n+2s) = n-s$ always holds, which proves the required inequality. $\square$

Netx, we find the distribution of distance Laplacian eigenvalues in the interval $(n, \partial_1^{\mathscr{L}}(G)]$ with respect to the specific structure of the graph.

THEOREM 4.2. *Let G be a connected graph with n vertices. If $n_d(G) = |\{v \in V(G) : d_G(v) = n-1\}|$, where $1 \leqslant n_d(G) \leqslant n-1$, then*

$$m_{\mathscr{DL}(G)}(n, \partial_1^{\mathscr{L}}(G)] \leqslant n - n_d(G) - 1. \tag{5}$$

Proof. Since G has $n_d(G)$ vertices with degree $n-1$, $\overline{G}$ has at least $n_d(G)+1$ components. By Lemma 2.2, we know that n is a distance Laplacian eigenvalue of G with multiplicity at least $n_d(G)$. It is known that 0 is an eigenvalue with multiplicity one. Thus,

$$m_{\mathscr{DL}(G)}(n, \partial_1^{\mathscr{L}}(G)] \leqslant n - n_d(G) - 1,$$

which completes the proof of Theorem 4.2. $\square$

THEOREM 4.3. *Let G be a connected graph of order n having clique number $\omega(G) \leqslant n-1$. If only one vertex of the corresponding maximum clique is adjacent to the vertices outside of the clique, then*

$$m_{\mathscr{DL}(G)}[n, 2n-\omega(G)) \leqslant n - \omega(G) + 1.$$

Proof. Assume that $S(G) = \{v_1, v_2, \ldots, v_{\omega(G)}\} \subseteq V(G)$ be the set of vertices of the maximum clique such that $v_{\omega(G)}$ is the only vertex having neighbours outside of $S(G)$. It's easy to see that the set of vertices $C(G) = \{v_1, v_2, \ldots, v_{\omega(G)-1}\}$ also form a clique such that every vertex of $C(G)$ is adjacent to $v_{\omega(G)}$ only outside of $C(G)$. Clearly, all the vertices belonging to $C(G)$ have the same transmission. For any $v_i \in C(G)$, $i = 1, 2, \ldots, \omega(G)-1$, we have

$$\partial = Tr_G(v_i) \geqslant \omega(G) - 1 + 2(n - \omega(G)) = 2n - \omega(G) - 1.$$

By Lemma 2.5, we observe that $\partial + 1$ is a distance Laplacian eigenvalue of G of multiplicity at least $\omega(G)-2$. So there are at least $\omega(G)-2$ distance Laplacian eigenvalues of G which are greater than or equal to $2n-\omega(G)$. In other words, we have

$$m_{\mathscr{DL}(G)}[2n-\omega(G), \partial_1^{\mathscr{L}}(G)] \geqslant \omega(G) - 2.$$

Therefore, by the above observation and Lemma 2.1, we obtain that

$$m_{\mathscr{D}\mathscr{L}(G)}[n, 2n-\omega(G)) \leqslant n-\omega(G)+1.$$

Thus the result is established. $\square$

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