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NOTE ON BOUNDS FOR SECOND EXTREME EIGENVALUES OF HERMITIAN MATRICES

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Abstract. We obtain some bounds for the second smallest and second largest eigenvalues of a Hermitian matrix. Some additional bounds for the second extreme eigenvalues of positive definite matrices are also discussed here.

1. Introduction

The bounds for the eigenvalues of a matrix have been studied extensively in literature. There are many inequalities which provide bounds for the extreme eigenvalues of a matrix when all its eigenvalues are real as in case of Hermitian matrices. For some recent results related to the extreme eigenvalues of the Hermitian and nonnegative matrices, see [18, 25]. The second smallest and the second largest eigenvalues are also important in some contexts, for instance, these eigenvalues have been studied in detail in spectral graph theory. See [1, 2, 11, 12, 17, 19, 21, 22, 23, 26, 28]. We here consider more general case and discuss some bounds for the second extreme eigenvalues of a Hermitian matrix. We use the Cauchy interlacing theorem and the lower bounds of the spread of a matrix to derive our main results.

Let $\mathbb{M}(n)$ denote the algebra of all complex $n\times n$ matrices. The eigenvalues of a Hermitian element $A\in\mathbb{M}(n)$ are all real and we assume that they are arranged in ascending order as

$$\lambda_1(A)\leqslant\lambda_2(A)\leqslant\cdots\leqslant\lambda_{n-1}(A)\leqslant\lambda_n(A). \tag{1.1}$$

So, $\lambda_2(A)$ is the second smallest and $\lambda_{n-1}(A)$ is the second largest eigenvalue of A. The diagonal entries of a Hermitian matrix $A=(a_{ij})\in\mathbb{M}(n)$ are all real and we enumerate them as

$$a_1\leqslant a_2\leqslant\cdots\leqslant a_n. \tag{1.2}$$

Wolkowicz and Styan [27] have discussed various bounds for the eigenvalues using traces. In particular, it is shown that if the eigenvalues of $A\in\mathbb{M}(n)$ are all real,

$$\frac{\mathrm{tr}A}{n}-\sqrt{\frac{n-2}{2}}S\leqslant\lambda_2(A)\leqslant\frac{\mathrm{tr}A}{n}+\sqrt{\frac{1}{n-1}}S \tag{1.3}$$

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and

$$\frac{\mathrm{tr}A}{n}-\sqrt{\frac{1}{n-1}}S\leqslant\lambda_{n-1}(A)\leqslant\frac{\mathrm{tr}A}{n}+\sqrt{\frac{n-2}{2}}S \tag{1.4}$$

where $\mathrm{tr}A$ denotes the trace of A and

$$S^2=\frac{\mathrm{tr}A^2}{n}-\left(\frac{\mathrm{tr}A}{n}\right)^2. \tag{1.5}$$

Sharma and Pal [24] have shown that for a positive definite matrix $A=(a_{ij})\in\mathbb{M}(n)$,

$$\frac{1}{4}\left(\alpha_1(2)-\sqrt{\beta_1(2)}\right)\leqslant\lambda_2(A)\leqslant\frac{1}{2(n-1)}\left(\alpha_1(n-1)+\sqrt{\beta_1(n-1)}\right) \tag{1.6}$$

and

$$\frac{1}{2(n-1)}\left(\alpha_1(n-1)-\sqrt{\beta_1(n-1)}\right)\leqslant\lambda_{n-1}(A)\leqslant\frac{1}{4}\left(\alpha_1(2)+\sqrt{\beta_1(2)}\right) \tag{1.7}$$

where

$$\alpha_1(r)=\mathrm{tr}A-\frac{n(n-2r)}{\mathrm{tr}A^{-1}}\text{ and }\beta_1(r)=\alpha_1^2(r)-4r^2\frac{\mathrm{tr}A}{\mathrm{tr}A^{-1}}.$$

Some simple bounds involving fewer matrix entries are also given in [24].

A real symmetric matrix is a special case of a Hermitian matrix. Further, $A=(a_{ij})\in\mathbb{M}(n)$ is nonnegative if $a_{ij}\geqslant 0$ for all i,j. The second smallest eigenvalue of a nonnegative symmetric matrix is less than or equals its third smallest diagonal entry,

$$\lambda_2(A)\leqslant a_3. \tag{1.8}$$

See [3, 9, 10].

If A is Hermitian and its off-diagonal entries are all purely imaginary, then in addition to (1.8), we also have, [9],

$$\lambda_{n-1}(A)\geqslant a_{n-2}.$$

A matrix $A=(a_{ij})\in\mathbb{M}(n)$ with $a_{ij}>0$ is called a positive matrix. For a positive matrix a result due to Hopf [13] says that for any eigenvalue λ of A

$$|\lambda|\leqslant\frac{b-a}{b+a}\lambda_n(A) \tag{1.9}$$

where $a=\min_{i,j}a_{ij}$ and $b=\max_{i,j}a_{ij}$. One can easily see from (1.9) that if $A\in\mathbb{M}(n)$ is both positive and positive definite, then

$$\lambda_{n-1}(A)\leqslant\frac{b-a}{b+a}\lambda_n(A).$$

We derive an upper bound for the kth smallest eigenvalue of a positive semidefinite matrix in terms of its diagonal entries, (Theorem 2.1). An upper bound for $\lambda_2(A)$ and a lower bound for $\lambda_{n-1}(A)$ are given in terms of expressions involving any two diagonal entries and the corresponding off-diagonal entries of the principal submatrix containing these two diagonal entries, (Theorem 2.2). Some more bounds for $\lambda_2(A)$ and $\lambda_{n-1}(A)$ are obtained for Hermitian and positive definite matrices by using the Cauchy interlacing principle and positive linear functionals, (Theorem 3.1–3.8). We show that the bounds for the spreads of Hermitian matrices also provide bounds for the second extreme eigenvalues. In particular, we use lower bounds for the spreads and derive lower (upper) bounds for $\lambda_2(A)$ $(\lambda_{n-1}(A))$, (Theorem 4.1–4.4).

2. Bounds using eigenvalues of 2×2 submatrices

We use the Cauchy interlacing theorem of Hermitian matrices to derive the bounds for the second extreme eigenvalues of Hermitian matrices. Let A_r be any $r\times r$ principal submatrix of A and let the eigenvalues of A and A_r be arranged as in (1.1). The Cauchy interlacing principle says that for any Hermitian element $A\in\mathbb{M}(n)$, we have,

$$\lambda_i(A)\leqslant\lambda_i(A_r)\leqslant\lambda_{i+n-r}(A). \tag{2.1}$$

For more details see [4, 14].

Note that there are $\binom{n}{r}$ principal submatrices of order r, we denote them by A_{r_i}, calculate $\mathrm{tr}A_{r_i}$ for $i=1,2,\dots,\binom{n}{r}$ and denote the smallest one by $\min_i \mathrm{tr}A_{r_i}$. If A is positive semidefinite and its diagonal entries are arranged as in (1.2), we have

$$\min_i \mathrm{tr}A_{r_i}=a_1+a_2+\cdots+a_r. \tag{2.2}$$

The simplest bound for the extreme eigenvalues using traces says that $\lambda_1(A)\leqslant\frac{1}{n}\mathrm{tr}A\leqslant\lambda_n(A)$. There is no analogous bound for $\lambda_2(A)$ or $\lambda_{n-1}(A)$. We show in the following theorem that the bounds for the eigenvalues of a positive semidefinite matrix can be obtained in terms of its diagonal entries only.

THEOREM 2.1. *Let $A\in\mathbb{M}(n)$ be positive semidefinite and let its eigenvalues and diagonal entries be enumerated as in* (1.1) *and* (1.2)*, respectively. Then, for $k\leqslant r$,*

$$\lambda_k(A)\leqslant\frac{a_1+a_2+\cdots+a_r}{r-k+1}. \tag{2.3}$$

Proof. A principal submatrix of a positive semidefinite matrix is positive semidefinite. Let A_{r_i} be a principal submatrix of A of order r. Then, $\lambda_j(A_{r_i})\geqslant 0$, and $\sum_{j=1}^{r}\lambda_j(A_{r_i})=\mathrm{tr}A_{r_i}$. Therefore, for $k\leqslant r$,

$$(r-k+1)\lambda_k(A_{r_i})\leqslant\lambda_k(A_{r_i})+\cdots+\lambda_r(A_{r_i})\leqslant\mathrm{tr}A_{r_i}. \tag{2.4}$$

The inequality (2.4) holds for all $i=1,2,\dots,\binom{n}{r}$. Therefore, from (2.4), we get that

$$\lambda_k(A_{r_i})\leqslant\frac{1}{r-k+1}\min_i \mathrm{tr}(A_{r_i}). \tag{2.5}$$

From (2.2) and (2.5), we have

$$\lambda_k(A_{r_i}) \leqslant \frac{1}{r-k+1}(a_1+a_2+\cdots+a_r). \tag{2.6}$$

By using (2.1), we have $\lambda_k(A) \leqslant \lambda_k(A_{r_i})$. So, (2.6) implies (2.3). $\square$

From (2.3), we have for $r=2,3,\ldots,n$,

$$\lambda_2(A) \leqslant \frac{a_1+a_2+\cdots+a_r}{r-1} \leqslant \frac{r}{r-1}a_r.$$

For $r=3$, this yields $\lambda_2(A) \leqslant \frac{3}{2}a_3$. In case, A is nonnegative and symmetric, the inequality (1.8) provides a stronger bound, $\lambda_2(A) \leqslant a_3$.

The eigenvalues of a 2×2 matrix can be expressed in terms of its trace and determinant. By using (2.1) for any 2×2 principal submatrix of A, we find an upper (lower) bound for $\lambda_2(A)$ $(\lambda_{n-1}(A))$.

THEOREM 2.2. *Let $A=(a_{ij}) \in \mathbb{M}(n)$ be Hermitian and let its eigenvalues be arranged as in* (1.1)*. Then*

$$\lambda_2(A) \leqslant \frac{1}{2}\min_{r\neq s}\left(\alpha_{rs}+\sqrt{\beta_{rs}}\right) \text{ and } \lambda_{n-1}(A) \geqslant \frac{1}{2}\max_{r\neq s}\left(\alpha_{rs}-\sqrt{\beta_{rs}}\right) \tag{2.7}$$

where $\alpha_{rs}=a_{rr}+a_{ss}$ and $\beta_{rs}=(a_{rr}-a_{ss})^2+4|a_{rs}|^2$.

Proof. For $i=r=2$, (2.1) gives $\lambda_2(A) \leqslant \lambda_2(A_2)$ and for $i=1,\ r=2,\ \lambda_{n-1}(A) \geqslant \lambda_1(A_2)$. The assertions of the theorem then follow by using the fact that for $A_2 = \begin{bmatrix} a_{rr} & a_{rs} \\ \overline{a_{rs}} & a_{ss}\end{bmatrix}$, we have

$$\lambda_1(A_2) = \frac{1}{2}\left(\alpha_{rs}-\sqrt{\beta_{rs}}\right) \text{ and } \lambda_2(A_2) = \frac{1}{2}\left(\alpha_{rs}+\sqrt{\beta_{rs}}\right). \quad \square$$

We note that if $a_{rs}=0$ for some $r\neq s$, then from (2.7),

$$\lambda_2(A) \leqslant \max\{a_{rr},a_{ss}\} \text{ and } \lambda_{n-1}(A) \geqslant \min\{a_{rr},a_{ss}\}. \tag{2.8}$$

It may be noted that (2.8) can be extended to the case when the off-diagonal entries of any principal submatrix are all zero.

THEOREM 2.3. *Let $A=(a_{ij}) \in \mathbb{M}(n)$ be Hermitian. Let $A_k = D_k + N_k$ be any $k\times k$ principal submatrix of A whose diagonal part is D_k. If $N_k = O$, then*

$$\lambda_j(A) \leqslant n_j(D_k) \leqslant \lambda_{n-k+j}(A)$$

where $n_j(D_k)$ is the jth smallest diagonal entry of D_k, $j=1,2,\ldots,k$.

Proof. If $N_k = O$, then $\lambda_j(A_k) = n_j(D_k)$, $j = 1,2,\ldots,k$. So by (2.1), $\lambda_j(A) \leqslant \lambda_j(A_k) = n_j(D_k) \leqslant \lambda_{n-k+j}(A)$. $\square$

EXAMPLE 2.1. (Example 4, [27]) Let

$$A = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix}.$$

Then, $a_{12} = 0$ and $a_{34} = 0$, therefore (2.7) gives $\lambda_2(A) \leqslant 5$ and $\lambda_3(A) \geqslant 6$. The matrix A is nonnegative therefore from (1.8), $\lambda_2(A) \leqslant 6$. The estimates of Wolkowicz and Styan [27] give $\lambda_2(A) \leqslant 7.158$ and $\lambda_3(A) \geqslant 3.842$.

3. Bounds using positive linear functionals

In the above Theorem 2.2, we have derived the bounds for the second extreme eigenvalues in terms of the eigenvalues of 2×2 principal submatrices which are easily calculable. It is natural to extend this for 3×3 principal submatrices. In case, if the exact value of $\lambda_2(A_3)$ is easily calculable, we can use the inequality $\lambda_2(A) \leqslant \lambda_2(A_3)$ to find the upper bound for $\lambda_2(A)$ and likewise the lower bound for $\lambda_{n-1}(A)$. But it is not always easy to calculate the eigenvalues of 3×3 matrices. So, to obtain some further simple estimates, we here first derive the bounds for the second smallest eigenvalue of a 3×3 matrix using positive linear functionals and determinant, and then use interlacing inequalities (2.1) to derive the bounds for $\lambda_2(A)$ and $\lambda_{n-1}(A)$.

We need the following basic results in the proofs of the subsequent theorems.

1. A linear map $\Phi : \mathbb{M}(n) \longrightarrow \mathbb{M}(k)$ is said to be positive if $\Phi(A)$ is positive semidefinite whenever A is positive semidefinite. It is unital if $\Phi(I_n) = I_k$. In the special case when $k = 1$ such a map is called linear functional and it is customary to denote it by φ, see Bhatia [5].

2. Kadison's inequality [16] says that if $\Phi : \mathbb{M}(n) \longrightarrow \mathbb{M}(k)$ is a positive unital linear map and A is any Hermitian element of $\mathbb{M}(n)$, then

$$\Phi\left(A^2\right) \geqslant \Phi(A)^2. \tag{3.1}$$

3. The inequality complementary to (3.1) due to Bhatia and Davis [6] says that if $\Phi : \mathbb{M}(n) \longrightarrow \mathbb{M}(k)$ is a positive unital linear map and A is any Hermitian element of $\mathbb{M}(n)$ whose spectrum is contained in the interval $[m, M]$, then

$$\Phi\left(A^2\right) - \Phi(A)^2 \leqslant (MI_k - \Phi(A))(\Phi(A) - mI_k). \tag{3.2}$$

For $A \in \mathbb{M}(3)$, $\det(A - cI_3) = (\lambda_1(A) - c)(\lambda_2(A) - c)(\lambda_3(A) - c) = 0$ if and only if $\lambda_i(A) = c$ for some $i = 1,2,3$ where $\det A$ denotes the determinant of A. We exclude this trivial case and assume in the following discussion that $\det(A - cI_3) \neq 0$.

THEOREM 3.1. *Let $A \in \mathbb{M}(3)$ and let its eigenvalues be all real and arranged as in (1.1). Let c be a real number such that $\lambda_1(A) < c < \lambda_3(A)$. Let $B = A - cI_3$. If* $\det B > 0$, *then*

$$\lambda_2(A) \leqslant c - 2\frac{\det B}{trB^2} \leqslant c + 2\frac{\det B}{trB^2} \leqslant \lambda_3(A). \tag{3.3}$$

If $\det B < 0$, *then*

$$\lambda_1(A) \leqslant c + 2\frac{\det B}{trB^2} \leqslant c - 2\frac{\det B}{trB^2} \leqslant \lambda_2(A). \tag{3.4}$$

Proof. Let $X \in \mathbb{M}(n)$. Then, $\lambda_i(X^2) = \lambda_i(X)^2$ and $\lambda_i(X + cI_n) = \lambda_i(X) + c$. It follows that eigenvalues of B^2 are $(\lambda_j(A) - c)^2$. Also, $\lambda_j(A)$ and c are real numbers and $(\lambda_j(A) - c)^2 > 0$ for $j = 1,2,3,$ therefore, we have

$$\mathrm{tr}B^2 = \sum_{j=1}^{3} (\lambda_j(A) - c)^2 \geqslant (c - \lambda_1(A))^2 + (\lambda_3(A) - c)^2. \tag{3.5}$$

For $\lambda_1(A) < c < \lambda_3(A)$, $x = c - \lambda_1(A) > 0$ and $y = \lambda_3(A) - c > 0$. By arithmetic mean-geometric mean inequality $x^2 + y^2 \geqslant 2xy$. Using this fact, we find from (3.5) that

$$\mathrm{tr}B^2 \geqslant 2(c - \lambda_1(A))(\lambda_3(A) - c). \tag{3.6}$$

We also have,

$$\det B = (\lambda_1(A) - c)(\lambda_2(A) - c)(\lambda_3(A) - c). \tag{3.7}$$

From (3.6) and (3.7), we get

$$\mathrm{tr}B^2 \geqslant 2\frac{\det B}{c - \lambda_2(A)}. \tag{3.8}$$

For $\lambda_1(A) < c < \lambda_3(A)$, $(\lambda_1(A) - c)(\lambda_3(A) - c) < 0$. So, from (3.7), if $\det B > 0$, then $\lambda_2(A) < c$. In this case (3.8) implies the first inequality (3.3).

Likewise for $\lambda_2(A) < c$, we have

$$\mathrm{tr}B^2 \geqslant (c - \lambda_1(A))^2 + (c - \lambda_2(A))^2 \geqslant 2(c - \lambda_1(A))(c - \lambda_2(A)) = \frac{2\det B}{\lambda_3(A) - c}. \tag{3.9}$$

We have $\lambda_3(A) > c$. Then, (3.9) implies the third inequality (3.3). The second inequality (3.3) is evident.

In case, $\det B < 0$, we have $\lambda_2(A) > c$ and (3.8) implies the third inequality (3.4). For $\lambda_2(A) > c$, we also have

$$\mathrm{tr}B^2 \geqslant (\lambda_2(A) - c)^2 + (\lambda_3(A) - c)^2 \geqslant 2(\lambda_2(A) - c)(\lambda_3(A) - c) = \frac{2\det B}{\lambda_1(A) - c}. \tag{3.10}$$

We have $\lambda_1(A) < c$, therefore (3.10) gives the first inequality (3.4). The second inequality (3.4) is immediate. $\square$

THEOREM 3.2. *Let $A \in \mathbb{M}(n)$ be Hermitian and let its eigenvalues be arranged as in (1.1). Let A_3 be any 3×3 principal submatrix of A and $B_3 = A_3 - cI_3$ where c is a real number in the interval $(\lambda_1(B_3), \lambda_3(B_3))$. If $\det B_3 > 0$, then*

$$\lambda_2(A) \leqslant c - 2\frac{\det B_3}{trB_3^2} \leqslant c + 2\frac{\det B_3}{trB_3^2} \leqslant \lambda_n(A). \tag{3.11}$$

If $\det B_3 < 0$, then

$$\lambda_1(A) \leqslant c + 2\frac{\det B_3}{trB_3^2} \leqslant c - 2\frac{\det B_3}{trB_3^2} \leqslant \lambda_{n-1}(A). \tag{3.12}$$

Proof. Any principal submatrix of a Hermitian matrix is Hermitian. So, eigenvalues $\lambda_j(A_3)$ of A_3 are all real and let they be arranged as in (1.1). From (2.1) for $i = 2$, $r = 3$, $\lambda_2(A) \leqslant \lambda_2(A_3)$ and for $i = 3$, $r = 3$, $\lambda_3(A_3) \leqslant \lambda_n(A)$. We therefore have

$$\lambda_2(A) \leqslant \lambda_2(A_3) \leqslant \lambda_3(A_3) \leqslant \lambda_n(A). \tag{3.13}$$

By applying (3.3) to B_3 and using (3.13), we immediately get (3.11).

Likewise, from (2.1) we have

$$\lambda_1(A) \leqslant \lambda_1(A_3) \leqslant \lambda_2(A_3) \leqslant \lambda_{n-1}(A). \tag{3.14}$$

Then, (3.12) follows by applying (3.4) to B_3 and using (3.14). □

COROLLARY 3.3. *Let $H \in \mathbb{M}(3)$ be a principal submatrix of a Hermitian element $A \in \mathbb{M}(n)$. Let $\varphi : \mathbb{M}(3) \longrightarrow \mathbb{C}$ be a positive unital linear functional. If $\det(H - \varphi(H)I_3) > 0$, then*

$$\lambda_2(A) \leqslant \varphi(H) - 2\frac{\det(H-\varphi(H)I_3)}{tr(H-\varphi(H)I_3)^2} \leqslant \varphi(H) + 2\frac{\det(H-\varphi(H)I_3)}{tr(H-\varphi(H)I_3)^2} \leqslant \lambda_n(A). \tag{3.15}$$

If $\det(H - \varphi(H)I_3) < 0$, then

$$\lambda_1(A) \leqslant \varphi(H) + 2\frac{\det(H-\varphi(H)I_3)}{tr(H-\varphi(H)I_3)^2} \leqslant \varphi(H) - 2\frac{\det(H-\varphi(H)I_3)}{tr(H-\varphi(H)I_3)^2} \leqslant \lambda_{n-1}(A). \tag{3.16}$$

Proof. It is clear that H is Hermitian and let its eigenvalues be arranged as $\lambda_1(H) \leqslant \lambda_2(H) \leqslant \lambda_3(H)$. So, $\lambda_1(H) \leqslant x^*Hx \leqslant \lambda_3(H)$ for all unit vectors $x \in \mathbb{C}^3$. So, $H - \lambda_1(H)I_3 \geqslant O$ and $\lambda_3(H)I_3 - H \geqslant O$. The functional φ is positive, therefore we have $\varphi(H - \lambda_1(H)I_3) \geqslant 0$ and $\varphi(\lambda_3(H)I_3 - H) \geqslant 0$. By using linearity and the fact that φ is unital we have $\lambda_1(H) \leqslant \varphi(H) \leqslant \lambda_3(H)$. The cases $\lambda_1(H) = \varphi(H)$ and $\lambda_3(H) = \varphi(H)$ are obvious. So, let $\lambda_1(H) < \varphi(H) < \lambda_3(H)$. The assertions of the corollary then follow from the Theorem 3.2 by choosing $c = \varphi(H)$. □

It may be noted that the Corollary 3.3 provides various bounds for $\lambda_2(A)$ and $\lambda_{n-1}(A)$ for different choices of φ. For example, $\varphi(H) = \frac{\mathrm{tr}H}{3}$ and $\varphi(H) = \frac{1}{3}\sum_{i,j}^{3} a_{ij}$ are positive unital linear functionals and by substituting these in (3.15) and (3.16) we can write down the corresponding inequalities. One such similar case provides a refinement of the inequality (1.8). We mention it in the following corollary.

COROLLARY 3.4. *Let $A \in \mathbb{M}(n)$ be nonnegative and symmetric and let its eigenvalues and diagonal entries be arranged as in (1.1) and (1.2), respectively. Let $X \in \mathbb{M}(3)$ be the principal submatrix of A whose diagonal entries are a_1, a_2, a_3. Then*

$$\lambda_2(A) \leqslant a_3 - 2\frac{\det(X - a_3 I_3)}{tr(X - a_3 I_3)^2} \leqslant a_3. \tag{3.17}$$

Proof. The matrix $X = (x_{ij}) \in \mathbb{M}(3)$ is nonnegative and symmetric. Its largest diagonal entry is $a_3 = \max_i x_{ii}$. Suppose $a_3 = x_{33}$. Then

$$\det(X - a_3 I_3) = 2x_{12}x_{23}x_{13} + (x_{33} - x_{11})x_{23}^2 + (x_{33} - x_{22})x_{13}^2.$$

So, $\det(X - a_3 I_3) \geqslant 0$ when X is nonnegative. The same is true if $a_3 = x_{11}$ or $a_3 = x_{22}$. Hence $\det(X - a_3 I_3) \geqslant 0$. The equality holds in this inequality if and only if a_3 is an eigenvalue of X. Further, $\varphi(X) = a_3$ is a positive unital linear functional. Then the first inequality (3.17) follows by applying the first inequality (3.15) to X. Since $\det(X - a_3 I_3) \geqslant 0$, the second inequality (3.17) is immediate. $\square$

COROLLARY 3.5. *Let $A \in \mathbb{M}(n)$ be positive definite and let its eigenvalues be arranged as in (1.1). Let $H \in \mathbb{M}(3)$ be any principal submatrix of A and $\varphi : \mathbb{M}(3) \longrightarrow \mathbb{C}$ be a positive unital linear functional. Denote $H - \frac{\varphi(H^2)}{\varphi(H)} I_3$ by X. Then, if $\det X > 0$,*

$$\lambda_2(A) \leqslant \frac{\varphi(H^2)}{\varphi(H)} - 2\frac{\det X}{trX^2} \leqslant \frac{\varphi(H^2)}{\varphi(H)} + 2\frac{\det X}{trX^2} \leqslant \lambda_n(A) \tag{3.18}$$

and if $\det X < 0$

$$\lambda_1(A) \leqslant \frac{\varphi(H^2)}{\varphi(H)} + 2\frac{\det X}{trX^2} \leqslant \frac{\varphi(H^2)}{\varphi(H)} - 2\frac{\det X}{trX^2} \leqslant \lambda_{n-1}(A). \tag{3.19}$$

Proof. The principal submatrix of a positive definite matrix is positive definite. Therefore, H is positive definite. Let the eigenvalues of H be arranged as in (1.1). The functional $\varphi : \mathbb{M}(3) \longrightarrow \mathbb{C}$ is positive, therefore, $0 < \lambda_1(H) \leqslant \varphi(H) \leqslant \lambda_3(H)$. Also, by (3.1), $\varphi(H^2) \geqslant \varphi(H)^2$. We thus have

$$\lambda_1(H)\varphi(H) \leqslant \varphi(H)^2 \leqslant \varphi(H^2). \tag{3.20}$$

Further, by the spectral theorem,

$$H = \lambda_1(H)P_1 + \lambda_2(H)P_2 + \lambda_3(H)P_3$$

where P_j's are orthogonal projections, $P_j^* = P_j \geqslant O$, $P_iP_j = O$ for $i \neq j$ and $P_1 + P_2 + P_3 = I_3$. So,

$$H^2 = \lambda_1^2(H)P_1 + \lambda_2^2(H)P_2 + \lambda_3^2(H)P_3. \tag{3.21}$$

By applying φ to both sides of (3.21), and using linearity of φ, we get

$$\varphi\left(H^2\right) = \sum_{j=1}^{3} \lambda_j^2(H)\,\varphi(P_j) \leqslant \lambda_3(H) \sum_{j=1}^{3} \lambda_j(H)\,\varphi(P_j) = \lambda_3(H)\,\varphi(H). \tag{3.22}$$

From (3.20) and (3.22), we get that

$$\lambda_1(H) \leqslant \frac{\varphi\left(H^2\right)}{\varphi(H)} \leqslant \lambda_3(H).$$

The assertions of the corollary now follow from the Theorem 3.2 by choosing $c = \frac{\varphi\left(H^2\right)}{\varphi(H)}$ and applying (3.11) and (3.12) to H. The arguments are same as in the proof of the Corollary 3.3. □

The Corollary 3.3 provides an upper bound for $\lambda_2(A)$ when $\det(H - \varphi(H) I_3) > 0$. In the following theorem we show that by using (3.2) we can derive an upper bound for $\lambda_2(A)$ when $\det(H - \varphi(H) I_3) < 0$.

THEOREM 3.6. *Let $H \in \mathbb{M}(3)$ be a principal submatrix of a Hermitian element $A \in \mathbb{M}(n)$. Let $\varphi : \mathbb{M}(3) \longrightarrow \mathbb{C}$ be positive and unital. If* $\det(H - \varphi(H) I_3) < 0$, *then for $\varphi\left(H^2\right) \neq \varphi(H)^2$, we have*

$$\lambda_2(A) \leqslant \varphi(H) - \frac{\det(H - \varphi(H) I_3)}{\varphi(H^2) - \varphi(H)^2} \tag{3.23}$$

and if $\det(H - \varphi(H) I_3) > 0$, *we have*

$$\lambda_{n-1}(A) \geqslant \varphi(H) - \frac{\det(H - \varphi(H) I_3)}{\varphi(H^2) - \varphi(H)^2}. \tag{3.24}$$

Proof. By applying (3.1) and (3.2) to positive unital linear functional $\varphi : \mathbb{M}(3) \longrightarrow \mathbb{C}$, we find that

$$0 < \varphi\left(H^2\right) - \varphi(H)^2 \leqslant (\lambda_3(H) - \varphi(H))(\varphi(H) - \lambda_1(H)). \tag{3.25}$$

Also, $\det(H - \varphi(H) I_3) = \prod_{j=1}^{3} (\lambda_j(H) - \varphi(H))$, therefore

$$(\lambda_3(H) - \varphi(H))(\varphi(H) - \lambda_1(H)) = -\frac{\det(H - \varphi(H) I_3)}{\lambda_2(H) - \varphi(H)}. \tag{3.26}$$

From (3.25) and (3.26),we get

$$0 < \varphi\left(H^2\right) - \varphi(H)^2 \leqslant -\frac{\det(H - \varphi(H) I_3)}{\lambda_2(H) - \varphi(H)}. \tag{3.27}$$

We have $\lambda_1(H) < \varphi(H) < \lambda_3(H)$, therefore, if $\det(H - \varphi(H) I_3) < 0$, then $\lambda_2(H) > \varphi(H)$ and (3.27) gives an upper bound for $\lambda_2(H)$ and by using $\lambda_2(A) \leqslant \lambda_2(H)$, we get (3.23).

Further, if $\det(H - \varphi(H) I_3) > 0$, then $\lambda_2(H) < \varphi(H)$ and (3.27) yields a lower bound for $\lambda_2(H)$. Then, by using $\lambda_{n-1}(H) \geqslant \lambda_2(H)$, we immediately get (3.24). □

The inequalities discussed in the above theorems are derived by using the condition that A is Hermitian and $\det(A_3 - cI_3)$ is positive or negative. By using (3.2) we now derive some bounds for $\lambda_2(A)$ and $\lambda_{n-1}(A)$ when A is positive definite.

THEOREM 3.7. *Let $H \in \mathbb{M}(3)$ be positive definite and let its eigenvalues be arranged as in (1.1). Then, for any positive unital linear functional $\varphi : \mathbb{M}(3) \longrightarrow \mathbb{C}$, we have*

$$\lambda_1(H) \leqslant \alpha - \beta \leqslant \lambda_2(H) \leqslant \alpha + \beta \leqslant \lambda_3(H) \tag{3.28}$$

where

$$\alpha = \frac{trH\varphi(H) - \varphi\left(H^2\right)}{2\varphi(H)} \quad and \quad \beta = \sqrt{\alpha^2 - \frac{\det H}{\varphi(H)}}. \tag{3.29}$$

Proof. The matrix $H \in \mathbb{M}(3)$ is positive definite, therefore $\lambda_i(H) > 0,\ i = 1,2,3$, we write,

$$\lambda_1(H)\lambda_3(H) = \frac{\det H}{\lambda_2(H)} \quad \text{and} \quad \lambda_1(H) + \lambda_3(H) = \mathrm{tr}H - \lambda_2(H). \tag{3.30}$$

From (3.2) and (3.30), we find that

$$\varphi\left(H^2\right) \leqslant (\lambda_1(H) + \lambda_3(H))\varphi(H) - \lambda_1(H)\lambda_3(H) = (\mathrm{tr}H - \lambda_2(H))\varphi(H) - \frac{\det H}{\lambda_2(H)}.$$

This gives a quadratic inequality in $\lambda_2(H)$,

$$\varphi(H)\lambda_2^2(H) + \left(\varphi\left(H^2\right) - \mathrm{tr}H\varphi(H)\right)\lambda_2(H) + \det H \leqslant 0. \tag{3.31}$$

The roots of the quadratic equation in (3.31) are $\alpha - \beta$ and $\alpha + \beta$ where α and β are as given in (3.29). Also $\varphi(H) > 0$. Therefore, we conclude that $\alpha - \beta \leqslant \lambda_2(H) \leqslant \alpha + \beta$. This gives second and third inequalities (3.28). It is clear that $\lambda_i(H)$ lies outside the interval $\left(\lambda_{j-1}(H), \lambda_j(H)\right)$. Also, projections P_i's are positive semidefinite, therefore for $i = 1,2,3$ and any fixed $j = 2,3$, we have

$$\left(\lambda_i(H) - \lambda_{j-1}(H)\right)\left(\lambda_i(H) - \lambda_j(H)\right)P_i \geqslant 0.$$

By adding these three inequalities for any fixed $j = 2,3$, we get

$$\sum_{i=1}^{3}\lambda_i^2(H)P_i - \left(\lambda_{j-1}(H) + \lambda_j(H)\right)\sum_{i=1}^{3}\lambda_i(H)P_i + \lambda_{j-1}(H)\lambda_j(H)I_3 \geqslant 0. \tag{3.32}$$

By applying φ to (3.32), we find that

$$\varphi\left(H^2\right) \geqslant (\lambda_1(H)+\lambda_2(H))\,\varphi(H)-\lambda_1(H)\lambda_2(H) = (\mathrm{tr}H-\lambda_3(H))\,\varphi(H)-\frac{\det H}{\lambda_3(H)} \tag{3.33}$$

and

$$\varphi\left(H^2\right) \geqslant (\lambda_2(H)+\lambda_3(H))\,\varphi(H)-\lambda_2(H)\lambda_3(H) = (\mathrm{tr}H-\lambda_1(H))\,\varphi(H)-\frac{\det H}{\lambda_1(H)}. \tag{3.34}$$

From (3.33) and (3.34), we find that for $k=1,3,$ we have

$$\varphi(H)\lambda_k^2(H)+\left(\varphi\left(H^2\right)-\mathrm{tr}H\varphi(H)\right)\lambda_k(H)+\det H \geqslant 0. \tag{3.35}$$

The discriminant of the quadratic in (3.35) is nonnegative and its roots are $\alpha-\beta$ and $\alpha+\beta$. We conclude that $\lambda_k(H)$ lies outside $(\alpha-\beta,\alpha+\beta)$. So,

$$\lambda_1(H)\leqslant\alpha-\beta \text{ or } \lambda_1(H)\geqslant\alpha+\beta \quad\text{and}\quad \lambda_3(H)\leqslant\alpha-\beta \quad\text{or}\quad \lambda_3(H)\geqslant\alpha+\beta.$$

But $\alpha-\beta\leqslant\lambda_2(H)\leqslant\alpha+\beta$. So, $\lambda_3(H)\nleqslant\alpha-\beta$ and $\lambda_1(H)\ngeqslant\alpha+\beta$. Hence, $\lambda_1(H)\leqslant\alpha-\beta$ and $\lambda_3(H)\geqslant\alpha+\beta$. □

THEOREM 3.8. *Let $\varphi:\mathbb{M}(n)\longrightarrow\mathbb{C}$ be a positive unital linear functional. Let $A\in\mathbb{M}(n)$, $n\geqslant 3$, be positive definite and let its eigenvalues be arranged as in (1.1). For any principal submatrix $H\in\mathbb{M}(3)$, we have*

$$\lambda_1(A)\leqslant\alpha-\beta\leqslant\lambda_{n-1}(A) \text{ and } \lambda_2(A)\leqslant\alpha+\beta\leqslant\lambda_n(A) \tag{3.36}$$

where α and β are given as in (3.29).

Proof. The principal submatrix $H\in\mathbb{M}(3)$ of positive definite matrix A is positive definite. By interlacing inequalities (2.1), we have

$$\lambda_1(A)\leqslant\lambda_1(H)\leqslant\lambda_2(H)\leqslant\lambda_{n-1}(A). \tag{3.37}$$

The first and second inequalities (3.36) follow on combining (3.37) and the first and second inequalities (3.28).

Likewise, from (2.1), we have $\lambda_2(A)\leqslant\lambda_2(H)\leqslant\lambda_3(H)\leqslant\lambda_n(A)$. Then, (3.28) yields the bounds for $\lambda_2(A)$ and $\lambda_{n-1}(A)$ in (3.36). □

The above inequalities and their applications are independent of each others. We illustrate and compare our results with those in literature by means of the following simple examples.

EXAMPLE 3.1. Let

$$A_1=\begin{bmatrix}1&1&i\\1&1&1\\-i&1&2\end{bmatrix},\quad A_2=\begin{bmatrix}1&1&i\\1&1&1\\-i&1&4\end{bmatrix}\quad\text{and}\quad A_3=\begin{bmatrix}1&-2&3\\-2&1&2\\3&2&1\end{bmatrix}.$$

Choose $\varphi(A_1) = a_{33} = 2$. Then, $\det(A_1 - a_{33}I_3) = 2 > 0$ and therefore from (3.15) and (3.24), $1 \leqslant \lambda_2(A_1) \leqslant 1.5$. From (1.3) and (1.4), we have $0.2792 \leqslant \lambda_2(A_1) \leqslant 2.387$.

Choose $\varphi(A_2) = a_{33} = 4$. Then, $\det(A_2 - a_{33}I_3) = 6 > 0$. Therefore, from (3.15) and (3.24), $1 \leqslant \lambda_2(A_2) \leqslant 3.5$. From (1.3) and (1.4), we have $0.585 \leqslant \lambda_2(A_2) \leqslant 3.414$.

Choose $\varphi(A_3) = a_{33} = 1$. Then, $\det(A_3 - a_{33}I_3) = -24 < 0$. Therefore, from (3.16) and (3.23), $2.411 \leqslant \lambda_2(A_3) \leqslant 2.846$. From (1.3) and (1.4), $-1.380 \leqslant \lambda_2(A_3) \leqslant 3.380$.

EXAMPLE 3.2. Let

$$A_4 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \quad \text{and} \quad A_5 = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 6 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The matrix A_4 is positive definite. Choose $\varphi(A_4) = a_{33} = 3$. Then, $c = \frac{\varphi(A_4^2)}{\varphi(A_4)} = \frac{10}{3}$ and $\det(A_4 - cI_3) = \frac{14}{27} > 0$. Therefore, from (3.18), $\lambda_2(A_4) \leqslant 3.160$ while from (1.3) and (1.7), $1.784 \leqslant \lambda_2(A_4) \leqslant 3.548$ and $1.441 \leqslant \lambda_2(A_4) \leqslant 3.506$, respectively. From (3.28), $1.131 \leqslant \lambda_2(A_4) \leqslant 3.535 \leqslant \lambda_3(A_4)$. The matrix A_5 is positive definite. Choose $\varphi(A_5) = a_{22} = 6$. Then $c = \frac{\varphi(A_5^2)}{\varphi(A_5)} = \frac{19}{3}$ and $\det(A_5 - cI_3) = \frac{179}{27} > 0$. Therefore, from (3.18), $\lambda_2(A_5) \leqslant 5.968$ while from (1.3) and (1.7), $2.174 \leqslant \lambda_2(A_5) \leqslant 5.825$ and $1.112 \leqslant \lambda_2(A_5) \leqslant 5.677$, respectively. From (3.28), $0.666 \leqslant \lambda_2(A_5) \leqslant 4.999$. From (3.15) and (3.24), $2 \leqslant \lambda_2(A_5) \leqslant 5.5$.

EXAMPLE 3.3. (Example 4.2, [29].) The Laplacian matrix of a graph in [29] is

$$A = \begin{bmatrix} 6 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & -1 & 0 & 0 & 0 \\ -1 & -1 & -1 & 3 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 & 3 \end{bmatrix}.$$

The best lower bound for the second largest eigenvalue of A in [29] is 2.9 that is $\lambda_6(A) \geqslant 2.9$. By (2.8), $\lambda_6(A) \geqslant 3$. Further, consider the principal submatrix

$$H = \begin{bmatrix} 6 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

of A and let $\varphi(H) = h_{22} = 3$. Then, from (3.16), $\lambda_6(A) \geqslant 3.6$. The best lower bound for the largest eigenvalue of A in [29] is 3.4 that is $\lambda_7(A) \geqslant 3.4$. For the choice $\varphi(H) = h_{11} = 6$ and by (3.15) we have $\lambda_7(A) \geqslant 6.3$.

4. Bounds using estimates of the spread

The spread of a matrix is the maximum distance between the eigenvalues of a matrix in the complex plane. For any Hermitian element $A \in \mathbb{M}(n)$, we have $\operatorname{spd}(A) = \lambda_n(A) - \lambda_1(A)$. The bounds for the spread have been studied extensively in literature. Jiang and Zhan [15] and Bhatia and Sharma [8] have discussed various lower bounds for the spreads of Hermitian matrices. We show here that these lower bounds for the spreads can be used to obtain lower (upper) bounds for $\lambda_2(A)$ $(\lambda_{n-1}(A))$.

We need following basic results in the proofs of the subsequent theorems.

1. Let $\Phi : \mathbb{M}(n) \longrightarrow \mathbb{M}(k)$ be a positive unital linear map and let A be any Hermitian element of $\mathbb{M}(n)$. Then, Bhatia and Davis [6] have proved that

$$\Phi\left(A^2\right) - \Phi(A)^2 \leqslant \frac{\operatorname{spd}(A)^2}{4} I_k. \tag{4.1}$$

Also, Bhatia and Sharma [7] have shown that

$$||\Phi_1(A) - \Phi_2(A)|| \leqslant \operatorname{spd}(A), \tag{4.2}$$

where $||.||$ denotes the operator norm, $||A|| = \max\limits_{x^*x=1} x^*A^*Ax,\ x \in \mathbb{C}^n$.

2. Let $x_1, x_2, \ldots, x_n$ denote n real numbers. Their arithmetic mean and variance are respectively the numbers,

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2.$$

The Nagy inequality [20] says that

$$s^2 \geqslant \frac{1}{2n}\left(\max_i x_i - \min_i x_i\right)^2. \tag{4.3}$$

LEMMA 4.1. *Let $\overline{x}$ and s^2 respectively denote the arithmetic mean and variance of n real numbers x_j's. Let s_{n-1}^2 be the variance of $n-1$ numbers obtained by excluding a number x_k from the numbers $x_1, x_2, \ldots, x_n$. Then,*

$$s^2 = \frac{n-1}{n} s_{n-1}^2 + \frac{1}{n-1}(\overline{x} - x_k)^2. \tag{4.4}$$

Proof. Denote $\frac{1}{n-1}\sum_{\substack{i=1 \\ i \neq k}}^{n} x_i$ by $\overline{y}$. Then, a simple calculation shows that

$$\overline{x} - x_k = \frac{n-1}{n}(\overline{y} - x_k), \tag{4.5}$$

$$s^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - x_k + x_k - \overline{x})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - x_k)^2 - (\overline{x} - x_k)^2 \tag{4.6}$$

and

$$\sum_{i=1}^{n}(x_i-x_k)^2=(n-1)\,s_{n-1}^2+(n-1)\,(\overline{y}-x_k)^2\,. \tag{4.7}$$

Combining (4.5)–(4.7), a little computation leads to (4.4). □

LEMMA 4.2. *For $x_1\leqslant x_i\leqslant x_n,\ i=1,2,\ldots,n,$ and with notations as above, we have*

$$s^2\geqslant\frac{(x_n-x_1)^2}{2n}+\frac{1}{n-1}\,(\overline{x}-x_k)^2 \tag{4.8}$$

for all $k=2,3,\ldots,n-1$.

Proof. Let $x_1\leqslant x_i\leqslant x_n,\ i=1,2,\ldots,n$. We exclude a number $x_k,\ k=2,\ldots,n-1$, from these n numbers x_j's. Then, $\max\limits_i x_i-\min\limits_i x_i=x_n-x_1$. Therefore, by applying (4.3) to these $n-1$ numbers, we get

$$s_{n-1}^2\geqslant\frac{(x_n-x_1)^2}{2\,(n-1)}. \tag{4.9}$$

Combining (4.4) and (4.9), we immediately get (4.8). □

THEOREM 4.1. *Let $\Phi_i:\mathbb{M}(n)\longrightarrow\mathbb{M}(k),\ i=1,2,$ be two positive unital linear maps and let the eigenvalues of any Hermitian element $A\in\mathbb{M}(n)$ be arranged as in (1.1). Let S^2 be defined as in (1.5). Then, for $j=2,3,\ldots,n-1$, we have*

$$\left|\lambda_j(A)-\frac{trA}{n}\right|I_k\leqslant\sqrt{n-1}\sqrt{S^2I_k-\frac{\|\Phi_1(A)-\Phi_2(A)\|^2}{2n}I_k}. \tag{4.10}$$

Proof. We have $\mathrm{tr}A^r=\sum_{j=1}^{n}\lambda_j(A)^r,\ r=1,2$. Therefore, from (1.5), with $\widetilde{\lambda}(A)=\frac{1}{n}\sum_{j=1}^{n}\lambda_j(A)$,

$$S^2=\frac{1}{n}\sum_{j=1}^{n}\lambda_j(A)^2-\left(\frac{1}{n}\sum_{j=1}^{n}\lambda_j(A)\right)^2=\frac{1}{n}\sum_{j=1}^{n}\left(\lambda_j(A)-\widetilde{\lambda}(A)\right)^2$$

is the variance of the eigenvalues $\lambda_j(A)$. It then follows from the Lemma 4.2 that

$$S^2I_k\geqslant\frac{(\lambda_n(A)-\lambda_1(A))^2}{2n}I_k+\frac{1}{n-1}\left(\lambda_j(A)-\frac{\mathrm{tr}A}{n}\right)^2I_k. \tag{4.11}$$

We have $\mathrm{spd}(A)=\lambda_n(A)-\lambda_1(A)$. So, by using (4.2) in (4.11), we immediately get (4.10). □

COROLLARY 4.2. *Let $\varphi_i : \mathbb{M}(n) \longrightarrow \mathbb{C}$ be positive unital linear functionals, $i = 1,2$. Then, under the conditions of the Theorem 4.1, we have*

$$\lambda_2(A) \geqslant \frac{trA}{n} - \sqrt{n-1}\sqrt{S^2 - \frac{(\varphi_1(A) - \varphi_2(A))^2}{2n}} \tag{4.12}$$

and

$$\lambda_{n-1}(A) \leqslant \frac{trA}{n} + \sqrt{n-1}\sqrt{S^2 - \frac{(\varphi_1(A) - \varphi_2(A))^2}{2n}}. \tag{4.13}$$

Proof. Note that for any Hermitian element $A \in \mathbb{M}(n)$, $\varphi(A)$ is a real number. Then, by applying (4.10) to φ and using the fact that $|x-a| \leqslant b$ if and only if $a-b \leqslant x \leqslant a+b$, we immediately get (4.12) and (4.13). □

THEOREM 4.3. *With notations and conditions as in the Theorem 4.1, we have for $j = 2,3,\ldots,n-1$,*

$$\left|\lambda_j(A) - \frac{trA}{n}\right| I_k \leqslant \sqrt{n-1}\sqrt{\left(S^2 - \frac{2}{n}\left\|\Phi(A^2) - \Phi(A)^2\right\|\right)} I_k. \tag{4.14}$$

Proof. For $P \geqslant Q$, we have $x^*Px \geqslant x^*Qx$ for all unit vectors $x \in \mathbb{C}^n$. Therefore, from (4.1), we find that

$$\operatorname{spd}(A)^2 \geqslant 4\left\|\Phi(A^2) - \Phi(A)^2\right\|. \tag{4.15}$$

Combining (4.11) and (4.15), we immediately get (4.14). □

COROLLARY 4.4. *With notations and conditions as in the Corollary 4.2, we have*

$$\lambda_2(A) \geqslant \frac{trA}{n} - \sqrt{n-1}\sqrt{S^2 - \frac{2}{n}\left(\varphi(A^2) - \varphi(A)^2\right)} \tag{4.16}$$

and

$$\lambda_{n-1}(A) \leqslant \frac{trA}{n} + \sqrt{n-1}\sqrt{S^2 - \frac{2}{n}\left(\varphi(A^2) - \varphi(A)^2\right)}. \tag{4.17}$$

Proof. The assertions of the corollary follow from the Theorem 4.3 by using arguments similar to those used in the proof of Corollary 4.2. □

Bhatia and Sharma [8] have shown that the various lower bounds obtained by Jiang and Zhan [15] also follow from the inequality (4.1). We borrow two examples from [15] to illustrate our results.

EXAMPLE 4.1. Let

$$A_6 = \begin{bmatrix} 2 & 0 & 1-2i & 1-2i \\ 0 & 1 & 1+2i & 1+2i \\ 1+2i & 1-2i & 1 & 0 \\ 1+2i & 1-2i & 0 & 1 \end{bmatrix} \quad \text{and} \quad A_7 = \begin{bmatrix} 5 & 1 & 2-i \\ 1 & 1 & 1+2i \\ 2+i & 1-2i & 3 \end{bmatrix}.$$

Here $\operatorname{spd}(A_6)^2 \geqslant 81$. Therefore from (4.16) and (4.17), we have $0.816 \leqslant \lambda_2(A_6) \leqslant \lambda_3(A_6) \leqslant 1.683$. From (1.3), $-1.931 \leqslant \lambda_2(A_6) \leqslant 3.0928$. Also, $\operatorname{spd}(A_7)^2 \geqslant 60$. Therefore, from (4.16) and (4.17), we have $\lambda_2(A_7) = 3$. From (1.3), $0.763 \leqslant \lambda_2(A_7) \leqslant 5.236$.

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