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SHARP GENERALIZED UNCERTAINTY PRINCIPLES VIA FACTORIZATIONS

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Abstract. Using the factorizations of suitable operators, we establish several identities that give simple and direct understandings as well as provide the remainders and optimizers of the sharp generalized uncertainty principles.

1. Introduction

In quantum mechanics, the well-known Heisenberg-Pauli-Weyl Uncertainty Principle (henceforth, HUP for short) can be mathematically stated as the following inequality

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} |x|^2 |u|^2 dx \geqslant \frac{N^2}{4} \left(\int_{\mathbb{R}^N} |u|^2 dx \right)^2, \; u \in C_0^\infty(\mathbb{R}^N). \tag{1.1}$$

It can also be extended to functions u in appropriate Sobolev spaces via standard density arguments. See, e.g., [26, 30].

The physical meaning of (1.1) is that if u is a wave function, i.e. $\|u\|_2 = 1$, then since $p = -i\nabla$ denotes the momentum operator, we have that the position $\|xu\|_2$ and the momentum $\|\nabla u\|_2 = \|pu\|_2$ cannot be small enough simultaneously because of the estimate $\|pu\|_2 \|xu\|_2 \geqslant \frac{N}{2}$. Therefore, the HUP asserts that the more precisely the position of a particle is given, the less precisely can one say what its momentum is, and vice versa. Actually, the HUP is one of the fundamental differences between quantum and classical mechanics.

It is well-known that the constant $\frac{N^2}{4}$ is optimal (see, e.g., [17]). Moreover, equality in (1.1) can be attained by the Gaussian profiles of the form $u(x) = \alpha e^{-\beta |x|^2}$, $\beta > 0$. We note that these optimizers are not in the space $C_0^\infty(\mathbb{R}^N)$, but in a larger space which is the Schwartz space $\mathscr{S}(\mathbb{R}^N)$.

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It is also worth mentioning the Hydrogen Uncertainty Principle (HyUP) that can be stated as follows: for any $u \in C_0^\infty(\mathbb{R}^N)$, there holds

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \int_{\mathbb{R}^N} |u|^2 dx \geqslant \frac{(N-1)^2}{4} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|} dx \right)^2. \tag{1.2}$$

HyUP is an uncertainty principle in the sense that localization in u at the origin (i.e., increasing the probability that the electron's position is close to the nucleus) together with the Coulomb potential $|x|^{-1}$ imply its momentum $\|\nabla u\|_2$ must be large. Therefore, one can immediately deduce that the quantum mechanical energy of the hydrogenic atom is finite (e.g., see [19, 26]).

The constant $\frac{(N-1)^2}{4}$ in (1.2) is also sharp and the optimizers are of the form $u(x) = \alpha e^{-\beta|x|}$, $\beta > 0$ (see, e.g. [18]). Notice that in this case the extremal functions are not in $\mathscr{S}(\mathbb{R}^N)$ but in a Sobolev space, namely, for our purpose, $W^{1,2}(\mathbb{R}^N)$.

Related to the HUP and HyUP is the classical Hardy inequality (HI): for any $u \in C_0^\infty(\mathbb{R}^N)$, there holds

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx \geqslant \left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx. \tag{1.3}$$

It is worthy to mention that the HI is one of the most used inequalities in analysis, and is studied intensively and extensively in the literature. We refer the interested reader to the celebrated paper [1] for some pioneering improvements of (1.3) and their applications.

Uncertainty principles such as HUP, HyUP and HI have several mathematical and physical applications. For instance, in mathematics, uncertainty principles may be used to study variable-coefficient differential operators (e.g., [15]) such as certain Schrödinger operators, and so on. In physics, uncertainty principles may be used for establishing stability of matter. In particular, in [25], stronger uncertainty principles have been established and used for studying stability for more general systems (e.g., a many-electron atom or many fermion systems).

The HUP, HyUP and HI belong to a more general family of inequalities known as the Caffarelli-Kohn-Nirenberg inequalities:

$$\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx \right)^{\frac{1}{2}} \geqslant C(N,a,b) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx, \ u \in C_0^\infty(\mathbb{R}^N \setminus \{0\}), \tag{1.4}$$

where $a, b \in \mathbb{R}$ are given constants. The sharp constant $C(N,a,b)$ in (1.4), which can naturally be defined by

$$C(N,a,b) := \inf_{u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})} \frac{\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx \right)^{\frac{1}{2}}}{\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx},$$

has first been investigated in [4, 9] using some technical tools such as the Emden-Fowler transformation, the spherical harmonics decomposition and the Kelvin-type transform. Recently, the authors in [7] provided a very simple way to compute the optimal constant $C(N,a,b)$. More precisely, the main results in [4, 7, 9] can be read as follows:

THEOREM 1.1. *We have*

$$C(N,a,b) = \max\left\{\frac{|N-(a+b+1)|}{2}, \frac{|N-(3b-a+3)|}{2}\right\}.$$

More precisely, according to the location of the points (a,b) *in the plane, we have that*

1. *If* $(a,b) \in \mathscr{A}$, *then the best constant is* $C(N,a,b) = \frac{|N-(a+b+1)|}{2}$ *and it is achieved by the functions* $u(x) = D\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, *with* $t<0$ *in* $\mathscr{A}_1$ *and* $t>0$ *in* $\mathscr{A}_2$, *and* D *a nonzero constant.*

2. *If* $(a,b) \in \mathscr{B}$, *then the best constant is* $C(N,a,b) = \frac{|N-(3b-a+3)|}{2}$ *and it is achieved by the functions* $u(x) = D|x|^{2(b+1)-N}\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, *with* $t>0$ *in* $\mathscr{B}_1$ *and* $t<0$ *in* $\mathscr{B}_2$.

3. *In addition, the only values of the parameters where the best constant is not achieved are those on the line* $a=b+1$, *where* $C(N,b+1,b) = \frac{|N-2(b+1)|}{2}$.

Here

$$\begin{cases}\mathscr{A}_1 := \{(a,b) \mid b+1-a>0,\ b \leqslant (N-2)/2\},\\ \mathscr{A}_2 := \{(a,b) \mid b+1-a<0,\ b \geqslant (N-2)/2\},\\ \mathscr{A} := \mathscr{A}_1 \cup \mathscr{A}_2,\\ \mathscr{B}_1 := \{(a,b) \mid b+1-a<0,\ b \leqslant (N-2)/2\},\\ \mathscr{B}_2 := \{(a,b) \mid b+1-a>0,\ b \geqslant (N-2)/2\},\\ \mathscr{B} := \mathscr{B}_1 \cup \mathscr{B}_2.\end{cases}$$

We also refer the interested reader to [2, 3, 5, 6, 11, 12, 14, 16, 27, 28, 29], to name just a few, for related results.

Our main motivation of this article is the approach in [20] in which Gesztesy and Littlejohn showed how factorizations of singular, even-order partial differential operators give simple proofs for several Hardy-Rellich type inequalities. We also mention here that factorizing differential equations was used in the setting of the classical Hardy inequality and its improvements. See, for instance, [13, 21, 22, 23, 24]. Moreover, as noted in [20], the method of factorization is not only elementary, but also quite flexible when it comes to studying remainder terms and higher-order operators.

The principal purpose of this note is to employ the factorization method to investigate the optimal constant $C(N,a,b)$ and the optimizers of the generalized uncertainty principles (1.4). Our goal is to get some estimates on the remainder of (1.4). Our strategy is as follows: First, by working with suitable differential linear operators, we obtain the following results:

THEOREM 1.2. *For all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, there holds*

1. *If $(a,b) \in \mathscr{A}_1$, then*

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx - (N-1-a-b) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx$$
$$= \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} \left| \nabla \left(u.e^{\frac{|x|^{b+1-a}}{b+1-a}} \right) \right|^2 e^{-\frac{2|x|^{b+1-a}}{b+1-a}} dx.$$

2. *If $(a,b) \in \mathscr{A}_2$, then*

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx - (a+b+1-N) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx$$
$$= \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} \left| \nabla \left(u.e^{-\frac{|x|^{b+1-a}}{b+1-a}} \right) \right|^2 e^{\frac{2|x|^{b+1-a}}{b+1-a}} dx.$$

3. *If $(a,b) \in \mathscr{B}_1$, then*

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx - (N-3b+a-3) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx$$
$$= \int_{\mathbb{R}^N} \frac{1}{|x|^{2N-2b-4}} \left| \nabla \left(u|x|^{N-2b-2} e^{-\frac{|x|^{b+1-a}}{b+1-a}} \right) \right|^2 e^{\frac{2|x|^{b+1-a}}{b+1-a}} dx.$$

4. *If $(a,b) \in \mathscr{B}_2$, then*

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx - (3b-a+3-N) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx$$
$$= \int_{\mathbb{R}^N} \frac{1}{|x|^{2N-2b-4}} \left| \nabla \left(u|x|^{N-2b-2} e^{\frac{|x|^{b+1-a}}{b+1-a}} \right) \right|^2 e^{-\frac{2|x|^{b+1-a}}{b+1-a}} dx.$$

We note that the four identities in Theorem 1.2 have also been established in [8, 10] by a different method. Also, each of them holds for any $(a,b) \in \mathscr{A} \cup \mathscr{B}$. Hence, for all $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx + \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx - 2C(N,a,b) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx \geqslant 0$$

with $C(N,a,b) = \max\left\{ \frac{|N-(a+b+1)|}{2}, \frac{|N-(3b-a+3)|}{2} \right\}$. Moreover, on $\mathscr{A}_1, \mathscr{A}_2, \mathscr{B}_1, \mathscr{B}_2$, the optimal constant $2C(N,a,b$ is $(N-1-a-b)$, $(a+b+1-N)$, $(N-3b+a-3)$, and $(N-3b+a-3)$ respectively.

Next, by using the standard scaling-invariant method, we deduce from Theorem 1.2 that

THEOREM 1.3. (Theorem 1.1) *Let* $(a,b) \in \mathscr{A} \cup \mathscr{B}$. *Then for all* $u \in C_0^\infty(\mathbb{R}^N \setminus \{0\})$, *there holds*

$$\left(\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}} dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}} dx\right)^{\frac{1}{2}} \geqslant C(N,a,b) \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}} dx. \tag{1.5}$$

Here

$$C(N,a,b) = \max\left\{\frac{|N-(a+b+1)|}{2}, \frac{|N-(3b-a+3)|}{2}\right\}.$$

Moreover,

1. *If* $(a,b) \in \mathscr{A}$, *then the best constant is* $C(N,a,b) = \frac{|N-(a+b+1)|}{2}$ *and all the nontrivial optimizers are* $u(x) = D\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, *with* $t<0$ *in* $\mathscr{A}_1$ *and* $t>0$ *in* $\mathscr{A}_2$, *and* D *a nonzero constant.*
2. *If* $(a,b) \in \mathscr{B}$, *then the best constant is* $C(N,a,b) = \frac{|N-(3b-a+3)|}{2}$ *and all the nontrivial optimizers are* $u(x) = D|x|^{2(b+1)-N}\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, *with* $t>0$ *in* $\mathscr{B}_1$ *and* $t<0$ *in* $\mathscr{B}_2$.

2. Proofs of main results

Proof of Theorem 1.2. Assume that our functions are in $C_0^\infty(\mathbb{R}^N \setminus \{0\}) \setminus \{0\}$ throughout this proof. Let us consider the differential operator $\mathbf{T} = |x|^{-b}\nabla + (|x|^{-a-1} + \gamma|x|^{-b-2})x$. Then

$$\begin{aligned}
\langle \mathbf{T}u, \mathbf{v}\rangle &= \int_{\mathbb{R}^N} |x|^{-b}\nabla u \cdot \overline{\mathbf{v}} + (|x|^{-a-1} + \gamma|x|^{-b-2})ux \cdot \overline{\mathbf{v}} dx \\
&= \int_{\mathbb{R}^N} -u\nabla \cdot (|x|^{-b}\overline{\mathbf{v}}) + \left(|x|^{-a-1} + \gamma|x|^{-b-2}\right) ux \cdot \overline{\mathbf{v}} dx \\
&= \int_{\mathbb{R}^N} -u\left[\nabla(|x|^{-b}) \cdot \overline{\mathbf{v}} + |x|^{-b}\nabla \cdot \overline{\mathbf{v}}\right] + \left(|x|^{-a-1} + \gamma|x|^{-b-2}\right) ux \cdot \overline{\mathbf{v}} dx \\
&= \int_{\mathbb{R}^N} u\left[b|x|^{-b-2}x \cdot \overline{\mathbf{v}} - |x|^{-b}\nabla \cdot \overline{\mathbf{v}} + (|x|^{-a-1} + \gamma|x|^{-b-2})x \cdot \overline{\mathbf{v}}\right] dx \\
&= \int_{\mathbb{R}^N} u\left[((b+\gamma)|x|^{-b-2} + |x|^{-a-1})x \cdot \overline{\mathbf{v}} - |x|^{-b}\nabla \cdot \overline{\mathbf{v}}\right] dx.
\end{aligned}$$

Hence, its formal adjoint operator is $T^* = ((b+\gamma)|x|^{-b-2} + |x|^{-a-1})x \cdot -|x|^{-b}\nabla \cdot$. Therefore,

$$\begin{aligned}
T^*\mathbf{T}u = &\left[((b+\gamma)|x|^{-b-2} + |x|^{-a-1})x - |x|^{-b}\nabla\right] \cdot \left[|x|^{-b}\nabla u + (|x|^{-a-1} + \gamma|x|^{-b-2})ux\right] \\
= &((b+\gamma)|x|^{-b-2} + |x|^{-a-1})x \cdot |x|^{-b}\nabla u \\
&+ ((b+\gamma)|x|^{-b-2} + |x|^{-a-1})x \cdot (|x|^{-a-1} + \gamma|x|^{-b-2})ux \\
&- |x|^{-b}\nabla \cdot (|x|^{-b}\nabla u) \\
&- |x|^{-b}\nabla \cdot (|x|^{-a-1} + \gamma|x|^{-b-2})ux
\end{aligned}$$

$$
\begin{aligned}
&= x\cdot\nabla u((b+\gamma)|x|^{-2b-2}+|x|^{-a-b-1}) \\
&\qquad + u[(b+2\gamma)|x|^{-a-b-1}+\gamma(b+\gamma)|x|^{-2b-2})+|x|^{-2a}] \\
&\qquad + b|x|^{-2b-2}x\cdot\nabla u-|x|^{-2b}\Delta u \\
&\qquad - u[(N-a-1)|x|^{-a-b-1}+\gamma(N-b-2)|x|^{-2b-2}] \\
&\qquad - [(|x|^{-a-b-1}+\gamma|x|^{-2b-2})x\cdot\nabla u].
\end{aligned}
$$

As a consequence, we have

$$
\begin{aligned}
\langle u,T^*\mathbf{T}u\rangle &= \int_{\mathbb{R}^N} x\cdot u\nabla\overline{u}[(b+\gamma)|x|^{-2b-2}+|x|^{-a-b-1}] \\
&\qquad + |u|^2[(b+2\gamma)|x|^{-a-b-1}+\gamma(b+\gamma)|x|^{-2b-2})+|x|^{-2a}] \\
&\qquad + b|x|^{-2b-2}x\cdot u\nabla\overline{u}-|x|^{-2b}u\Delta\overline{u} \\
&\qquad - |u|^2[(N-a-1)|x|^{-a-b-1}+\gamma(N-b-2)|x|^{-2b-2}] \\
&\qquad - u[(|x|^{-a-b-1}+\gamma|x|^{-2b-2})x\cdot\nabla\overline{u}]dx \\
&= \int_{\mathbb{R}^N} |u|^2[(b-(N-a-1)+2\gamma)|x|^{-a-b-1} \\
&\qquad + (\gamma^2+b\gamma-\gamma(N-b-2))|x|^{-2b-2}+|x|^{-2a}] \\
&\qquad + |\nabla u|^2|x|^{-2b}dx \\
&= \int_{\mathbb{R}^N} |\nabla u|^2|x|^{-2b}+|u|^2|x|^{-2a} \\
&\qquad - ((N-a-b-1)-2\gamma)|u|^2|x|^{-a-b-1} \\
&\qquad + |u|^2[\gamma^2-\gamma(N-2b-2)]|x|^{-2b-2}dx.
\end{aligned}
$$

Now, if we choose $\gamma=0$, then by noting that $\langle u,T^*\mathbf{T}u\rangle=\|\mathbf{T}u\|_2^2$, we get

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^2|x|^{-2b}+|u|^2|x|^{-2a}dx-(N-a-b-1)\int_{\mathbb{R}^N}|u|^2|x|^{-a-b-1}dx \\
&= \int_{\mathbb{R}^N}\left||x|^{-b}\nabla u+|x|^{-a-1}xu\right|^2dx \\
&= \int_{\mathbb{R}^N}\frac{1}{|x|^{2b}}\left|\nabla\left(u.e^{\frac{|x|^{b+1-a}}{b+1-a}}\right)\right|^2e^{-\frac{2|x|^{b+1-a}}{b+1-a}}dx.
\end{aligned}
$$

Similarly, choosing $\gamma=(N-2b-2)$ yields

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^2|x|^{-2b}+|u|^2|x|^{-2a}dx+(N+a-3b-3)\int_{\mathbb{R}^N}|u|^2|x|^{-a-b-1}dx \\
&= \int_{\mathbb{R}^N}\left||x|^{-b}\nabla u+(|x|^{-a-1}+(N-2b-2)|x|^{-b-2})xu\right|^2dx \\
&= \int_{\mathbb{R}^N}\frac{1}{|x|^{2N-2b-4}}\left|\nabla\left(u|x|^{N-2b-2}e^{\frac{|x|^{b+1-a}}{b+1-a}}\right)\right|^2e^{-\frac{2|x|^{b+1-a}}{b+1-a}}dx.
\end{aligned}
$$

Now, let us define $\mathbf{S}u = |x|^{-b}\nabla u - [|x|^{-a-1} + \gamma|x|^{-b-2}]ux$. Then

$$\begin{aligned}
\langle \mathbf{S}u, \mathbf{v}\rangle &= \int_{\mathbb{R}^N} \left[|x|^{-b}\nabla u - (|x|^{-a-1} + \gamma|x|^{-b-2})ux\right] \cdot \overline{\mathbf{v}}\, dx \\
&= \int_{\mathbb{R}^N} \nabla u \cdot |x|^{-b}\overline{\mathbf{v}} - (|x|^{-a-1} + \gamma|x|^{-b-2})ux \cdot \overline{\mathbf{v}}\, dx \\
&= \int_{\mathbb{R}^N} -u\nabla \cdot \left(|x|^{-b}\overline{\mathbf{v}}\right) - \left[|x|^{-a-1} + \gamma|x|^{-b-2}\right] ux \cdot \overline{\mathbf{v}}\, dx \\
&= \int_{\mathbb{R}^N} -u[\nabla \left(|x|^{-b}\right) \cdot \overline{\mathbf{v}} + |x|^{-b}\nabla \cdot \overline{\mathbf{v}}] - [|x|^{-a-1} + \gamma|x|^{-b-2}]ux \cdot \overline{\mathbf{v}}\, dx \\
&= \int_{\mathbb{R}^N} u \left[((b-\gamma)|x|^{-b-2} - |x|^{-a-1})x - |x|^{-b}\nabla\right] \cdot \overline{\mathbf{v}}\, dx.
\end{aligned}$$

Hence, its formal adjoint operator is $S^* = [(b-\gamma)|x|^{-b-2} - |x|^{-a-1}]x\cdot - |x|^{-b}\nabla\cdot$. Therefore,

$$\begin{aligned}
S^*\mathbf{S}u &= \left[[(b-\gamma)|x|^{-b-2} - |x|^{-a-1}]x - |x|^{-b}\nabla\right] \cdot \left[|x|^{-b}\nabla u - [|x|^{-a-1} + \gamma|x|^{-b-2}]ux\right] \\
&= ((b-\gamma)|x|^{-b-2} - |x|^{-a-1})x \cdot |x|^{-b}\nabla u \\
&\qquad + ((b-\gamma)|x|^{-b-2} - |x|^{-a-1})x \cdot (-xu(|x|^{-a-1} + \gamma|x|^{-b-2})) \\
&\qquad + (-|x|^{-b}\nabla) \cdot (|x|^{-b}\nabla u) \\
&\qquad + (-|x|^{-b}\nabla) \cdot (-xu(|x|^{-a-1} + \gamma|x|^{-b-2})) \\
&= x \cdot \nabla u((b-\gamma)|x|^{-2b-2} - |x|^{-a-b-1}) \\
&\qquad + u((2\gamma - b)|x|^{-a-b-1} - \gamma(b-\gamma)|x|^{-2b-2} + |x|^{-2a}) \\
&\qquad + b|x|^{-2a-2}x \cdot \nabla u - |x|^{-2b}\Delta u \\
&\qquad + u((N-a-1)|x|^{-a-b-1} \\
&\qquad + \gamma(N-b-2)|x|^{-2b-2}) + \nabla u \cdot x(|x|^{-a-b-1} + \gamma|x|^{-2b-2}).
\end{aligned}$$

This implies that

$$\begin{aligned}
\langle u, S^*\mathbf{S}u\rangle &= \int_{\mathbb{R}^N} u\big[(x \cdot \nabla\overline{u}((b-\gamma)|x|^{-2b-2} - |x|^{-a-b-1}) \\
&\qquad + \overline{u}((2\gamma - b)|x|^{-a-b-1} - \gamma(b-\gamma)|x|^{-2b-2} + |x|^{-2a}) \\
&\qquad + b|x|^{-2a-2}x \cdot \nabla\overline{u} - |x|^{-2b}\Delta\overline{u} \\
&\qquad + \overline{u}((N-a-1)|x|^{-a-b-1} + \gamma(N-b-2)|x|^{-2b-2}) \\
&\qquad + \nabla\overline{u} \cdot x(|x|^{-a-b-1} + \gamma|x|^{-2b-2})\big]\, dx \\
&= \int_{\mathbb{R}^N} |\nabla u|^2|x|^{-2b} + |u|^2|x|^{-2a} + (N-a-b-1+2\gamma)|u|^2|x|^{-a-b-1} \\
&\qquad + |u|^2|x|^{-2b-2}(\gamma^2 + \gamma(N-2b-2))\, dx.
\end{aligned}$$

Now, if we choose $\gamma = 0$, then by noting that $\langle u, S^*\mathbf{S}u\rangle = \|\mathbf{S}u\|_2^2$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^2|x|^{-2b} + |u|^2|x|^{-2a}dx - (1+a+b-N)\int_{\mathbb{R}^N} |u|^2|x|^{-a-b-1}dx \\
&= \int_{\mathbb{R}^N} \left| |x|^{-b}\nabla u - |x|^{-a-1}xu\right|^2 dx \\
&= \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} \left|\nabla\left(u.e^{-\frac{|x|^{b+1-a}}{b+1-a}}\right)\right|^2 e^{\frac{2|x|^{b+1-a}}{b+1-a}} dx.
\end{aligned}$$

Similarly, choosing $\gamma = -(N-2b-2)$ yields

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u|^2|x|^{-2b} + |u|^2|x|^{-2a}dx - (N+a-3b-3)\int_{\mathbb{R}^N} |u|^2|x|^{-a-b-1}dx \\
&= \int_{\mathbb{R}^N} \left| |x|^{-b}\nabla u - (|x|^{-a-1} - (N-2b-2)|x|^{-b-2})xu\right|^2 dx \\
&= \int_{\mathbb{R}^N} \frac{1}{|x|^{2N-2b-4}} \left|\nabla\left(u|x|^{N-2b-2}e^{-\frac{|x|^{b+1-a}}{b+1-a}}\right)\right|^2 e^{\frac{2|x|^{b+1-a}}{b+1-a}} dx. \quad \square
\end{aligned}$$

Proof of Theorem 1.3 (Alternative proof of Theorem 1.1). Let $u \in C_0^\infty(\mathbb{R}^N\setminus\{0\})\setminus\{0\}$ and $\lambda = \left(\frac{\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx}{\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}dx}\right)^{\frac{1}{2(b+1-a)}}$. Assume that $(a,b) \in \mathscr{A}_1$. Recall that since $(a,b) \in \mathscr{A}_1$, for all $v \in C_0^\infty(\mathbb{R}^N\setminus\{0\})\setminus\{0\}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|\nabla v|^2}{|x|^{2b}}dx + \int_{\mathbb{R}^N} \frac{|v|^2}{|x|^{2a}}dx - (N-1-a-b)\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^{a+b+1}}dx \\
&= \int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} \left|\nabla\left(v.e^{\frac{|x|^{b+1-a}}{b+1-a}}\right)\right|^2 e^{-\frac{2|x|^{b+1-a}}{b+1-a}} dx. \qquad (2.1)
\end{aligned}$$

Now, if we choose $v(x) = u(\lambda x)$, then $\nabla v(x) = \lambda(\nabla u)(\lambda x)$. Therefore, by making change of variables, we obtain that

$$\int_{\mathbb{R}^N} \frac{|\nabla v|^2}{|x|^{2b}}dx = \lambda^{(2+2b-N)}\int_{\mathbb{R}^N} \frac{|(\nabla u)(\lambda x)|^2}{|\lambda x|^{2b}}d(\lambda x) = \lambda^{(2+2b-N)}\int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2b}}dx,$$

$$\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^{2a}}dx = \lambda^{(2a-N)}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{2a}}dx,$$

$$\int_{\mathbb{R}^N} \frac{|v|^2}{|x|^{a+b+1}}dx = \lambda^{(a+b+1-N)}\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^{a+b+1}}dx,$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} \left|\nabla\left(v.e^{\frac{|x|^{b+1-a}}{b+1-a}}\right)\right|^2 e^{-\frac{2|x|^{b+1-a}}{b+1-a}} dx \\
&= \lambda^{2+2b-N}\int_{\mathbb{R}^N} \frac{1}{|x|^{2b}} \left|\nabla\left(ue^{\frac{|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}\right)\right|^2 e^{-\frac{2|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}} dx.
\end{aligned}$$

Therefore, (2.1) becomes

$$\lambda^{b-a+1}\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}dx+\lambda^{a-b-1}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx-(N-1-a-b)\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx$$
$$=\lambda^{b-a+1}\int_{\mathbb{R}^N}\frac{1}{|x|^{2b}}\left|\nabla\left(ue^{\frac{|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}\right)\right|^2 e^{-\frac{2|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}dx. \tag{2.2}$$

By choosing $\lambda=\left(\frac{\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx}{\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}dx}\right)^{\frac{1}{2(b+1-a)}}$, we obtain

$$\left(\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}dx\right)^{\frac12}\left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx\right)^{\frac12}-\left|\frac{N-a-b-1}{2}\right|\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx$$
$$=\frac12\lambda^{b-a+1}\int_{\mathbb{R}^N}\frac{1}{|x|^{2b}}\left|\nabla\left(ue^{\frac{|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}\right)\right|^2 e^{-\frac{2|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}dx.$$

Similarly, if $(a,b)\in\mathscr{A}_2$, then

$$\left(\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}dx\right)^{\frac12}\left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx\right)^{\frac12}-\left|\frac{a+b+1-N}{2}\right|\left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx\right)$$
$$=\frac12\lambda^{b-a+1}\int_{\mathbb{R}^N}\frac{1}{|x|^{2b}}\left|\nabla\left(ue^{-\frac{|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}\right)\right|^2 e^{\frac{2|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}dx.$$

If $(a,b)\in\mathscr{B}_1$, then

$$\left(\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}dx\right)^{\frac12}\left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx\right)^{\frac12}-\left|\frac{N-3b+a-3}{2}\right|\left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx\right)$$
$$=\frac12\lambda^{b-a+1}\int_{\mathbb{R}^N}\frac{1}{|x|^{2N-2b-4}}\left|\nabla\left(u|x|^{N-2b-2}e^{-\frac{|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}\right)\right|^2 e^{\frac{2|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}dx.$$

If $(a,b)\in\mathscr{B}_2$, then

$$\left(\int_{\mathbb{R}^N}\frac{|\nabla u|^2}{|x|^{2b}}dx\right)^{\frac12}\left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2a}}dx\right)^{\frac12}-\left|\frac{3b-a+3-N}{2}\right|\left(\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{a+b+1}}dx\right)$$
$$=\frac12\lambda^{b-a+1}\int_{\mathbb{R}^N}\frac{1}{|x|^{2N-2b-4}}\left|\nabla\left(u|x|^{N-2b-2}e^{\frac{|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}\right)\right|^2 e^{-\frac{2|x|^{b+1-a}}{(b+1-a)\lambda^{b-a+1}}}dx.$$

From the above identities, it is easy to deduce that if $(a,b)\in\mathscr{A}$, then the best constant is $C(N,a,b)=\frac{|N-(a+b+1)|}{2}$ and it is achieved only by the functions $u(x)=D\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, with $t<0$ in $\mathscr{A}_1$ and $t>0$ in $\mathscr{A}_2$. Also, if $(a,b)\in\mathscr{B}$, then the best constant is $C(N,a,b)=\frac{|N-(3b-a+3)|}{2}$ and it is achieved only by the functions $u(x)=D|x|^{2(b+1)-N}\exp(\frac{t|x|^{b+1-a}}{b+1-a})$, with $t>0$ in $\mathscr{B}_1$ and $t<0$ in $\mathscr{B}_2$. It is also worthy to note that these optimizers are not in $C_0^\infty(\mathbb{R}^N\setminus\{0\})$. However, we refer the interested reader to [7] for the arguments that they belong to the suitable Sobolev spaces, and therefore are truly the optimizers of the L^2-Caffarelli-Kohn-Nirenberg inequality (1.5). □

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