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A GENERALIZATION OF THE WEIGHTED ALGEBRAIC NUMERICAL RADIUS ON C^*–ALGEBRAS

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Abstract. Let $N(\cdot)$ be a norm on a unital C^*-algebra $\mathfrak{A}$. For s and t are both nonnegative reals and $s+t>0$, we introduce a family of non-negative real-valued functions on $\mathfrak{A}$, defined by

$$v_{(N,(s,t))}(x)=\sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right),\quad (x\in\mathfrak{A}).$$

Here, $\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)=se^{i\theta}x+t(e^{i\theta}x)^*$ for all $x\in\mathfrak{A}$. Some basic properties and other useful characterizations of this family of functions are presented. As a special case of this family of functions, some results involving the weighted algebraic numerical radius are obtained. Additionally, we establish the equivalence between the numerical radius $v(x)$ and the norm $v_{(s,t)}(x)$.

1. Introduction and preliminaries

Let $\mathfrak{A}$ be a unital C^*-algebra with unit denoted by e. We denote by $\mathfrak{A}^*$ the topological dual space of $\mathfrak{A}$. A linear functional $\varphi\in\mathfrak{A}^*$ is said to be positive, and write $\varphi\geqslant 0$, if $\varphi(a)\geqslant 0$ for every positive element $a\in\mathfrak{A}$. A state is a positive linear functional whose norm is equal to one. Let $\mathscr{S}(\mathfrak{A})$ denote the set of all states on $\mathfrak{A}$. The algebraic numerical range of $x\in\mathfrak{A}$ is defined by

$$V(x)=\{\varphi(x):\varphi\in S(\mathfrak{A})\}.$$

It is a nonempty compact and convex set of the complex plane $\mathbb{C}$, and its maximum modulus is the algebraic numerical radius of $x\in\mathfrak{A}$, defined by

$$v(x)=\sup\{|\xi|:\xi\in V(x)\}.$$

It is known that $v(\cdot)$ defines a norm on $\mathfrak{A}$, which is equivalent to the C^*-norm $\|\cdot\|$. In fact, for every $x\in\mathfrak{A}$, the following inequalities hold:

$$\frac{1}{2}\|x\|\leqslant v(x)\leqslant\|x\|.$$

For more detailed information about C^*-algebras, the reader is referred to [17,23,24] and the references therein.

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There are many generalizations of the classical numerical range and numerical radius, and there has been a great deal of interest in their systematic properties and applications. For instance, in [4], authors introduced and studied A-numerical range and A-numerical radius for Semi-Hilbertian space operators. Then more properties of these concepts were established, see [3,11,20]. Bourhim and Mabrouk in [9] introduced and studied a-numerical range and a-numerical radius of elements in C^*-algebras. More basic properties of these concepts were established in [2, 16]. Sheikhhosseini, Khosravi and Sababheh in [22] introduced and studied the weighted numerical radius of Hilbert space operators. In addition, the relationship between weighted numerical radius and classical numerical radius was also investigated. In [14], the weighted A-numerical radius for Semi-Hilbertian space operators was introduced and some basic properties and inequalities about it were obtained. For more about numerical range and numerical radius, we refer the reader to [5–7,10,12,13,15,18,19,21] and the references therein.

Throughout this paper, $\mathfrak{A}$ denotes a unital C^*-algebra with unit denoted by e, and s and t are both nonnegative real numbers such that their sum is positive. Let $N(\cdot)$ be a norm on $\mathfrak{A}$. The norm $N(\cdot)$ is said to be an algebra norm if $N(xy) \leqslant N(x)N(y)$ for every $x,y \in \mathfrak{A}$, and is called self-adjoint if $N(x) = N(x^*)$ for every $x \in \mathfrak{A}$.

Any element $x \in \mathfrak{A}$ can be represented by the Cartesian decomposition as $x = \mathfrak{R}(x) + i\mathfrak{I}(x)$, where $\mathfrak{R}(x) = \frac{1}{2}(x+x^*)$ and $\mathfrak{I}(x) = \frac{1}{2i}(x-x^*)$ are the real and imaginary parts of x, respectively. Recently, Mabrouk and Zamani in [16] defined the weighted real and imaginary parts of $x \in \mathfrak{A}$, by

$$\mathfrak{R}_{(s,t)}(x) = sx + tx^* \text{ and } \mathfrak{I}_{(s,t)}(x) = s(-ix) + t(-ix)^*,$$

respectively, where s and t are both nonnegative reals and $s+t>0$. When $s=\gamma$, $t = 1-\gamma$, where $0 \leqslant \gamma \leqslant 1$, we can see that $\mathfrak{R}_\gamma(x) = \gamma x + (1-\gamma)x^*$ and $\mathfrak{I}_\gamma(x) = \gamma(-ix) + (1-\gamma)(-ix)^*$. When $s=t=\frac{1}{2}$, we can see that $\mathfrak{R}_{\frac{1}{2}}(x) = \mathfrak{R}(x)$ and $\mathfrak{I}_{\frac{1}{2}}(x) = \mathfrak{I}(x)$. In addition, the authors also defined the following norm:

$$v_{(s,t)}(x) = \sup_{\theta\in\mathbb{R}} \left\| \mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right) \right\| \tag{1.1}$$

for all $x \in \mathfrak{A}$, which generalizes the C^*-norm and algebraic numerical radius.

Some interesting relationships about $\mathfrak{R}_{(s,t)}(x)$, $\mathfrak{I}_{(s,t)}(x)$ and $\mathfrak{R}(x)$, $\mathfrak{I}(x)$ are contained in the following result.

PROPOSITION 1.1. *Let* $x \in \mathfrak{A}$. *Then*
(i) $\mathfrak{R}_{(s,t)}(x) = \mathfrak{I}_{(s,t)}(ix)$.
(ii) $\mathfrak{R}_{(s,t)}(x) = (s+t)\mathfrak{R}(x) + i(s-t)\mathfrak{I}(x)$.
(iii) $\mathfrak{I}_{(s,t)}(x) = (s+t)\mathfrak{I}(x) + i(t-s)\mathfrak{R}(x)$.
(iv) $\mathfrak{R}_{(s,t)}(x) - i\mathfrak{R}_{(s,t)}(ix) = 2sx$.

Proof. (i) According to the definition of weighted real and imaginary parts of $x \in \mathfrak{A}$, we have $\mathfrak{I}_{(s,t)}(ix) = s\left(-i^2x\right) + t\left(-i^2x\right)^* = sx + tx^* = \mathfrak{R}_{(s,t)}(x)$.

(ii) According to the definition of weighted real part of $x \in \mathfrak{A}$, we have

$$\begin{aligned}\mathfrak{R}_{(s,t)}(x) = sx + tx^* &= \left(\frac{s+t}{2} + \frac{s-t}{2}\right)x + \left(\frac{s+t}{2} - \frac{s-t}{2}\right)x^* \\ &= (s+t)\frac{x+x^*}{2} + i(s-t)\frac{x-x^*}{2i} \\ &= (s+t)\mathfrak{R}(x) + i(s-t)\mathfrak{I}(x).\end{aligned}$$

(iii) By (i) and (ii), we can derive $\mathfrak{I}_{(s,t)}(x) = \mathfrak{R}_{(s,t)}(-ix) = (s+t)\mathfrak{R}(-ix) + i(s-t)\mathfrak{I}(-ix)$. And because $\mathfrak{R}(-ix) = \mathfrak{I}(x)$, $\mathfrak{I}(-ix) = -\mathfrak{R}(x)$, we have $\mathfrak{I}_{(s,t)}(x) = (s+t)\mathfrak{I}(x) + i(t-s)\mathfrak{R}(x)$.

(iv) By (ii), we have $\mathfrak{R}_{(s,t)}(ix) = (s+t)\mathfrak{R}(ix) + i(s-t)\mathfrak{I}(ix)$. Hence,

$$\begin{aligned}&\mathfrak{R}_{(s,t)}(x) - i\mathfrak{R}_{(s,t)}(ix) \\ &= (s+t)\mathfrak{R}(x) + i(s-t)\mathfrak{I}(x) - i\left[(s+t)\mathfrak{R}(ix) + i(s-t)\mathfrak{I}(ix)\right] \\ &= (s+t)\mathfrak{R}(x) + i(s-t)\mathfrak{I}(x) - i\left[(s+t)(-\mathfrak{I}(x)) + i(s-t)\mathfrak{R}(x)\right] \\ &= 2s(\mathfrak{R}(x) + i\mathfrak{I}(x)) = 2sx. \quad \square\end{aligned}$$

Inspired by the weighted numerical radius of Hilbert space operators, let $s = \gamma$, $t = 1-\gamma$ as in (1.1), where $0 \leqslant \gamma \leqslant 1$, the authors defined the weighted algebraic numerical radius of elements in C^*-algebras in [16]. As shown below.

DEFINITION 1.2. Let $x \in \mathfrak{A}$ and $0 \leqslant \gamma \leqslant 1$. Then the weighted algebraic numerical radius on $\mathfrak{A}$ is denoted by

$$v_\gamma(x) = \sup_{\theta \in \mathbb{R}} \left\| \mathfrak{R}_\gamma\left(e^{i\theta}x\right) \right\|. \tag{1.2}$$

The remainder of the paper is organized as follows. In Section 2, inspired by [25], we introduce a family of non-negative real-valued functions on a C^*-algebra $\mathfrak{A}$, which generalizes the norm $v_{(s,t)}(x)$ and the weighted algebraic numerical radius $v_\gamma(x)$. Some of the fundamental properties of this family of functions are presented. We also present some other useful characterizations of this family of functions. Further, we draw some conclusions about the norm $v_{(s,t)}(x)$ and the weighted algebraic numerical radius. In Section 3, some results of $v_{(s,t)}(x)$ are given. In particular, we prove the equivalence between the numerical radius $v(x)$ and the norm $v_{(s,t)}(x)$.

2. A generalization of $v_{(s,t)}(\cdot)$ and $v_\gamma(\cdot)$

In this section, we introduce a norm on $\mathfrak{A}$, which generalizes the norm $v_{(s,t)}(x)$ and the weighted algebraic numerical radius $v_\gamma(x)$. Further, some basic properties of this norm will be established.

DEFINITION 2.1. Let $N(\cdot)$ be a norm on $\mathfrak{A}$. The function $v_{(N,(s,t))}(\cdot):\mathfrak{A}\to[0,+\infty)$ is defined by

$$v_{(N,(s,t))}(x)=\sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right),\quad x\in\mathfrak{A}.$$

REMARK 2.2. When $N(\cdot)$ is the C^*-norm $\|\cdot\|$ in Definition 2.1, the function $v_{(N,(s,t))}(\cdot)$ is denoted by (1.1). When $N(\cdot)$ is the C^*-norm $\|\cdot\|$ and $s=\gamma$, $t=1-\gamma$ in Definition 2.1, where $0\leqslant\gamma\leqslant 1$, the function $v_{(N,(s,t))}(\cdot)$ represents the weighted algebraic numerical radius, and is denoted by (1.2).

THEOREM 2.3. *Let $N(\cdot)$ be a norm on $\mathfrak{A}$ and $x\in\mathfrak{A}$. Then*

$$v_{(N,(s,t))}(x)=\sup_{\theta\in\mathbb{R}} N\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right).$$

Proof. To prove this result, first of all, we need to prove that

$$v_{(N,(s,t))}(\lambda x)=|\lambda|\,v_{(N,(s,t))}(x)\tag{2.1}$$

for all $\lambda\in\mathbb{C}$. We may assume that $\lambda\neq 0$, otherwise (2.1) trivially holds. For every nonzero $\lambda\in\mathbb{C}$, there exists $\phi\in\mathbb{R}$ such that $\lambda=|\lambda|e^{i\phi}$, we get

$$\begin{aligned}
v_{(N,(s,t))}(\lambda x)&=\sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}\lambda x\right)\right)=\sup_{\theta\in\mathbb{R}} N\left(se^{i\theta}\lambda x+te^{-i\theta}\bar{\lambda}x^*\right)\\
&=\sup_{\theta\in\mathbb{R}} N\left(s|\lambda|e^{i(\theta+\phi)}x+t|\lambda|e^{-i(\theta+\phi)}x^*\right)\\
&=|\lambda|\sup_{\theta,\phi\in\mathbb{R}} N\left(se^{i(\theta+\phi)}x+te^{-i(\theta+\phi)}x^*\right)\\
&=|\lambda|\,v_{(N,(s,t))}(x).
\end{aligned}$$

Then by using Proposition 1.1 (i), we have

$$\begin{aligned}
v_{(N,(s,t))}(x)=v_{(N,(s,t))}(ix)&=\sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(ie^{i\theta}x\right)\right)\\
&=\sup_{\theta\in\mathbb{R}} N\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right).\quad\square
\end{aligned}$$

REMARK 2.4. When $N(\cdot)$ is the C^*-norm $\|\cdot\|$ in Theorem 2.3, we have

$$v_{(s,t)}(x)=\sup_{\theta\in\mathbb{R}}\left\|\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right\|$$

for all $x\in\mathfrak{A}$.

REMARK 2.5. In Remark 2.4, let $s=\gamma$, $t=1-\gamma$, where $0\leqslant\gamma\leqslant 1$. Then

$$v_\gamma(x)=\sup_{\theta\in\mathbb{R}}\left\|\mathfrak{I}_\gamma\left(e^{i\theta}x\right)\right\|$$

for all $x\in\mathfrak{A}$.

In the following result, we present some properties of $v_{(N,(s,t))}(\cdot)$.

PROPOSITION 2.6. *Let $N(\cdot)$ be a norm on $\mathfrak{A}$ and $x \in \mathfrak{A}$. Then*
(i) $v_{(N,(s,t))}(x) = v_{(N,(t,s))}(x^*)$.
(ii) *If x is self-adjoint, then $v_{(N,(s,t))}(x) = (s+t)N(x)$.*

Proof. (i) can be obtained by the definition of the $v_{(N,(s,t))}(\cdot)$, as follows

$$\begin{aligned} v_{(N,(s,t))}(x) &= \sup_{\theta \in \mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right) = \sup_{\theta \in \mathbb{R}} N\left(se^{i\theta}x + te^{-i\theta}x^*\right) \\ &= \sup_{\theta \in \mathbb{R}} N\left(te^{-i\theta}x^* + se^{i\theta}(x^*)^*\right) \\ &= \sup_{\theta \in \mathbb{R}} N\left(\mathfrak{R}_{(t,s)}\left(e^{-i\theta}x^*\right)\right) \\ &= v_{(N,(t,s))}(x^*). \end{aligned}$$

Next, we prove (ii). Since x is self-adjoint, we have

$$\begin{aligned} v_{(N,(s,t))}(x) &= \sup_{\theta \in \mathbb{R}} N\left(se^{i\theta}x + te^{-i\theta}x^*\right) = \sup_{\theta \in \mathbb{R}} N\left(\left(se^{i\theta} + te^{-i\theta}\right)x\right) \\ &= \sup_{\theta \in \mathbb{R}} \left|se^{i\theta} + te^{-i\theta}\right| N(x) \\ &= \sup_{\theta \in \mathbb{R}} \sqrt{s^2 + t^2 + 2st\left(\cos^2\theta - \sin^2\theta\right)} N(x) \\ &= (s+t)N(x). \quad \square \end{aligned}$$

REMARK 2.7. When $N(\cdot)$ is the C^*-norm $\|\cdot\|$ in Proposition 2.6, for every $x \in \mathfrak{A}$ the following statements hold:

1. $v_{(s,t)}(x) = v_{(t,s)}(x^*)$.
2. If x is self-adjoint, then $(s+t)v(x) = v_{(s,t)}(x) = (s+t)\|x\|$.

REMARK 2.8. In Remark 2.7, let $s = \gamma$, $t = 1-\gamma$, where $0 \leqslant \gamma \leqslant 1$. Then for every $x \in \mathfrak{A}$, the following statements hold:

1. $v_\gamma(x) = v_{1-\gamma}(x^*)$.
2. If x is self-adjoint, then $v(x) = v_\gamma(x) = \|x\|$.

In the next theorem, we prove that $v_{(N,(s,t))}(\cdot)$ is a norm on $\mathfrak{A}$, and the bounds of the norm $v_{(N,(s,t))}(\cdot)$ are presented.

THEOREM 2.9. *Let $N(\cdot)$ be a norm on $\mathfrak{A}$. Then $v_{(N,(s,t))}(\cdot)$ is a norm on $\mathfrak{A}$ and*

$$\max\{sN(x), tN(x^*)\} \leqslant v_{(N,(s,t))}(x) \leqslant (s+t)\max\{N(x), N(x^*)\} \tag{2.2}$$

for all $x \in \mathfrak{A}$.

Proof. Let $N(\cdot)$ be a norm on $\mathfrak{A}$ and $x \in \mathfrak{A}$. Since $v_{(N,(s,t))}(x) \geqslant 0$, we know that non-negativity is ture. Next, we show that positive definiteness is also true. Let us assume that $v_{(N,(s,t))}(x) = 0$. If $s = 0$, we have $v_{(N,(s,t))}(x) = tN(x^*) = 0$. Since $s+t > 0$, then $t \neq 0$. Thus $x^* = 0$, or equivalently $x = 0$. Similarly, if $t = 0$, we have $v_{(N,(s,t))}(x) = sN(x) = 0$. Thus $x = 0$. Hence, we may assume that $s \neq 0$ and $t \neq 0$. Then by Definition 2.1, we can see that $\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right) = 0$ for every $\theta \in \mathbb{R}$. Taking $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively, we obtain

$$\mathfrak{R}_{(s,t)}(x) = \mathfrak{R}_{(s,t)}(ix) = 0.$$

Applying Proposition 1.1 (iv), we get

$$x = \frac{\mathfrak{R}_{(s,t)}(x) - i\mathfrak{R}_{(s,t)}(ix)}{2s} = 0.$$

In Theorem 2.3, we have proved $v_{(N,(s,t))}(\lambda x) = |\lambda|\, v_{(N,(s,t))}(x)$ for all $\lambda \in \mathbb{C}$. That is, positive homogeneity is satisfied. Therefore, to show that $v_{(N,(s,t))}(\cdot)$ is a norm on $\mathfrak{A}$, it suffices to show that $v_{(N,(s,t))}(\cdot)$ is subadditive. Let $y,z \in \mathfrak{A}$, we have

$$\begin{aligned}
v_{(N,(s,t))}(y+z) &= \sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}(y+z)\right)\right) \\
&= \sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}y\right) + \mathfrak{R}_{(s,t)}\left(e^{i\theta}z\right)\right) \\
&\leqslant \sup_{\theta\in\mathbb{R}} \left(N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}y\right)\right) + N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}z\right)\right)\right) \\
&\leqslant \sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}y\right)\right) + \sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}z\right)\right) \\
&= v_{(N,(s,t))}(y) + v_{(N,(s,t))}(z).
\end{aligned}$$

In addition, by using Proposition 1.1 (i) and (iv), we have

$$\mathfrak{R}_{(s,t)}(x) + i\mathfrak{I}_{(s,t)}(x) = 2sx. \tag{2.3}$$

By the Definition 2.1, we have

$$v_{(N,(s,t))}(x) \geqslant N\left(se^{i\theta}x + te^{-i\theta}x^*\right), \quad \theta \in \mathbb{R}.$$

Setting $\theta = 0$ and $\theta = -\pi/2$ in the above inequality, respectively. We get

$$v_{(N,(s,t))}(x) \geqslant N\left(\mathfrak{R}_{(s,t)}(x)\right) \text{ and } v_{(N,(s,t))}(x) \geqslant N\left(\mathfrak{I}_{(s,t)}(x)\right). \tag{2.4}$$

Then applying (2.3) and (2.4), we have that

$$\begin{aligned}
v_{(N,(s,t))}(x) &\geqslant \frac{N\left(\mathfrak{R}_{(s,t)}(x)\right) + N\left(\mathfrak{I}_{(s,t)}(x)\right)}{2} \\
&\geqslant \frac{N\left(\mathfrak{R}_{(s,t)}(x) + i\mathfrak{I}_{(s,t)}(x)\right)}{2} \\
&= \frac{N(2sx)}{2} = sN(x).
\end{aligned}$$

Hence, we have

$$sN(x) \leqslant v_{(N,(s,t))}(x). \tag{2.5}$$

In (2.5), let s and t replace each other and let x be replaced by x^*. We have

$$tN(x^*) \leqslant v_{(N,(t,s))}(x^*). \tag{2.6}$$

Since $v_{(N,(s,t))}(x) = v_{(N,(t,s))}(x^*)$ (see Proposition 2.6 (i)), we deduce that

$$tN(x^*) \leqslant v_{(N,(s,t))}(x). \tag{2.7}$$

By combining (2.5) together with (2.7), we get

$$\max\{sN(x), tN(x^*)\} \leqslant v_{(N,(s,t))}(x). \tag{2.8}$$

On the other hand, by the triangle inequality for the norm $N(\cdot)$, we have

$$\begin{aligned} v_{(N,(s,t))}(x) &= \sup_{\theta\in\mathbb{R}} N\left(se^{i\theta}x + te^{-i\theta}x^*\right) \leqslant sN(x) + tN(x^*) \\ &\leqslant s\max\{N(x), N(x^*)\} + t\max\{N(x), N(x^*)\} \\ &= (s+t)\max\{N(x), N(x^*)\}. \end{aligned}$$

Hence, we obtain

$$v_{(N,(s,t))}(x) \leqslant (s+t)\max\{N(x), N(x^*)\}. \tag{2.9}$$

By combining (2.8) together with (2.9), we get (2.2) as required. $\square$

As a consequence of the preceding theorem, we have the following result.

COROLLARY 2.10. *If $N(\cdot)$ is a self-adjoint norm on $\mathfrak{A}$, then $v_{(N,(s,t))}(\cdot)$ is self-adjoint. Furthermore, $v_{(N,(s,t))}(\cdot)$ is equivalent to $N(\cdot)$ and for every $x \in \mathfrak{A}$ the following inequalities hold:*

$$\max\{s,t\}N(x) \leqslant v_{(N,(s,t))}(x) \leqslant (s+t)N(x). \tag{2.10}$$

Proof. Since $N(\cdot)$ is a self-adjoint norm on $\mathfrak{A}$, we have

$$\begin{aligned} v_{(N,(s,t))}(x^*) &= \sup_{\theta\in\mathbb{R}} N\left(se^{i\theta}x^* + te^{-i\theta}x\right) \\ &= \sup_{\theta\in\mathbb{R}} N\left(se^{-i\theta}x + te^{i\theta}x^*\right) \\ &= v_{(N,(s,t))}(x). \end{aligned}$$

Therefor, $v_{(N,(s,t))}(\cdot)$ is self-adjoint. Furthermore, since $N(\cdot)$ is a self-adjoint norm on $\mathfrak{A}$, from (2.2) it follows that

$$\max\{s,t\}N(x) \leqslant v_{(N,(s,t))}(x) \leqslant (s+t)N(x).$$

So $v_{(N,(s,t))}(\cdot)$ is equivalent to $N(\cdot)$. $\square$

REMARK 2.11. If $N(\cdot)$ is a self-adjoint norm on $\mathfrak{A}$. By Corollary 2.10, we know that $v_{(N,(s,t))}(\cdot)$ is self-adjoint. In view of Proposition 2.6 (i), thus we have

$$v_{(N,(s,t))}(x) = v_{(N,(t,s))}(x^*) = v_{(N,(t,s))}(x).$$

REMARK 2.12. Since C^*-norm $\|\cdot\|$ is a self-adjoint norm on $\mathfrak{A}$, by Remark 2.11, we have

$$v_{(s,t)}(x) = v_{(t,s)}(x^*) = v_{(t,s)}(x).$$

Furthermore, $v_{(s,t)}(\cdot)$ is equivalent to C^*-norm $\|\cdot\|$ and for every $x \in \mathfrak{A}$ the following inequalities hold:

$$\max\{s,t\}\|x\| \leqslant v_{(s,t)}(x) \leqslant (s+t)\|x\|. \tag{2.11}$$

REMARK 2.13. In Remark 2.12, let $s = \gamma$, $t = 1-\gamma$, where $0 \leqslant \gamma \leqslant 1$. It is clear that $v_\gamma(x)$ is self-adjoint. We have

$$v_\gamma(x) = v_{1-\gamma}(x^*) = v_{1-\gamma}(x).$$

Furthermore, $v_\gamma(x)$ is equivalent to C^*-norm $\|\cdot\|$ and for every $x \in \mathfrak{A}$, the following inequalities hold:

$$\max\{\gamma, 1-\gamma\}\|x\| \leqslant v_\gamma(x) \leqslant \|x\|. \tag{2.12}$$

REMARK 2.14. It should be mentioned here that in Corollary 2.10, when $\mathfrak{A}$ is the set of all bounded linear operators on a complex Hilbert space, and with $s = t = \frac{1}{2}$, Abu-Omar and Kittaneh have proved this case in [1, Theorem 2].

The next theorem shows that when $N(\cdot)$ is a self-adjoint norm on $\mathfrak{A}$ and the sum of s and t is fixed, the maximum and minimum values of the norm $v_{(N,(s,t))}(\cdot)$ can be obtained.

THEOREM 2.15. *If $N(\cdot)$ is a self-adjoint norm on $\mathfrak{A}$ and λ is a positive real number. Then for every $x \in \mathfrak{A}$, the function $f(s) = v_{(N,(s,\lambda-s))}(x)$ is convex continuous function on $[0,\lambda]$, and that the minimum of f is $f\left(\frac{\lambda}{2}\right)$ and the maximum of f is $f(0), f(\lambda)$.*

Proof. For every $x \in \mathfrak{A}$ and $\xi \in [0,1]$, we have

$$\begin{aligned}
&f(\xi s + (1-\xi)t) \\
&= v_{(N,(\xi s+(1-\xi)t, \lambda-(\xi s+(1-\xi)t)))}(x) \\
&= \sup_{\theta\in\mathbb{R}} N\left((\xi s + (1-\xi)t)e^{i\theta}x + (\lambda - (\xi s + (1-\xi)t))e^{-i\theta}x^*\right) \\
&= \sup_{\theta\in\mathbb{R}} N\left(\xi s e^{i\theta}x + \xi(\lambda - s)e^{-i\theta}x^* + (1-\xi)te^{i\theta}x + (1-\xi)(\lambda - t)e^{-i\theta}x^*\right)
\end{aligned}$$

$$\begin{aligned}
&\leqslant \xi \sup_{\theta\in\mathbb{R}} N\left(se^{i\theta}x+(\lambda-s)\,e^{-i\theta}x^*\right)+(1-\xi)\sup_{\theta\in\mathbb{R}} N\left(te^{i\theta}x+(\lambda-t)\,e^{-i\theta}x^*\right)\\
&=\xi\, v_{(N,(s,\lambda-s))}(x)+(1-\xi)\, v_{(N,(t,\lambda-t))}(x)\\
&=\xi f(s)+(1-\xi)\, f(t).
\end{aligned}$$

Therefore, f is convex on $[0,\lambda]$. From property of convex function it follows that f is continuous on $(0,\lambda)$. In view of (2.10), we have

$$0\leqslant \lambda N(x)-v_{(N,(s,\lambda-s))}(x)\leqslant \frac{\lambda-|2s-\lambda|}{2}N(x).$$

Hence f is continuous at $s=0$ and $s=\lambda$, i.e. it is continuous on $[0,\lambda]$. By Remark 2.11, we have

$$v_{(N,(s,\lambda-s))}(x)=v_{(N,(\lambda-s,s))}(x^*)=v_{(N,(\lambda-s,s))}(x).$$

It follows that $f(s)=f(\lambda-s)$. Thus f is symmetric about $s=\frac{\lambda}{2}$. In other words, f is decreasing on $[0,\frac{\lambda}{2}]$ and increasing on $[\frac{\lambda}{2},\lambda]$. Therefore, we get that f attains its minimum at $s=\frac{\lambda}{2}$ and its maximum at $s=0$, $s=\lambda$. This completes the proof. □

As an immediate consequence of Theorem 2.15, we obtain the following result.

COROLLARY 2.16. *If $N(\cdot)$ is a self-adjoint norm on $\mathfrak{A}$, λ is a positive real number. Then for every $x\in\mathfrak{A}$, $v_{(N,(s,\lambda-s))}(x)\leqslant v_{(N,(t,\lambda-t))}(x)$ if and only if $\left|s-\frac{\lambda}{2}\right|\leqslant \left|t-\frac{\lambda}{2}\right|$.*

Proof. From Theorem 2.15, f is symmetric about $s=\frac{\lambda}{2}$, i.e. f is decreasing on $[0,\frac{\lambda}{2}]$ and increasing on $[\frac{\lambda}{2},\lambda]$. Thus Corollary 2.16 is obviously true. □

REMARK 2.17. Let $N(\cdot)$ is the C^*-norm $\|\cdot\|$ in Theorem 2.15. Since C^*-norm $\|\cdot\|$ is a self-adjoint norm on $\mathfrak{A}$. Thus the function $f(s)=v_{(s,\lambda-s)}(x)$ is convex continuous function on $[0,\lambda]$, and that the minimum of f is $f\left(\frac{\lambda}{2}\right)$ and the maximum of f is $f(0),f(\lambda)$. Then we have

$$\lambda v(x)\leqslant v_{(s,\lambda-s)}(x)\leqslant \lambda\,\|x\|.$$

REMARK 2.18. In Remark 2.17, let $s=\gamma$, $t=1-\gamma$, where $0\leqslant\gamma\leqslant 1$. Hence, the function $f(\gamma)=v_\gamma(x)$ is convex continuous function on $[0,1]$, and the following inequality holds:

$$v(x)\leqslant v_\gamma(x)\leqslant \|x\|.$$

Now, we are in a position to state other useful representations of $v_{(N,(s,t))}(\cdot)$.

THEOREM 2.19. *Let $N(\cdot)$ be a norm on $\mathfrak{A}$ and $x \in \mathfrak{A}$. Then we have*

$$v_{(N,(s,t))}(x) = \frac{1}{2} \sup_{\theta,\phi\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(\left(e^{i\theta} - ie^{i\phi}\right)x\right)\right).$$

Proof. Let $N(\cdot)$ be a norm on $\mathfrak{A}$ and $x \in \mathfrak{A}$. We have

$$\begin{aligned}
v_{(N,(s,t))}(x) &= \frac{1}{2} \sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right) + \mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right) \\
&= \frac{1}{2} \sup_{\theta\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right) + \mathfrak{I}_{(s,t)}\left(e^{i\left(\frac{\pi}{2}+\theta\right)}x\right)\right) \\
&\leqslant \frac{1}{2} \sup_{\theta,\phi\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right) + \mathfrak{I}_{(s,t)}\left(e^{i\phi}x\right)\right) \\
&= \frac{1}{2} \sup_{\theta,\phi\in\mathbb{R}} N\left(s\left(e^{i\theta} - ie^{i\phi}\right)x + t\left(\left(e^{i\theta} - ie^{i\phi}\right)x\right)^*\right) \\
&= \frac{1}{2} \sup_{\theta,\phi\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(\left(e^{i\theta} - ie^{i\phi}\right)x\right)\right) \\
&\leqslant \frac{1}{2} \sup_{\theta,\phi\in\mathbb{R}} v_{(N,(s,t))}\left(\left(e^{i\theta} - ie^{i\phi}\right)x\right) \\
&= \frac{1}{2} \sup_{\theta,\phi\in\mathbb{R}} \left|e^{i\theta} - ie^{i\phi}\right| v_{(N,(s,t))}(x) \\
&= \frac{v_{(N,(s,t))}(x)}{2} \sup_{\theta,\phi\in\mathbb{R}} \sqrt{2 - 2\sin(\theta - \phi)} \\
&= v_{(N,(s,t))}(x).
\end{aligned}$$

Hence, we have

$$v_{(N,(s,t))}(x) = \frac{1}{2} \sup_{\theta,\phi\in\mathbb{R}} N\left(\mathfrak{R}_{(s,t)}\left(\left(e^{i\theta} - ie^{i\phi}\right)x\right)\right). \quad \square$$

REMARK 2.20. When $N(\cdot)$ is the C^*-norm $\|\cdot\|$ in Theorem 2.19, for every $x \in \mathfrak{A}$ the following equality holds:

$$v_{(s,t)}(x) = \frac{1}{2} \sup_{\theta,\phi\in\mathbb{R}} \left\|\mathfrak{R}_{(s,t)}\left(\left(e^{i\theta} - ie^{i\phi}\right)x\right)\right\|.$$

REMARK 2.21. In Remark 2.20, let $s = \gamma$, $t = 1-\gamma$, where $0 \leqslant \gamma \leqslant 1$. Then for every $x \in \mathfrak{A}$, the following equality holds:

$$v_\gamma(x) = \frac{1}{2} \sup_{\theta,\phi\in\mathbb{R}} \left\|\mathfrak{R}_\gamma\left(\left(e^{i\theta} - ie^{i\phi}\right)x\right)\right\|.$$

In the following theorem, we present an expression of $v_{(N,(s,t))}(x)$ in terms of the real and imaginary of $x \in \mathfrak{A}$.

THEOREM 2.22. *Let $N(\cdot)$ be a norm on $\mathfrak{A}$ and $x \in \mathfrak{A}$. For $\alpha, \beta \in \mathbb{R}$, we have*

$$\begin{aligned} & v_{(N,(s,t))}(x) \\ & = \sup_{\alpha^2+\beta^2=1} N\left(\alpha \mathfrak{R}_{(s,t)}(x) + \beta \mathfrak{I}_{(s,t)}(x)\right) \\ & = \sup_{\alpha^2+\beta^2=1} N\left((s+t)\left(\alpha \mathfrak{R}(x) + \beta \mathfrak{I}(x)\right) + i(s-t)\left(\alpha \mathfrak{I}(x) - \beta \mathfrak{R}(x)\right)\right). \end{aligned}$$

Proof. Let $\theta \in \mathbb{R}$. Put $\alpha = \cos\theta$ and $\beta = -\sin\theta$, we have

$$\begin{aligned} se^{i\theta}x + te^{-i\theta}x^* &= s(\cos\theta + i\sin\theta)x + t(\cos\theta - i\sin\theta)x^* \\ &= \cos\theta\,(sx + tx^*) - \sin\theta\,(-six + tix^*) \\ &= \alpha \mathfrak{R}_{(s,t)}(x) + \beta \mathfrak{I}_{(s,t)}(x). \end{aligned}$$

Therefor, we have

$$N(se^{i\theta}x + te^{-i\theta}x^*) = N\left(\alpha \mathfrak{R}_{(s,t)}(x) + \beta \mathfrak{I}_{(s,t)}(x)\right).$$

Taking the supremum over $\theta \in \mathbb{R}$ in the above equality, we obtain

$$v_{(N,(s,t))}(x) = \sup_{\alpha^2+\beta^2=1} N\left(\alpha \mathfrak{R}_{(s,t)}(x) + \beta \mathfrak{I}_{(s,t)}(x)\right).$$

Using Proposition 1.1 (ii) and (iii), we get

$$\begin{aligned} & v_{(N,(s,t))}(x) \\ & = \sup_{\alpha^2+\beta^2=1} N\left(\alpha \mathfrak{R}_{(s,t)}(x) + \beta \mathfrak{I}_{(s,t)}(x)\right) \\ & = \sup_{\alpha^2+\beta^2=1} N\left((s+t)\left(\alpha \mathfrak{R}(x) + \beta \mathfrak{I}(x)\right) + i(s-t)\left(\alpha \mathfrak{I}(x) - \beta \mathfrak{R}(x)\right)\right). \quad \square \end{aligned}$$

REMARK 2.23. When $N(\cdot)$ is the C^*-norm $\|\cdot\|$ in Theorem 2.22, for every $x \in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} & v_{(s,t)}(x) \\ & = \sup_{\alpha^2+\beta^2=1} \left\|\alpha \mathfrak{R}_{(s,t)}(x) + \beta \mathfrak{I}_{(s,t)}(x)\right\| \\ & = \sup_{\alpha^2+\beta^2=1} \left\|(s+t)\left(\alpha \mathfrak{R}(x) + \beta \mathfrak{I}(x)\right) + i(s-t)\left(\alpha \mathfrak{I}(x) - \beta \mathfrak{R}(x)\right)\right\|. \end{aligned}$$

REMARK 2.24. In Remark 2.23, let $s = \gamma$, $t = 1-\gamma$, where $0 \leqslant \gamma \leqslant 1$. Then for every $x \in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} & v_\gamma(x) \\ & = \sup_{\alpha^2+\beta^2=1} \left\|\alpha \mathfrak{R}_\gamma(x) + \beta \mathfrak{I}_\gamma(x)\right\| \\ & = \sup_{\alpha^2+\beta^2=1} \left\|\alpha \mathfrak{R}(x) + \beta \mathfrak{I}(x) + i(2\gamma-1)\left(\alpha \mathfrak{I}(x) - \beta \mathfrak{R}(x)\right)\right\|. \end{aligned}$$

In the following result, a upper bound of $v_{(N,(s,t))}(x)$ in terms of the real and imaginary of $x \in \mathfrak{A}$ is given.

THEOREM 2.25. *Let $N(\cdot)$ be a norm on $\mathfrak{A}$ and $x \in \mathfrak{A}$. Then*

$$v_{(N,(s,t))}(x) \leqslant \sqrt{2(s^2+t^2)} \inf_{\theta \in \mathbb{R}} \sqrt{N^2(\mathfrak{R}(e^{i\theta}x)) + N^2(\mathfrak{I}(e^{i\theta}x))}.$$

Proof. Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha^2+\beta^2 = 1$. Then clearly

$$\begin{aligned} & N((s+t)(\alpha\mathfrak{R}(x)+\beta\mathfrak{I}(x)) + i(s-t)(\alpha\mathfrak{I}(x)-\beta\mathfrak{R}(x))) \\ & = N(m\mathfrak{R}(x)+n\mathfrak{I}(x)), \end{aligned}$$

where $m = \alpha(s+t) - i\beta(s-t)$ and $n = \beta(s+t) + i\alpha(s-t)$. By using triangle inequality and Cauchy-Buniakowsky-Schwarz inequality, respectively, we get

$$\begin{aligned} N(m\mathfrak{R}(x)+n\mathfrak{I}(x)) & \leqslant |m| N(\mathfrak{R}(x)) + |n| N(\mathfrak{I}(x)) \\ & \leqslant \sqrt{|m|^2+|n|^2} \sqrt{N^2(\mathfrak{R}(x)) + N^2(\mathfrak{I}(x))}. \end{aligned}$$

By simple calculations, we obtain $|m|^2+|n|^2 = 2\left(s^2+t^2\right)$. Hence we get

$$\begin{aligned} & N((s+t)(\alpha\mathfrak{R}(x)+\beta\mathfrak{I}(x)) + i(s-t)(\alpha\mathfrak{I}(x)-\beta\mathfrak{R}(x))) \\ & \leqslant \sqrt{2(s^2+t^2)} \sqrt{N^2(\mathfrak{R}(x)) + N^2(\mathfrak{I}(x))}. \end{aligned}$$

Taking the supremum over all $\alpha, \beta \in \mathbb{R}$ such that $\alpha^2+\beta^2 = 1$, and taking Theorem 2.22 into account, we get

$$v_{(N,(s,t))}(x) \leqslant \sqrt{2(s^2+t^2)} \sqrt{N^2(\mathfrak{R}(x)) + N^2(\mathfrak{I}(x))}. \tag{2.13}$$

Replacing x by $e^{i\theta}x$ in (2.13), we get the desired result. $\square$

REMARK 2.26. When $N(\cdot)$ is the C^*-norm $\|\cdot\|$ in Theorem 2.25, for $x \in \mathfrak{A}$, we have

$$v_{(s,t)}(x) \leqslant \sqrt{2(s^2+t^2)} \inf_{\theta \in \mathbb{R}} \sqrt{\|\mathfrak{R}(e^{i\theta}x)\|^2 + \|\mathfrak{I}(e^{i\theta}x)\|^2}.$$

REMARK 2.27. Let $s = \gamma$, $t = 1-\gamma$ in Remark 2.26, where $0 \leqslant \gamma \leqslant 1$. We get

$$v_\gamma(x) \leqslant \sqrt{4\gamma(\gamma-1)+2} \inf_{\theta \in \mathbb{R}} \sqrt{\|\mathfrak{R}(e^{i\theta}x)\|^2 + \|\mathfrak{I}(e^{i\theta}x)\|^2}.$$

In the next theorem, we give a lower bound for the $v_{(N,(s,t))}(x)$.

THEOREM 2.28. *Let $N(\cdot)$ be an algebra norm on $\mathfrak{A}$ and $x \in \mathfrak{A}$. Then*

$$stN\left(xx^*+x^*x\right)+\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|N^2\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right)-N^2\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right)\right|\leqslant v^2_{(N,(s,t))}\left(x\right).$$

Proof. Let $\theta\in\mathbb{R}$. By simple calculations, we obtain

$$\mathfrak{R}^2_{(s,t)}\left(e^{i\theta}x\right)+\mathfrak{I}^2_{(s,t)}\left(e^{i\theta}x\right)=2st\left(xx^*+x^*x\right).$$

It follows from Definition 2.1 and Theorem 2.3 that

$$v_{(N,(s,t))}\left(x\right)\geqslant\max\left\{N\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right),N\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right)\right\}.$$

Thus, we have

$$\begin{aligned}
&v^2_{(N,(s,t))}\left(x\right)\\
&\geqslant\max\left\{N^2\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right),N^2\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right)\right\}\\
&=\frac{N^2\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right)+N^2\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right)}{2}+\frac{\left|N^2\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right)-N^2\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right)\right|}{2}\\
&\geqslant\frac{N\left(\mathfrak{R}^2_{(s,t)}\left(e^{i\theta}x\right)\right)+N\left(\mathfrak{I}^2_{(s,t)}\left(e^{i\theta}x\right)\right)}{2}+\frac{\left|N^2\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right)-N^2\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right)\right|}{2}\\
&\qquad\text{(since } N(\cdot) \text{ is an algebra norm)}\\
&\geqslant\frac{N\left(\mathfrak{R}^2_{(s,t)}\left(e^{i\theta}x\right)+\mathfrak{I}^2_{(s,t)}\left(e^{i\theta}x\right)\right)}{2}+\frac{\left|N^2\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right)-N^2\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right)\right|}{2}\\
&=stN\left(xx^*+x^*x\right)+\frac{\left|N^2\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right)-N^2\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right)\right|}{2}.
\end{aligned}$$

Whence

$$stN\left(xx^*+x^*x\right)+\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|N^2\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right)-N^2\left(\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right)\right|\leqslant v^2_{(N,(s,t))}\left(x\right).\quad\square$$

REMARK 2.29. When $N(\cdot)$ is the C^*-norm $\|\cdot\|$ in Theorem 2.28, for $x\in\mathfrak{A}$, we have

$$st\left\|xx^*+x^*x\right\|+\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|\left\|\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right\|^2-\left\|\mathfrak{I}_{(s,t)}\left(e^{i\theta}x\right)\right\|^2\right|\leqslant v^2_{(s,t)}\left(x\right).$$

REMARK 2.30. Let $s=\gamma$, $t=1-\gamma$ in Remark 2.29, where $0\leqslant\gamma\leqslant 1$. We get

$$\gamma\left(1-\gamma\right)\left\|xx^*+x^*x\right\|+\frac{1}{2}\sup_{\theta\in\mathbb{R}}\left|\left\|\mathfrak{R}_{\gamma}\left(e^{i\theta}x\right)\right\|^2-\left\|\mathfrak{I}_{\gamma}\left(e^{i\theta}x\right)\right\|^2\right|\leqslant v^2_{\gamma}\left(x\right).$$

3. Additional results of $v_{(s,t)}(\cdot)$

In this Section, some results for $v_{(s,t)}(\cdot)$ are given. As a special case of norm $v_{(s,t)}(\cdot)$, some results involving the weighted algebraic numerical radius $v_\gamma(\cdot)$ are also presented.

Below we give a useful lemma for proving the latter theorem.

LEMMA 3.1. *Let φ be a state over $\mathfrak{A}$. For $x \in \mathfrak{A}$, the following statements hold.*
(i) $\sup\limits_{\theta\in\mathbb{R}} \left|\mathfrak{R}_{(s,t)}\left(e^{i\theta}\varphi(x)\right)\right| = (s+t)|\varphi(x)|$.
(ii) $\sup\limits_{\theta\in\mathbb{R}} \left|\mathfrak{I}_{(s,t)}\left(e^{i\theta}\varphi(x)\right)\right| = (s+t)|\varphi(x)|$.

Proof. We may assume that $\varphi(x) \neq 0$ otherwise (i) and (ii) trivially hold.

1. Put $e^{i\theta_0} = \frac{\overline{\varphi(x)}}{|\varphi(x)|}$. Then we have

$$(s+t)|\varphi(x)| = \left|\mathfrak{R}_{(s,t)}\left(e^{i\theta_0}\varphi(x)\right)\right| \leqslant \sup_{\theta\in\mathbb{R}} \left|\mathfrak{R}_{(s,t)}\left(e^{i\theta}\varphi(x)\right)\right|$$
$$\leqslant (s+t)\sup_{\theta\in\mathbb{R}}\left|e^{i\theta}\varphi(x)\right| = (s+t)|\varphi(x)|.$$

Therefore, we obtain

$$\sup_{\theta\in\mathbb{R}} \left|\mathfrak{R}_{(s,t)}\left(e^{i\theta}\varphi(x)\right)\right| = (s+t)|\varphi(x)|.$$

2. Replacing x in (i) by ix. By using Proposition 1.1 (i), we get

$$\sup_{\theta\in\mathbb{R}} \left|\mathfrak{I}_{(s,t)}\left(e^{i\theta}\varphi(x)\right)\right| = \sup_{\theta\in\mathbb{R}} \left|\mathfrak{R}_{(s,t)}\left(e^{i\theta}\varphi(ix)\right)\right|$$
$$= (s+t)|\varphi(ix)| = (s+t)|\varphi(x)|.$$

Hence we have

$$\sup_{\theta\in\mathbb{R}} \left|\mathfrak{I}_{(s,t)}\left(e^{i\theta}\varphi(x)\right)\right| = (s+t)|\varphi(x)|. \quad \square$$

REMARK 3.2. What we need to mention here is that Lemma 3.1 generalizes [23, Lemma 2.1]

The next result establishes that $v_{(s,t)}(\cdot)$ and $v(\cdot)$ are two equivalent norm on $\mathfrak{A}$.

THEOREM 3.3. *Let $x \in \mathfrak{A}$, the following inequalities hold.*

$$(s+t)v(x) \leqslant v_{(s,t)}(x) \leqslant 2\max\{s,t\}v(x).$$

Proof. Let $x \in \mathfrak{A}$. By Lemma 3.1, we have

$$\begin{aligned}(s+t)\,v(x) &= (s+t)\sup_{\varphi\in\mathscr{S}(\mathfrak{A})}|\varphi(x)| = \sup_{\varphi\in\mathscr{S}(\mathfrak{A})}\sup_{\theta\in\mathbb{R}}\left|\mathfrak{R}_{(s,t)}\left(e^{i\theta}\varphi(x)\right)\right| \\ &= \sup_{\varphi\in\mathscr{S}(\mathfrak{A})}\sup_{\theta\in\mathbb{R}}\left|\varphi\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right)\right| \\ &= \sup_{\theta\in\mathbb{R}}\sup_{\varphi\in\mathscr{S}(\mathfrak{A})}\left|\varphi\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right)\right| \\ &= \sup_{\theta\in\mathbb{R}} v\left(\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right) \leqslant \sup_{\theta\in\mathbb{R}}\left\|\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right\| = v_{(s,t)}(x),\end{aligned}$$

and hence

$$(s+t)\,v(x) \leqslant v_{(s,t)}(x). \tag{3.1}$$

On the other hand, using Proposition 1.1 (ii) , we have

$$\begin{aligned}v_{(s,t)}(x) &= \sup_{\theta\in\mathbb{R}}\left\|\mathfrak{R}_{(s,t)}\left(e^{i\theta}x\right)\right\| \\ &= \sup_{\theta\in\mathbb{R}}\left\|(s+t)\,\mathfrak{R}\left(e^{i\theta}x\right)+i\,(s-t)\,\mathfrak{I}\left(e^{i\theta}x\right)\right\| \\ &\leqslant (s+t)\sup_{\theta\in\mathbb{R}}\left\|\mathfrak{R}\left(e^{i\theta}x\right)\right\| + |s-t|\sup_{\theta\in\mathbb{R}}\left\|\mathfrak{I}\left(e^{i\theta}x\right)\right\| \\ &= (s+t)\,v(x)+|s-t|\,v(x) \\ &= (s+t+|s-t|)\,v(x) = 2\max\{s,t\}\,v(x).\end{aligned}$$

Therefore, we obtain

$$v_{(s,t)}(x) \leqslant 2\max\{s,t\}\,v(x). \tag{3.2}$$

Now, from (3.1) and (3.2), we deduce the desired result. □

REMARK 3.4. Let $s=\gamma$, $t=1-\gamma$ in Theorem 3.3, where $0\leqslant\gamma\leqslant 1$. We obtain inequalities involving the weighted algebraic numerical radius, that is,

$$v(x) \leqslant v_\gamma(x) \leqslant 2\max\{\gamma, 1-\gamma\}\,v(x).$$

Recall that a character on a commutative C^*-algebra $\mathfrak{A}$ is a non-zero homomorphism $\varphi:\mathfrak{A}\to\mathbb{C}$. We denote by $\widehat{\mathfrak{A}}$ the set of characters. Now, we prove a lemma that we need in what follows.

LEMMA 3.5. *Let $\mathfrak{A}$ be a commutative C^*-algebra. For $x\in\mathfrak{A}$, we have*

$$(s+t)\,\|x\| = v_{(s,t)}(x) = \sup\{(s+t)\,|\varphi(x)| : \varphi\in\widehat{\mathfrak{A}}\}.$$

Proof. Since $\mathfrak{A}$ is commutative, then $v(x) = \|x\|$ for any $x \in \mathfrak{A}$. This together with (2.11) and Theorem 3.3 allow us to conclude. In fact, one can say a little bit more that: $v_{(s,t)}(x) = (s+t)\|x\|$ for any normal element $x \in \mathfrak{A}$ without assuming that $\mathfrak{A}$ is commutative. □

Analogously to the usual numerical index [8], we define weighted algebra numerical index of $\mathfrak{A}$ by

$$n_\gamma(\mathfrak{A}) = \inf\{v_\gamma(x) : x \in \mathfrak{A}, \|x\| = 1\}.$$

We can state the following results.

THEOREM 3.6. *The following statements hold.*
(i) $\max\{\gamma, 1-\gamma\} \leqslant n_\gamma(\mathfrak{A}) \leqslant 1$.
(ii) *If $\mathfrak{A}$ is commutative then $n_\gamma(\mathfrak{A}) = 1$.*

Proof. (i) According to (2.12), it is easy to deduce the desired result.
(ii) It follows immediately from Lemma 3.5. □

In [24], the numerical radius Crawford number of $x \in \mathfrak{A}$ is defined by

$$\mathscr{C}(x) = \inf\{|\varphi(x)| : \varphi \in \mathscr{S}(\mathfrak{A})\}.$$

The following Proposition 3.7, as a generalization of [24, Proposition 2], gives a large family of elements satisfying $\mathscr{C}(x) > 0$.

PROPOSITION 3.7. *Let $x \in \mathfrak{A}$ with $v_{(s,t)}(x) < s+t$. If $y = e+x$, then $\mathscr{C}(y) > 0$.*

Proof. Since $v_{(s,t)}(x) < s+t$, thus there is a positive number $\sigma \in [0,1)$ such that $v_{(s,t)}(x) \leqslant (s+t)\sigma < s+t$. From the first inequality of Theorem 3.3, for $\varphi \in \mathscr{S}(\mathfrak{A})$, we have

$$|\varphi(y)| = |\varphi(e) + \varphi(x)| \geqslant |\varphi(e)| - |\varphi(x)| \geqslant 1 - \frac{v_{(s,t)}(x)}{s+t} \geqslant 1 - \sigma.$$

Taking the infimum over $\varphi \in \mathscr{S}(\mathfrak{A})$, we get $\mathscr{C}(y) \geqslant 1-\sigma$, Thus $\mathscr{C}(y) > 0$. □

Let $s = \gamma$, $t = 1-\gamma$ in Proposition 3.7, where $0 \leqslant \gamma \leqslant 1$. We have the following remark.

REMARK 3.8. Let $x \in \mathfrak{A}$ with $v_\gamma(x) < 1$. If $y = e+x$, then $\mathscr{C}(y) > 0$.

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