Operators and Matrices
Volume 19, Number 1 (2025), 49–67
doi:10.7153/oam-2025-19-04

MIXED PRODUCTS OF TOEPLITZ AND HANKEL OPERATORS ON THE FOCK–SOBOLEV SPACES

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(*Communicated by C.-K. Ng*)

Abstract. For entire functions f and g, we study the problem of when the mixed products of Toeplitz and Hankel operators $H_{\overline{f}}T_{\overline{g}}$ is bounded or compact on the Fock-Sobolev spaces $F^{p,m}(\mathbb{C}^n)$ with $1 \leqslant p < \infty$. This is a companion to Sarason's Toeplitz product problem which was completely solved for the Fock space by Cho-Park-Zhu in 2014. Our results here completely characterize the bounded and compact mixed product $H_{\overline{f}}T_{\overline{g}}$ on the Fock-Sobolev space.

1. Introduction

Let $\mathbb{C}^n$ be n dimensional complex vector space and dv be the ordinary volume measure on $\mathbb{C}^n$. If $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ are points in $\mathbb{C}^n$, we write

$$z \cdot \overline{w} = \sum_{j=1}^{n} z_j \overline{w_j}, \quad |z| = (z \cdot \overline{z})^{\frac{1}{2}}.$$

For any $1 \leqslant p < \infty$, L^p_g denotes the space of Lebesgue measurable functions f on $\mathbb{C}^n$ such that the function $f(z)e^{-\frac{|z|^2}{2}}$ is in $L^p(\mathbb{C}^n, dv)$. The Fock space F^p is a subspace of $L^p_g(\mathbb{C}^n, dv)$ consisting of all entire functions f on $\mathbb{C}^n$ such that

$$\|f\|_p = \left((\frac{p}{2\pi})^n \int_{\mathbb{C}^n} |f(z)e^{-\frac{|z|^2}{2}}|^p dv(z) \right)^{\frac{1}{p}} < \infty.$$

For an $n-$tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of non-negative integers and $z \in \mathbb{C}^n$, we write

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad \partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}, \quad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n},$$

where ∂_j denotes the partial differentiation with respect to the $j-$th component.

For any non-negative integer m, we consider the Fock-Sobolev space $F^{p,m}$ consisting of all entire functions f on $\mathbb{C}^n$ such that

$$\|f\|_{p,m} := \sum_{|\alpha| \leqslant m} \|\partial^\alpha f\|_p < \infty,$$

Mathematics subject classification (2020): Primary 46A65, 47A15; Secondary 47B38.
Keywords and phrases: Toeplitz operators, Hankel operators, Fock-Sobolev space, boundedness.
This research is supported by the Grant of Postdoctoral Research Foundation of China (Grant Nos. 2022TQ0007) and National Natural Science Foundation of China (Grant Nos. 12031002 and 12301146 and 12271075) and XLYC2403054.
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where $\|\cdot\|_p$ is the norm in F^p and $\partial^\alpha f$ is the α-th order derivative of f.

Let $L_g^{p,m}$ denote the space of Lebesgue measurable functions f on $\mathbb{C}^n$ such that the function $|z|^m f$ is in L_g^p. It follows from the mean-value property of subharmonic functions (see [26] for example) that $F^{p,m}$ is a closed subspace of $L_g^{p,m}$. The orthogonal projection $P_m : L_g^{p,m} \to F^{p,m}$ is defined by

$$P_m f(w) = \omega_{2,m,n} \int_{\mathbb{C}^n} f(z) K_m(w,z) e^{-|z|^2} |z|^{2m} dv(z),$$

where $K_m(z,w)$ is the reproducing kernel of the Fock-Sobolev space $F^{2,m}$. It is proved in [6] that P_m is a bounded projection from $L_g^{p,m}$ onto $F^{p,m}$. There are some research about Fock-Sobolev spaces, see [2–4, 6, 9, 11, 23]

Moreover, for any $1 \leqslant p < \infty$ it is easily checked $K_m(z,w)$ as a function of z is also in $F^{p,m}(\mathbb{C}^n)$. It yields from [6] that the set of all finite linear combinations of kernel functions is dense in $F^{p,m}(\mathbb{C}^n)$. Then for $\varphi \in F^{p,m}(\mathbb{C}^n)$, the Toeplitz operator T_φ can be densely defined as

$$T_\varphi f(w) = \omega_{2,m,n} \int_{\mathbb{C}^n} \varphi(z) f(z) K_m(w,z) e^{-|z|^2} |z|^{2m} dv(z), \quad z \in \mathbb{C}^n, \quad f \in F^{p,m},$$

and the Hankel operator $H_\varphi : F^{2,m} \longrightarrow (F^{2,m})^\perp$ is also densely defined by

$$H_\varphi = (I - P_m)(\varphi f).$$

The study of Hankel and Toeplitz operators is an active area of research over the past few decades, see [8, 17, 26]. Because researchers often consider the operators T_φ and H_φ side-by-side, a new term has been invented to refer to mixed products of Toeplitz and Hankel operators, that is, “Haplitz operators” or “Ha-plitz operators”.

Originally, Sarason’s Toeplitz product problem was raised by Sarason in [18]: characterize functions f and g in the Hardy space H^2 such that the Toeplitz product $T_f T_{\overline{g}}$ is bounded on H^2. Sarason’s Toeplitz product conjecture was offered in [18], in terms of the boundedness of the function

$$\widetilde{|f|^2}(z)\widetilde{|g|^2}(z),$$

where $\widetilde{f}$ denotes the so-called Berezin transform of f. It was eventually shown to be false, both for Hardy space and Bergman space, see [1, 15]. Furthermore, Sarason’s Toeplitz product problem was partially solved for the Hardy and Bergman spaces in [1, 15, 21, 25]. Somewhat surprisingly, Sarason’s conjecture is true for classical Fock space and Fock-sobolev space, see [5, 7, 19]. Cho, Park, and Zhu completely solved Sarason’s product problem for the classical Fock space F_α^2 and general Fock spaces F_α^p in [5], which was also generalized to the Fock-sobolev space in [7].

As companions to Sarason’s Toeplitz product problem are the following two analogous problems, which have been studied in the literature (see [2, 14, 19, 22, 24]):

(P1) characterize functions f and g in each of the Hardy, Bergman, Fock, and Fock-Sobolev spaces such that the Hankel product $H_{\overline{f}}^* H_{\overline{g}}$ is bounded (or compact). It is natural to conjecture that $H_{\overline{f}}^* H_{\overline{g}}$ is bounded if and only if the function

$$[\widetilde{|f|^2}(z) - |f(z)|^2][\widetilde{|g|^2}(z) - |g(z)|^2]$$

is bounded.

(P2) characterize functions f and g in each of the Hardy, Bergman, Fock, and Fock-Sobolev spaces such that the mixed product $H_{\bar{f}}T_{\bar{g}}$ is bounded. It is also natural to conjecture that $H_{\bar{f}}T_{\bar{g}}$ is bounded if and only if the function

$$\mathscr{D}(f,g)(z) := |g(z)|^2[\widetilde{|f|^2}(z) - |f(z)|^2]$$

is bounded.

Some scholars guess that Sarason's conjecture for Hankel products is probably false for the Hardy and Bergman and Fock spaces in [1,15]. Ma-Yan-Zheng-Zhu in [14] discovered that the conjecture for Hankel product problem was false for the Fock space. Although the conjecture for Haplitz product on Hardy and Bergman is still open, it is completely solved on the Fock spaces. In [22], Stroethoff and Zheng study the problem for the mixed product $H_{\bar{f}}T_{\bar{g}}$ on Bergman space. Also they gave a sufficient condition and a necessary condition, and proposed the conjecture in (P2) for the Bergman space.

In [13], the elementary explicit condition was given for the symbol functions f and g in $F^2_\alpha(\mathbb{C})$ such that $H_{\bar{f}}T_{\bar{g}}$ is bounded on $F^2_\alpha(\mathbb{C})$. The analogue problem to Fock-Sobolev space $F^{2,m}(\mathbb{C}^n)$ was proposed in [16].

The main contribution of this paper is to show that Sarason's conjecture for the Haplitz product problem is true on the Fock-Sobolev space $F^{p,m}(\mathbb{C}^n)$ with $1 \leqslant p < \infty$. The most critical part in our proof is that we make full use of some properties of the Fock-Sobolev space $F^{p,m}(\mathbb{C}^n)$ and find that there exist $t_1, t_2 > 0$ such that $F^{2,m}_{t_1} \subseteq F^{p,m} \subseteq F^{2,m}_{t_2}$. Our main result is stated as follows.

THEOREM 1. (Main) *Given $1 \leqslant p < \infty$. Suppose that f and g are functions in the Fock-Sobolev space $F^{p,m}(\mathbb{C}^n)$. Then the Haplitz product $H_{\bar{f}}T_{\bar{g}}$ is bounded on $F^{p,m}(\mathbb{C}^n)$ if and only if one of the following conditions holds.*

(a) f is constant.

(b) g is identically zero.

(c) f is a linear polynomial, and g is a nonzero constant.

(d) There exist a nonzero complex constant C and a complex linear polynomial q such that

$$f = e^q, \quad g = Ce^{-q}.$$

2. Preliminaries

Throughout the paper, we write $X \lesssim Y$ for non-negative quantities X and Y whenever there is a constant $C > 0$ (independent of the parameters in X and Y) such that $X \leqslant CY$. Similarly, we write $X \approx Y$ if $X \lesssim Y$ and $Y \lesssim X$.

For one fixed non-negative integer m, in the rest of the paper we simply write

$$K_m(z,w) = K_z(w) = K(z,w), \quad k_m(z,w) = k_z(w) = k(z,w),$$

where $k_m(z,w) = \frac{K_m(z,w)}{\|K_m(z,w)\|}$.

If T is a linear operator (not necessarily bounded) on $F^{2,m}$ whose domain contains the set of all finite linear combinations of reproducing kernel functions in $F^{2,m}$, then the Berezin transform of T is defined by

$$\widetilde{T}(z) = \langle Tk_z, k_z \rangle, \quad z \in \mathbb{C}^n.$$

If f is a Lebesgue measurable function on $\mathbb{C}$ satisfying

$$\int_{\mathbb{C}^n} |f(w)||w|^m |K_m(z,w)| e^{-|w|^2} dv(w) < \infty,$$

then the Berezin transform of f is defined by

$$\widetilde{f}(z) = \langle f k_z, k_z \rangle_m, \quad z \in \mathbb{C}^n,$$

or explicitly,

$$\widetilde{f}(z) = \omega_{2,m,n} \int_{\mathbb{C}} f(w) |k_z(w)|^2 |w|^{2m} e^{-|w|^2} dv(w).$$

Actually, $\widetilde{f} = f$ if f is an entire function.

It is shown in [6] that $f \in F^{p,m}(\mathbb{C}^n)$ if and only if the function $z^\alpha f(z)$ is in F^p for all multi-indices α with $|\alpha| = m$, then the norm in $F^{p,m}$ is defined by

$$\|f\|_{p,m}^p = \omega_{p,m,n} \int_{\mathbb{C}^n} ||z|^m |f(z)| e^{-\frac{|z|^2}{2}}|^p dv(z),$$

where

$$\omega_{p,m,n} = \left(\frac{p}{2}\right)^{(\frac{pm}{2})+n} \frac{(n-1)!}{\pi^n \Gamma((\frac{pm}{2})+n)}$$

is a normalizing constant so that the constant function 1 has norm 1 in $F^{p,m}$. Specially, $F^{2,m}$ is a Hilbert space with the inner product

$$\langle f, g \rangle_m = w_{2,m,n} \int_{\mathbb{C}^n} f(z)\overline{g(z)} |z|^{2m} e^{-|z|^2} dv(z), \quad f, g \in F^{2,m}.$$

In what follows, we give some lemmas which will be used to prove our main result.

LEMMA 1. [6] *Given* $f \in F^{2,m}$, $z, w \in \mathbb{C}^n$. *Then*

$$|f(z)| \lesssim \|f\|_{2,m} \frac{e^{\frac{1}{2}|z|^2}}{(1+|z|)^m}$$

and

$$|K_m(z,w)| \lesssim \frac{e^{\frac{1}{2}|z|^2 + \frac{1}{2}|w|^2 - \frac{1}{8}|z-w|^2}}{(1+|z||w|)^m}.$$

More specifically, there is a positive constant ε *such that*

$$\begin{aligned}&|K_m(z,w)|e^{-\frac{1}{2}(|z|^2+|w|^2)}(1+|z|^2)^{\frac{m}{2}}(1+|w|^2)^{\frac{m}{2}}\\ \gtrsim&|K_m(z,z)|e^{-|z|^2}(1+|z|^2)^m\\ \gtrsim&1\end{aligned}$$

whenever $|z-w|<\varepsilon$.

The above lemma implies that each point evaluation is a bounded linear functional on $F^{2,m}$.

REMARK 1. Lemma 1 shows that if a nonvanishing function f is in $F^{p,m}(\mathbb{C}^n)$ for $1\leqslant p<\infty$, then $f=e^q$, where q is a complex polynomial with $deg(q)\leqslant 2$.

The following result derives from [6], which is a basic auxiliary.

LEMMA 2. [6] *Given* $p>0$, $b>0$, *and* $a>0$. *Then there exists a constant* $C=C(a,b)>0$ *such that*

$$\int_{\mathbb{C}^n}|f(w)|^p e^{-b|w|^2}\leqslant C\int_{\mathbb{C}^n}|f(w)|^p|w|^a e^{-b|w|^2}dv(w)$$

for all entire functions f *on* $\mathbb{C}^n$.

In what follows, we give a relationship between the function in $F^{2,m}(\mathbb{C}^n)$ and its Berezin transform.

LEMMA 3. *Suppose* $f\in F^{2,m}$. *Then*

$$|f(z)|^2\lesssim\widetilde{|f|^2}(z)$$

for all $z\in\mathbb{C}^n$.

Proof. It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}|f(z)|^2\lesssim&\int_{\mathbb{C}^n}|f(w)|^2|w|^{2m}|k_m(z,w)|^2e^{-|w|^2}d\nu(w)\int_{\mathbb{C}^n}|w|^{2m}|k_m(z,w)|^2e^{-|w|^2}d\nu(w)\\ =&\widetilde{|f|^2}(z).\end{aligned}$$

It ends the proof. □

For Sarason's Toeplitz product problem on Fock-Sobolev space $F^{2,m}(\mathbb{C}^n)$, Chen-Wang in [7] gave a complete solution as the following.

LEMMA 4. [7] *Suppose that* f *and* g *are two nonzero functions in the Fock-Sobolev space* $F^{2,m}(\mathbb{C}^n)$. *Then the following conditions are equivalent:*

(i) The Toeplitz product $T_fT_{\overline{g}}$ *is bounded on* $F^{2,m}(\mathbb{C}^n)$.

(ii) *There exists a complex linear polynomial* $q(z)$ *on* $\mathbb{C}^n$ *such that* $f = e^q$ *and* $g = Ce^{-q}$*, where* C *is a nonzero complex constant.*

(iii) *The product* $\widetilde{|f|^2}\widetilde{|g|^2}$ *is a bounded function on the complex space* $\mathbb{C}^n$*.*

The next lemma is a key ingredient in our proof of the main theorem.

LEMMA 5. *Suppose that* $f \in F^{2,m}(\mathbb{C}^n)$*. Then*

$$\|H_{\bar{f}}k_z\|^2 = \widetilde{|f|^2}(z) - |f(z)|^2, \quad z \in \mathbb{C}^n.$$

Proof. For any $f \in F^{2,m}(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$, we have $H_{\bar{f}}k_z = (I-P)(\bar{f}k_z) = (\bar{f} - \overline{f(z)})k_z$. Then

$$\begin{aligned}\|H_{\bar{f}}k_z\|^2 =& \langle |f - \overline{f(z)}|^2 k_z, k_z\rangle \\ =& \langle |f|^2 k_z, k_z\rangle + \langle |f(z)|^2 k_z, k_z\rangle \\ & - \langle f(z)\bar{f}k_z, k_z\rangle - \langle \overline{f(z)} f k_z, k_z\rangle \\ =& \widetilde{|f|^2}(z) - |f(z)|^2.\end{aligned} \tag{1}$$

This ends the proof. □

The identity (1) above will be freely used in the next section without being explicitly mentioned. It directly yields the following result.

COROLLARY 1. *Suppose* f *is a linear polynomial and* g *is a constant function. Then for all* $z \in \mathbb{C}^n$

$$\mathscr{D}(f,g)(z) = C\|H_{\bar{f}}k_z\|^2,$$

where C *is a complex constant.*

Qin-Wang gave the following result for $F^{2,m}(\mathbb{C}^n)$.

LEMMA 6. (Proposition 3.4 in [16]) *Given* $f, g \in F^{2,m}(\mathbb{C}^n)$*. Then* $\mathscr{D}(f,g)$ *is bounded on* $\mathbb{C}^n$ *if and only if one of the following conditions holds.*

(a) f *is constant.*

(b) g *is identically zero.*

(c) f *is a linear polynomial, and* g *is a nonzero constant.*

(d) *There are a nonzero constant* C *and a constant* B*, and there is a linear polynomial* q *such that*

$$f = Ce^q + B, \quad g = e^{-q}.$$

We also need the following additional lemma to study the compactness of the Haplitz products.

LEMMA 7. [16] *Given* $f,g \in F^{2,m}(\mathbb{C}^n)$. *Then the following conclusions are equivalent.*

(a) $H_{\bar{f}}T_{\bar{g}}$ *is compact.*

(b) $H_{\bar{f}}T_{\bar{g}} = 0$.

(c) f *is constant or* $g = 0$.

3. Bounded Haplitz products

In this section we will show that Sarason's conjecture is true for the Haplitz products on Fock-Sobolev space $F^{p,m}(\mathbb{C}^n)$ when $1 \leqslant p < \infty$.

Firstly, we introduce some corresponding basic knowledge which will be used in this section. For any positive parameter t, the weighted Gaussian measure is $d\lambda_t(z) = e^{-t|z|^2}dv(z)$, and the reproducing kernel for F_t^2 is $K_t(z,w) = e^{tz\cdot\overline{w}}$. Let $L_t^{p,m}$ denote the space of Lebesgue measurable functions f on $\mathbb{C}^n$ such that the function $|z|^m f(z)$ is in $L^p(\mathbb{C}^n, e^{-\frac{tp}{2}|z|^2}dv(z))$. And the corresponding Fock-Sobolev spaces $F_t^{p,m}$ consists of all entire functions in $L_t^{p,m}$. The orthogonal projection $P_m : L_t^{p,m} \to F_t^{p,m}$ is given by

$$P_m f(w) = \omega_{2,n,m,t} \int_{\mathbb{C}^n} f(z) K_{m,t}(w,z) e^{-t|z|^2} |z|^{2m} dv(z),$$

where $K_{m,t}(z,w) = K_m(tz,w)$ is the reproducing kernel of $F_t^{2,m}$. Let $k_{t,z}$ denote the normalization of the kernel $K_{t,z}$. Then for any $z \in \mathbb{C}^n, t > 0$, we define

$$\mathscr{D}_t(f,g)(z) = |g(z)|^2[\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2],$$

where $\widetilde{f}(z) = \langle f k_{t,z}, k_{t,z} \rangle_m$.

The following lemma describes that $\|H_{\bar{f}}k_z\|$ is bounded when f is a linear polynomial.

LEMMA 8. *Given* $t > 0$, $\gamma \neq 0$ *and* $f = z^\gamma$. *If* $|\gamma| = 1$, *then* $\|H_{\bar{f}}k_z\|_{2,m,t}$ *is bounded.*

Proof. Note that $\omega_{2,m,n,t} \int_{\mathbb{C}^n} |w|^{2m} e^{-t|w|^2} dv(w) = 1$, and a simple calculation gives that $\omega_{2,m,n,t} = \frac{t^{m+n}}{\pi^n \Gamma(m+n)}$. By the definition of $H_{\bar{f}}k_z$ and the properties of reproducing kernel k_z, we have that

$$\begin{aligned}
&\|H_{\bar{f}}k_z\|^2_{2,m,n,t} \\
=&\omega_{n,m,t} \int_{\mathbb{C}^n} |f(w) - f(z)|^2 |k_z(w)|^2 |w|^{2m} e^{-t|w|^2} dv(w) \\
=&\frac{t^{m+n}}{K_z(z)} \Big(\sum_{k=0}^{\infty} \sum_{k=|\alpha|} \sum_{l=0}^{\infty} \sum_{l=|\beta|}^{\infty} \frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!} \frac{(n+l-1)!}{\beta!(n+l+m-1)!\pi^n} t^{|\alpha|} z^\alpha t^{|\beta|} \overline{z}^\beta \\
&\times \int_{\mathbb{C}^n} |w_1 - z_1|^2 \overline{w}^\alpha w^\beta |w|^{2m} e^{-t|w|^2} dv(w)
\end{aligned}$$

$$
\begin{aligned}
=&\frac{t^{m+n}}{K_z(z)}\Big(\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\sum_{l=0}^{\infty}\sum_{l=|\beta|}^{\infty}\frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!}\frac{(n+l-1)!}{\beta!(n+l+m-1)!\pi^n}t^{|\alpha|}z^{\alpha}t^{|\beta|}\overline{z}^{\beta}\\
&\times 2n\int_0^{\infty}r^{2n-1}dr\int_S|r\xi^{\gamma}-z_1|^2r^{2m+k+1}e^{-tr^2}\overline{\xi}^{\alpha}\xi^{\beta}d\sigma(\xi)\Big)\\
=&\frac{2nt^{m+n}}{K_z(z)}\Big(\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\sum_{l=0}^{\infty}\sum_{l=|\beta|}^{\infty}\frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!}\frac{(n+l-1)!}{\beta!(n+l+m-1)!\pi^n}t^{|\alpha|}z^{\alpha}t^{|\beta|}\overline{z}^{\beta}\\
&\times\int_0^{\infty}r^{2n+k+l+2m-1}e^{-tr^2}dr\int_S\big(r^2|\xi^{\gamma}|^2-\overline{z_1}r\xi^{\gamma}-z_1r\overline{\xi}^{\gamma}+|z_1|^2\big)\overline{\xi}^{\alpha}\xi^{\beta}d\sigma(\xi)\Big)\\
=&\frac{t^{m+n}}{K_z(z)}\Big(\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!}\frac{(n+k+m)(\alpha_1+1)}{(n+k)}t^{2k}|z^{\alpha}|^2\\
&-\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!}\frac{\alpha_1(n+k+m-1)}{(n+k-1)}t^{2k}|z^{\alpha}|^2\\
&-\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!}t^{2k}|z^{\alpha}|^2|z_1|^2\\
&+\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!}t^{2k}|z^{\alpha}|^2|z_1|^2\Big)\\
=&\frac{t^{m+n}}{K_z(z)}\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!}t^{2k}|z^{\alpha}|^2\\
&\times\Big(\frac{(n+k+m)(\alpha_1+1)}{(n+k)}-\frac{\alpha_1(n+k+m-1)}{(n+k-1)}\Big)\\
=&\frac{t^{m+n}}{K_z(z)}\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!}t^{2k}|z^{\alpha}|^2\\
&\times\Big((\alpha_1+1)(1+\frac{m}{n+k})-\alpha_1(1+\frac{m}{n+k-1})\Big)\\
=&\frac{t^{m+n}}{K_z(z)}\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\frac{(n+m-1)!(n+k-1)!}{(n-1)!\alpha!(n+k+m-1)!}t^{2k}|z^{\alpha}|^2\Big(1+\frac{m(n+k-\alpha_1-1)}{(n+k-1)(n+k)}\Big)\\
=&\frac{t^{m+n}}{K_z(z)}\frac{(n+m-1)!}{(n-1)!}\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\frac{(n+k-1)!}{\alpha!(n+k+m-1)!}t^{2k}|z^{\alpha}|^2\Big(1+\frac{m(n+k-\alpha_1-1)}{(n+k-1)(n+k)}\Big),
\end{aligned}
\tag{2}
$$

where the above equation yields from the fact that $\Gamma(s)=\int_0^{\infty}x^{s-1}e^{-x}dx$ and $\Gamma(n+1)=n!$. Since

$$1\leqslant 1+\frac{m(n+k-\alpha_1-1)}{(n+k-1)(n+k)}\leqslant 1+\frac{m}{n+k}\leqslant 1+m,$$

then (2) shows that

$$\|H_{\overline{f}}k_z\|^2\leqslant\frac{t^{m+n}}{K_z(z)}\frac{(n+m-1)!}{(n-1)!}\sum_{k=0}^{\infty}\sum_{k=|\alpha|}\frac{(n+k-1)!}{\alpha!(n+k+m-1)!}t^{2k}|z^{\alpha}|^2(1+m).\tag{3}$$

Note that

$$K_z(z) = \frac{(n+m-1)!}{(n-1)!} \sum_{k=0}^{\infty} \sum_{k=|\alpha|} \frac{(n+k-1)!}{\alpha!(n+k+m-1)!} t^{2k} |z^{\alpha}|^2.$$

Thus (3) shows that

$$\|H_{\overline{f}} k_z\|_{2,m,t}^2 \leqslant t^{m+n}(1+m),$$

which implies that $\|H_{\overline{f}} k_z\|_{2,m,t}$ is bounded. □

Similar to the proof of Lemma 3.4 in [16], we get the following conclusion.

PROPERTY 1. *Given $t > 0$ and $f, g \in F_t^{2,m}(\mathbb{C}^n)$. Then $\mathscr{D}_t(f,g)$ is bounded on $\mathbb{C}^n$ if and only if one of the following conditions holds:*

(a) f is constant.

(b) g is identically zero.

(c) f is a linear polynomial, and g is a nonzero constant.

(d)

$$f = e^q, \quad g = Ce^{-q},$$

where C is a nonzero complex constant and q is a complex linear polynomial.

Proof. It is obvious that $\mathscr{D}_t(f,g)(z) = 0$ if (a) or (b) holds. From Lemma 5 and Corollary 1 and Lemma 8, we have (c) implies that $\mathscr{D}_t(f,g)$ is bounded on $\mathbb{C}^n$. Assume $f(z) = e^{q(z)}$, $g(z) = e^{-q(z)}$. Given any $t > 0$. By Lemma 1, for any $z, w \in \mathbb{C}^n$ it is easily seen that

$$|K_{m,t}(z,w)| \lesssim \frac{e^{\frac{t}{2}|z|^2} + \frac{t}{2}|w|^2 - \frac{1}{8}|z-w|^2}{(1+|z||w|)^m}, \quad K_{m,t}(z,z) = \frac{e^{t|z|^2}}{1+|z|^{2m}}. \tag{4}$$

Then by a simple calculation, we have that

$$\begin{aligned}\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2 =& \omega_{2,n,m,t} \int_{\mathbb{C}^n} |e^{q(w)} - e^{q(z)}|^2 |k_z(w)|^2 |w|^{2m} e^{-t|w|^2} dv(w) \\ \lesssim& |e^{q(z)}|^2 \int_{\mathbb{C}^n} |e^{-q(w)} - 1|^2 e^{-\frac{1}{4}|w|^2} dv(w),\end{aligned}$$

where the last equality yields from Lemma 1 and (4). It is easily got that

$$\int_{\mathbb{C}^n} |e^{-q(w)} - 1|^2 e^{-\frac{1}{4}|w|^2} dv(w) < \infty,$$

then

$$\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2 \lesssim |e^{q(z)}|^2.$$

By the same way, we get that

$$\begin{aligned}\widetilde{|g|^2}(z) =&\omega_{2,n,m,t}\int_{\mathbb{C}^n}|e^{-q(w)}|^2|k_z(w)|^2|w|^{2m}e^{-t|w|^2}dv(w)\\ \lesssim&|e^{-q(z)}|^2\int_{\mathbb{C}^n}|e^{q(w)}-1|^2e^{-\frac{1}{4}|w|^2}dv(w)\\ \lesssim&|e^{-q(z)}|^2.\end{aligned}$$

Then

$$\widetilde{|g|^2}(z)(\widetilde{|f|^2}(z)-|\widetilde{f}(z)|^2)\lesssim|e^{q(z)}|^2|e^{-q(z)}|^2<\infty.$$

Moreover, Lemma 3 implies that $|g(z)|^2\lesssim\widetilde{|g|^2}(z)$. Then

$$\begin{aligned}\mathscr{D}_t(f,g)(z) =&|g(z)|^2(\widetilde{|f|^2}(z)-|\widetilde{f}(z)|^2)\\ \lesssim&\widetilde{|g|^2}(z)(\widetilde{|f|^2}(z)-|\widetilde{f}(z)|^2)\\ <&\infty.\end{aligned}$$

Thus $\mathscr{D}_t(f,g)$ is bounded on $\mathbb{C}^n$ if (d) holds.

The proof of the necessity is completely the same as Lemma 6 which was proved in [16]. It ends the proof. □

From Lemma 2, for the pointwise estimates of the functions in $F_t^{2,m}$, we get the following result, as one would expect.

PROPERTY 2. *Given $t>0$ and $0<p<\infty$. If f is an entire function, then*

$$|f(z)|\lesssim\|f\|_{p,m,t}\frac{e^{\frac{t|z|^2}{2}}}{(1+|z|)^m}$$

holds for any $z\in\mathbb{C}^n$.

Proof. From the subharmonicity of the function $w\mapsto|f(z+w)e^{-aw\cdot\overline{z}}|^p$, we have that

$$\begin{aligned}|f(z)|^p\lesssim&\int_{|w|<r}|f(z+w)e^{-tw\cdot\overline{z}}|^pe^{-\frac{pt}{2}|w|^2}dv(w)\\ =&e^{\frac{pt}{2}|z|^2}\int_{|w-z|<r}|f(w)|^pe^{-\frac{pt}{2}|w|^2}dv(w).\end{aligned}$$

As $|z|<r+|w|$ for $|w-z|<r$, we have

$$\begin{aligned}|f(z)|^p(1+|z|)^{pm}e^{-\frac{pt}{2}|z|^2}\leqslant&C\int_{|w-z|<r}|f(w)|^p(1+r+|w|)^{pm}e^{-\frac{pt}{2}|w|^2}dv(w)\\ \lesssim&\int_{|w-z|<r}|f(w)|^p(1+|w|)^{pm}e^{-\frac{pt}{2}|w|^2}dv(w)\\ \lesssim&\int_{\mathbb{C}^n}|f(w)|^p(1+|w|)^{pm}e^{-\frac{pt}{2}|w|^2}dv(w),\end{aligned}\tag{5}$$

where the suppressed constant depends only m,r. Since $(1+|w|)^{pm} \leqslant C(1+|w|^{pm})$, then it yields from (5) and Lemma 2 that

$$\begin{aligned} |f(z)|^p(1+|z|)^{pm}e^{-\frac{pt}{2}|z|^2} \lesssim & \int_{\mathbb{C}^n} |f(w)|^p e^{-\frac{pt}{2}|w|^2} dv(w) \\ & + \int_{\mathbb{C}^n} |f(w)|^p |w|^{pm} e^{-\frac{pt}{2}|w|^2} dv(w) \\ \leqslant & C_1 \int_{\mathbb{C}^n} |f(w)|^p |w|^{pm} e^{-\frac{pt}{2}|w|^2} dv(w) \\ & + C_2 \int_{\mathbb{C}^n} |f(w)|^p |w|^{pm} e^{-\frac{pt}{2}|w|^2} dv(w) \\ \leqslant & C_3 \int_{\mathbb{C}^n} |f(w)|^p |w|^{pm} e^{-\frac{pt}{2}|w|^2} dv(w), \end{aligned}$$

where $C_3 = \max\{C_1, C_2\}$. This implies that

$$|f(z)|^p(1+|z|)^{pm}e^{-\frac{pt}{2}|z|^2} \lesssim \|f\|_{p,m,t}.$$

Then the desired result follows that

$$|f(z)| \lesssim \|f\|_{p,m,t} \frac{e^{\frac{t|z|^2}{2}}}{(1+|z|)^m}.$$

It ends the proof. $\square$

Moreover, the following lemma gives a relationship between that the norm of functions in $L^{p,m}$ and the norm of functions in $L_t^{2,m}$.

PROPERTY 3. *Given $1 \leqslant p < \infty$. Then there exist $0 < t_1 < 1$ and $t_2 > 1$ such that*

$$\|f\|_{2,m,t_2} \lesssim \|f\|_{p,m} \lesssim \|f\|_{2,m,t_1}, \quad f \in F^{p,m}.$$

Proof. For any $t > 0$, Proposition 2 implies that the following pointwise estimate

$$|f(z)| \lesssim \|f\|_{2,m,t} \frac{e^{\frac{t|z|^2}{2}}}{(1+|z|)^m}$$

for any entire function f. Then, it follows that

$$\begin{aligned} & \omega_{n,p,m} \int_{\mathbb{C}^n} ||z|^m |f(z)| e^{-\frac{|z|^2}{2}}|^p dv(z) \\ \lesssim & \omega_{n,p,m} \int_{\mathbb{C}^n} \|f\|_{2,m,t}^p \frac{e^{\frac{pt}{2}|z|^2}}{(1+|z|)^{mp}} |z|^{mp} e^{-\frac{p|z|^2}{2}} dv(z) \\ \lesssim & \omega_{n,p,m} \|f\|_{2,m,t}^p \int_{\mathbb{C}^n} (\frac{|z|}{1+|z|})^{mp} e^{\frac{(t-1)p|z|^2}{2}} dv(z) \\ \lesssim & \omega_{n,p,m} \|f\|_{2,m,t}^p \int_{\mathbb{C}^n} e^{\frac{(t-1)p|z|^2}{2}} dv(z), \end{aligned} \tag{6}$$

where the second inequality yields from Lemma 2. Take $0<t_1=t<1$, then

$$\int_{\mathbb{C}^n} e^{\frac{(t_1-1)p|z|^2}{2}}dv(z)<\infty,$$

which coupled with (6) imply that

$$\begin{aligned}\|f\|_{p,m}^p =&\omega_{n,p,m}\int_{\mathbb{C}^n}||z|^m|f(z)|e^{-\frac{|z|^2}{2}}|^p dv(z)\\ \lesssim&\|f\|_{2,m,t}^p.\end{aligned}$$

Thus it yields that there exist $0<t_1<1$ and a positive constant $C_1>0$ such that

$$\|f\|_{p,m}\leqslant C_1\|f\|_{2,m,t_1}$$

for all entire functions f.

Moreover, using Proposition 2 again, we also have that

$$\begin{aligned}&\omega_{n,2,m}\int_{\mathbb{C}^n}||z|^m f(z)|^2 e^{-t_2|z|^2}dv(z)\\ \lesssim&\omega_{n,2,m}\int_{\mathbb{C}^n}|z|^{2m}\|f\|_{p,m}^2\frac{e^{|z|^2}}{(1+|z|)^{2m}}e^{-t_2|z|^2}dv(z)\\ \lesssim&\|f\|_{p,m}^2\int_{\mathbb{C}^n}e^{(1-t_2)|z|^2}dv(z).\end{aligned}$$

Taking $t_2>1$, then $\int_{\mathbb{C}^n}e^{(1-t_2)|z|^2}dv(z)<\infty$. Thus

$$\|f\|_{2,m,t_2}\leqslant C_2\|f\|_{p,m}.$$

This ends the proof. □

The following result follows directly from Proposition 3.

COROLLARY 2. *If* $1\leqslant p<\infty$, *then there exist* $0<t_1<1$ *and* $t_2>1$ *such that*

$$F_{t_1}^{2,m}\subseteq F^{p,m}\subseteq F_{t_2}^{2,m}.$$

Using Lemma 4, we have the following result for Sarason's Toeplitz product problem on Fock-Sobolev space $F^{p,m}(\mathbb{C}^n)$.

PROPERTY 4. *Suppose that* f *and* g *are two nonzero functions in the Fock-Sobolev space* $F^{p,m}(\mathbb{C}^n)$. *Then the Toeplitz product* $T_fT_{\overline{g}}$ *is bounded on* $F^{p,m}(\mathbb{C}^n)$ *if and only if there exists a complex linear polynomial* q *on* $\mathbb{C}^n$ *and a nonzero complex scalar* C *such that* $f=e^q$ *and* $g=Ce^{-q}$.

Proof. If $T_f T_{\overline{g}}$ is bounded on $F^{p,m}(\mathbb{C}^n)$, then the Berezin transform $\widetilde{T_f T_{\overline{g}}}(z)$ is also bounded on $\mathbb{C}^n$. It is easily seen that $T_f^* = T_{\overline{f}}$. More specially, for any given function f which belongs to the Fock-Sobolev space $F^{2,m}(\mathbb{C}^n)$, we have that

$$\begin{aligned} T_{\overline{f}} K_z(w) =& \langle T_{\overline{f}} K_z, K_w \rangle_m \\ =& \langle K_z, T_f K_w \rangle_m \\ =& \overline{\langle f K_w, K_z \rangle_m} \\ =& \overline{f(z)} K_z(w). \end{aligned} \tag{7}$$

Then it follows that

$$\begin{aligned} \widetilde{T_f T_{\overline{g}}}(z) =& \langle T_f T_{\overline{g}} k_z, k_z \rangle_m \\ =& \overline{g(z)} \langle f k_z, k_z \rangle_m \\ =& f(z) \overline{g(z)}. \end{aligned}$$

Since each k_z is a unit vector, it follows from the Cauchy-Schwartz inequality that

$$|f(z)\overline{g(z)}| \leqslant \|T_f T_{\overline{g}}\|_\infty$$

for all $z \in \mathbb{C}^n$. Then Liouville's theorem shows that there exists a constant C such that $fg = C$. Since neither f nor g is identically zero, we have $C \neq 0$. Then, both f and g are non-vanishing. By Remark 1, there exists a complex polynomial $q(z)$ on $\mathbb{C}^n$ with $deg(q) \leqslant 2$ such that $f = e^q$ and $g = Ce^{-q}$. We claim that $deg(q) \leqslant 1$. Indeed, if we assume $deg(q) = 2$, then we can write $q(z) = bz^2 + az + c$ for convenience, where $b \neq 0$. In the following we hope that we can reach a contradiction.

By the boundedness of $T_f T_{\overline{g}}$ on $F^{p,m}(\mathbb{C}^n)$, the function

$$T(z,w) = \frac{\langle T_f T_{\overline{g}} K_z, K_w \rangle_m}{\sqrt{K_z(z)}\sqrt{K_w(w)}}$$

is also bounded on $\mathbb{C}^n \times \mathbb{C}^n$. Then we will show that this is impossible when $deg(q) = 2$. Combining the fact that $T_f^* = T_{\overline{f}}$ with (7), we have that

$$\langle T_f T_{\overline{g}} K_z, K_w \rangle_m = f(w)\overline{g(z)} \langle K_z, K_w \rangle_m.$$

Thus,

$$|T(z,w)| = |C||e^{q(w) - \overline{q(z)}}| \frac{K_z(w)}{\sqrt{K_z(z)}\sqrt{K_w(w)}}.$$

Therefore, from the above equality and the properties of reproducing kernel, there exists a constant $\varepsilon_0 > 0$ independent of z, w such that for any $z, w \in \mathbb{C}^n$ satisfying $|w - z| < \varepsilon_0$ we have

$$|T(z,w)| \backsimeq |e^{q(w) - \overline{q(z)}}|.$$

Moreover, in view of homogeneous property of $b \neq 0$, we let $q_2(z) = bz^2$ and $q_1(z) = az + c$. Fix two points ζ and η in the unit sphere of $\mathbb{C}^n$, then $b\zeta \cdot \eta \neq 0$. Let $z = r\zeta$ and $w = r\zeta + \frac{\varepsilon_0}{2}\eta$, where r is any real positive number. It follows that

$$\begin{aligned} q_2(w) - q_2(z) =& b\left(z + \frac{\varepsilon_0}{2}\eta\right)^2 - bz^2 \\ =& br\varepsilon_0\zeta \cdot \eta + \frac{\varepsilon_0^2\eta^2}{4}. \end{aligned}$$

Then we have that there exists a positive constant L dependent ε_0 but not r such that

$$|e^{q(w) - \overline{q(z)}}| = L \cdot \exp(br\varepsilon_0\zeta \cdot \eta).$$

The fact that $b\zeta \cdot \eta \neq 0$ implies that $T(z,w)$ cannot be a bounded function on $\mathbb{C}^n \times \mathbb{C}^n$ as $r \to \infty$. Then the contradiction demostrates that $b = 0$ and $deg(q) \leqslant 1$. Thus the polynomial q must be linear.

Moreover, by a simple calculation, it is easily yielded that $T_f T_{\overline{g}}$ is bounded on $F^{p,m}(\mathbb{C}^n)$ if $f = e^q$ and $g = Ce^{-q}$. $\square$

COROLLARY 3. *Suppose $f = e^q$ and $g = e^{-q}$ with q is a complex linear polynomial. Then the Haplitz product $H_{\overline{f}} T_{\overline{g}}$ is bounded on $F^{p,m}(\mathbb{C}^n)$.*

Proof. It is easily seen that $f, g \in F^{p,m}$. We begin with the identity

$$H^*_{\overline{f}} H_{\overline{f}} = T_{f\overline{f}} - T_f T_{\overline{f}},$$

which is well-known (see [26] for example) and can be verified easily. An application of this identity gives

$$\begin{aligned} (H_{\overline{f}} T_{\overline{g}})^* H_{\overline{f}} T_{\overline{g}} =& T_g H^*_{\overline{f}} H_{\overline{f}} T_{\overline{g}} \\ =& T_g (T_{f\overline{f}} - T_f T_{\overline{f}}) T_{\overline{g}} \\ =& T_g T_{\overline{f}} T_f T_{\overline{g}} - T_{fg} T_{\overline{fg}} \\ =& (T_f T_{\overline{g}})^* T_f T_{\overline{g}} - T_{fg} T_{\overline{fg}}. \end{aligned} \tag{8}$$

Since $fg = 1$, then it follows that T_{fg} and $T_{\overline{fg}}$ are bounded as they are identity operators on $F^{p,m}(\mathbb{C}^n)$. Moreover, Proposition 4 implies that $T_f T_{\overline{g}}$ is also bounded on $F^{p,m}(\mathbb{C}^n)$. Hence, together with (8) we conclude that $H_{\overline{f}} T_{\overline{g}}$ is bounded on $F^{p,m}(\mathbb{C}^n)$. This ends the proof. $\square$

Next, let us prove our main theorem, which will give the result of Haplitz product on $F^{p,m}(\mathbb{C}^n)$ for $1 \leqslant p < \infty$.

THEOREM 2. *Given $1 \leqslant p < \infty$ and $f, g \in F^{p,m}(\mathbb{C}^n)$. Then the Haplitz product $H_{\overline{f}} T_{\overline{g}}$ is bounded on $F^{p,m}(\mathbb{C}^n)$ if and only if one of the following conditions holds.*

(a) *f is constant.*

(b) g *is identically zero.*

(c) f *is a linear polynomial, and* g *is a nonzero constant.*

(d) There exist a nonzero complex constant C *and a complex linear polynomial* q *such that*

$$f = e^q, g = Ce^{-q}.$$

Proof. That condition (a) or (b) implies $H_{\bar{f}}T_{\bar{g}}$ is bounded on $F^{p,m}$ follows from the fact that $H_{\bar{f}} = 0$ for constant functions f and $H_{\bar{f}}T_{\bar{g}} = 0$ for $g = 0$. From Corollary 8.6 in [26], we have that (c) implies that $H_{\bar{f}}T_{\bar{g}}$ is bounded. Moreover, Corollary 3 shows that (d) implies that $H_{\bar{f}}T_{\bar{g}}$ is bounded.

Finally, it is sufficient for us to prove the necessity. We divide it into two cases. Assume $H_{\bar{f}}T_{\bar{g}}$ is bounded on $F^{p,m}(\mathbb{C}^n)$ for $1 \leqslant p \leqslant 2$. As $K_z \in F^{p,m}(\mathbb{C}^n)$, thus for all $z \in \mathbb{C}^n$ we have $k_z \in F^{p,m}(\mathbb{C}^n)$, then

$$\|H_{\bar{f}}T_{\bar{g}}k_z\|_{p,m} < \infty. \tag{9}$$

It follows from the fact that $F^{p,m}(\mathbb{C}^n) \subseteq F^{2,m}(\mathbb{C}^n)$ for $1 \leqslant p \leqslant 2$ that

$$\|H_{\bar{f}}T_{\bar{g}}k_z\|_{2,m} \leqslant \|H_{\bar{f}}T_{\bar{g}}k_z\|_{p,m}. \tag{10}$$

Therefore, $\|H_{\bar{f}}T_{\bar{g}}k_z\|_{2,m} < \infty$ directly follows from (9) and (10). On the other hand, (7) shows that $T_{\bar{f}}k_z = \overline{f(z)}k_z$ for any $z \in \mathbb{C}^n$. These together with Lemma 5 show that

$$\begin{aligned}
\|H_{\bar{f}}T_{\bar{g}}k_z\|^2 =& \|\overline{g(z)}H_{\bar{f}}k_z\|^2 = |g(z)|^2\|H_{\bar{f}}k_z\|^2 \\
=& |g(z)|^2[\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2] \\
=& \mathscr{D}(f,g)(z) \\
<& \infty.
\end{aligned}$$

Then Lemma 6 implies that (a) or (b) or (c) or (d) holds.

Assume that Haplitz product $H_{\bar{f}}T_{\bar{g}}$ is bounded on $F^{p,m}(\mathbb{C}^n)$ for $2 \leqslant p < \infty$. As $k_z \in F^{p,m}(\mathbb{C}^n)$ for all $z \in \mathbb{C}^n$, then $\|H_{\bar{f}}T_{\bar{g}}k_z\|_{p,m}$ is bounded. Proposition 3 and Corollary 2 show that there exists $t_2 > 0$ such that

$$\|H_{\bar{f}}T_{\bar{g}}k_z\|_{2,m,t_2} \leqslant C_2\|H_{\bar{f}}T_{\bar{g}}k_z\|_{p,m} < \infty,$$

which implies that $\|H_{\bar{f}}T_{\bar{g}}k_z\|_{2,m,t_2} < \infty$. Combining Lemma 5 with (7), we can get that

$$\begin{aligned}
\|H_{\bar{f}}T_{\bar{g}}k_{t_2,z}\|_{2,m,t_2} =& \|\overline{g(z)}H_{\bar{f}}k_{t_2,z}\|^2 = |g(z)|^2\|H_{\bar{f}}k_{t_2,z}\|^2 \\
=& |g(z)|^2[\widetilde{|f|^2}(z) - |\tilde{f}(z)|^2] \\
=& \mathscr{D}_{t_2}(f,g)(z) \\
<& \infty.
\end{aligned}$$

Then $\mathscr{D}_{t_2}(f,g)(z) < \infty$. Thus Proposition 1 implies that (a) or (b) or (c) or (d) holds. Hence the necessity is proved. This completes the proof. □

The following result about the compactness of $H_{\bar{f}}T_{\bar{g}}$ follows from Lemma 7.

COROLLARY 4. *Given $1 \leqslant p < \infty$ and $f,g \in F^{p,m}$. Then the following conditions are equivalent:*

(a) $H_{\bar{f}}T_{\bar{g}}$ is compact.

(b) $H_{\bar{f}}T_{\bar{g}} = 0$.

(c) f is constant or $g = 0$.

Inspired by the result of Theorem 2, we can generalize the result to the corresponding Haplitz product problem on the Fock-Sobolv spaces $F_t^{p,m}(\mathbb{C}^n)$ for any $1 \leqslant p < \infty$ and any $t > 0$. It is obvious that Theorem 2 is the special case that $t = 1$. Moreover, similar to the above analysis process for Theorem 2, we can also give the explicit characterization for the Haplitz product problem on $F_t^{p,m}(\mathbb{C}^n)$ for $1 \leqslant p < \infty$, which is stated as follows.

COROLLARY 5. *Given $t > 0$ and $1 \leqslant p < \infty$, $f,g \in F_t^{p,m}(\mathbb{C}^n)$. Then the following conditions are equivalent:*

(i) The Haplitz product $H_{\bar{f}}T_{\bar{g}}$ is bounded on $F_t^{p,m}(\mathbb{C}^n)$.

(ii) At least one of the following conditions holds:

(a) f is constant.

(b) g is identically zero.

(c) f is a linear polynomial, and g is a nonzero constant.

(d) There exist a nonzero complex constant C and a complex linear polynomial q such that

$$f = e^q, \quad g = Ce^{-q}.$$

4. Further results

If we generalize the above conclusions to a more general situation when its weight becomes a more general weight, we have the following results.

Let dA be the Lebesgue area measure on the complex plane $\mathbb{C}$. Suppose $\varphi \in C^2(\mathbb{C})$ is a real-valued function and there are two positive numbers M_1 and M_2 such that

$$M_1\omega_0 \leqslant dd^c\varphi \leqslant M_2\omega_0,$$

where $\omega_0 = dd^c|z|^2,\ d^c = \frac{\sqrt{-1}}{4}(\bar{\partial} - \partial)$. For $1 \leqslant p < \infty$, the space $L^p(\varphi)$ is the family of all Lebesgue measurable functions f on $\mathbb{C}$ such that

$$\|f\|_{p,\varphi} = \left(\int_{\mathbb{C}^n} |f(z)e^{-\varphi(z)}|^p d\nu(z) \right)^{\frac{1}{p}} < \infty.$$

For $0 < p < \infty$, $L^p(\varphi) = L^p(\mathbb{C}, e^{-p\varphi}dA)$. Moreover, $(L^p(\varphi), \|\cdot\|_{p,\varphi})$ is a Banach space for $1 \leqslant p \leqslant \infty$, and a quasi-Banach space for $0 < p < 1$.

We consider the one-dimensional situation here. Let $H(\mathbb{C})$ be the set of all holomorphic functions on $\mathbb{C}$. The weighted Fock space [12] is defined to be

$$F^p(\varphi) = L^p(\varphi) \cap H(\mathbb{C})$$

with the norm $\|\cdot\|_{(p,\varphi)}$. Notice that $F^p(\varphi)$ is a closed subspace of $L^p(\varphi)$, and $F^p(\varphi)$ is a Banach space for $1 \leqslant p \leqslant \infty$. This type of weighted Fock spaces was studied by many authors, see [10, 11, 20] and the references therein. If $\varphi(z) = \frac{\alpha}{2}|z|^2, \alpha > 0$, the standard Fock space is obtained. And when $\varphi(z) = -m\ln(A + |z|^2) + |z|^2$ with some suitable $A > 0$ and positive integer m, $F^2(\varphi)$ is just the Fock-Sobolev space $F^{2,m}$.

Let K_z be the reproducing kernel of $F^2(\varphi)$, and let k_z be the normalized reproducing kernel, that is $k_z(\cdot) = \frac{K_z(\cdot)}{\sqrt{K(z,z)}}$. From [20], the following conclusions hold on $F^p(\varphi)$:

(i) There exist C and $\theta > 0$ such that

$$|K(z,w)|e^{-\varphi(z)}e^{-\varphi(w)} \leqslant Ce^{-\theta|z-w|}$$

for $z, w \in \mathbb{C}$.

(ii) There exists some $r > 0$ such that

$$|K(z,w)|e^{-\varphi(z)}e^{-\varphi(w)} \simeq 1$$

whenever $w \in B(z,r)$ and $z \in \mathbb{C}$.

(iii) For $1 \leqslant p < \infty$,

$$\|K_z(\cdot)\|_{p,\varphi} \simeq e^{\varphi(z)} \simeq \sqrt{K(z,z)}, \quad z \in \mathbb{C}.$$

The orthogonal projection P from $L^2(\varphi)$ to $F^2(\varphi)$ can be represented as

$$Pf(z) = \int_{\mathbb{C}} f(w)K(z,w)e^{-2\varphi(w)}dA(w).$$

With this expression, P can be extended to a bounded linear operator from $L^p(\varphi))$ to $F^p(\varphi)$ for $1 \leqslant p < \infty$. Moreover, $Pf = f$ for all $f \in F^p(\varphi)$. The set span $\{K_z : z \in \mathbb{C}\}$ is dense in $F^p(\varphi)$ for $1 \leqslant p < \infty$. See [20] for more details.

Similar to Corollary 2, for any $1 \leqslant p < \infty$, it yields that there exist $0 < t_1 < 1$ and $t_2 > 1$ such that $F^2_{t_1}(\varphi) \subseteq F^p(\varphi) \subseteq F^2_{t_2}(\varphi)$. Then the following result holds.

THEOREM 3. *Given* $1 \leqslant p < \infty$ *and* $f, g \in F^p(\varphi)(\mathbb{C})$. *Then the following conditions are equivalent.*

(i) The Haplitz product $H_{\overline{f}}T_{\overline{g}}$ *is bounded on* $F^p(\varphi)(\mathbb{C})$.

(ii) At least one of the following conditions holds:

(a) f *is constant.*

(b) g *is identically zero.*

(c) f *is a linear polynomial, and* g *is a nonzero constant.*

(d) There are constants a, b, c, *and* A *such that*

$$f(z) = e^{az+b} + A, \qquad g(z) = e^{-az+c}.$$

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(Received October 7, 2024)

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