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JACOBSON'S LEMMA FOR THE OPERATORS ADMITTING SAPHAR DECOMPOSITIONS

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Abstract. We extend Jacobson's lemma to the operators of Saphar type and the operators admitting the generalized Saphar decomposition and the generalized Saphar-Riesz decomposition.

1. Introduction and notations

Let $L(X)$ denote the Banach algebra of all bounded linear operators acting on an infinite-dimensional Banach space X. We denote by $\mathbb{N}$ ($\mathbb{N}_0$) the set of all positive (non-negative) integers and by $\mathbb{C}$ the set of all complex numbers. If $K \subset \mathbb{C}$, we denote by $\operatorname{acc} K$ the set of all accumulation points of K.

We say that an operator T is completely reduced by the pair (M,N), denoted by $(M,N) \in \operatorname{Red}(T)$, if there exist two closed, T-invariant subspaces M and N such that $M+N = X$ and $M \cap N = \{0\}$, or $M \oplus N = X$ for short. In this case we write $T = T_M \oplus T_N$ and say that T is the direct sum of T_M and T_N. A closed subspace M of X is said to be complemented if there is a closed subspace N of X such that $X = M \oplus N$.

For $T \in L(X)$ we use $N(T)$ and $R(T)$, respectively, to denote the null-space and the range of T. It is well-known that $T \in L(X)$ is left invertible if and only if T is injective and $R(T)$ is a complemented subspace of X. Meanwhile, $T \in L(X)$ is right invertible if and only if T is onto and $N(T)$ is a complemented subspace of X. We use $G_l(X)$ and $G_r(X)$, respectively, to denote the semigroups of left and right invertible operators on X and $\sigma_l(T)$ and $\sigma_r(T)$ to denote left and right spectrum of $T \in L(X)$. An operator T is invertible if it is left and right invertible and by $\sigma(T)$ we denote the spectrum of T.

Nullity of $T \in L(X)$ is defined by $\alpha(T) = \dim N(T)$ in case of a finite dimensional null-space and by $\alpha(T) = \infty$ when $N(T)$ is infinite dimensional. Similarly, defect of T is defined as $\beta(T) = \dim Y/R(T) = \operatorname{codim} R(T)$ if $Y/R(T)$ is finite dimensional, and $\beta(T) = \infty$ otherwise. An operator $T \in L(X)$ is called upper semi-Fredholm if $\alpha(T) < \infty$ and $R(T)$ is closed, while T is called lower semi-Fredholm if $\beta(T) < \infty$.

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We say that an operator is Fredholm if it is both upper and lower semi-Fredholm and by $\Phi(X)$ we denote the set of all Fredholm operators $T \in L(X)$. If either $\alpha(T)$ or $\beta(T)$ is finite, we define the index of T by $i(T) = \alpha(T) - \beta(T)$. We say that $T \in L(X)$ is a Weyl operator and write $T \in W(X)$, if it is a Fredholm operator with index equal to 0.

An operator $T \in L(X)$ is relatively regular (or g-invertible) if there exists $S \in L(X)$ such that $TST = T$. An operator $T \in L(X)$ is left Fredholm, or $T \in \Phi_l(X)$ for short, if T is relatively regular upper semi-Fredholm. Also, $T \in L(X)$ is right Fredholm, or $T \in \Phi_r(X)$, if T is relatively regular lower semi-Fredholm. An operator $T \in L(X)$ is left (right) Weyl if T is left (right) Fredholm operator with non-positive (non-negative) index. We use $W_l(X)$ ($W_r(X)$) to denote the set of all left (right) Weyl operators. An operator $T \in L(X)$ is left (right) Browder if it is left (right) Fredholm and $0 \notin \operatorname{acc} \sigma_l(T)$ ($0 \notin \operatorname{acc} \sigma_r(T)$). We use $B_l(X)$ ($B_r(X)$) to denote the set of all left (right) Browder operators.

We denote the left Fredholm, the right Fredholm, the Fredholm, the left Weyl, the right Weyl, the Weyl, the left Browder and the right Browder spectrum of $T \in L(X)$ by $\sigma_{\Phi_l}(T)$, $\sigma_{\Phi_r}(T)$, $\sigma_{\Phi}(T)$, $\sigma_{W_l}(T)$, $\sigma_{W_r}(T)$, $\sigma_W(T)$, $\sigma_{B_l}(T)$ and $\sigma_{B_r}(T)$.

We say that $T \in L(X)$ is Saphar if it is relatively regular and $N(T) \subset R(T^n)$ for every $n \in \mathbb{N}$. An operator $T \in L(X)$ is nilpotent if there exists some $n \in \mathbb{N}$ such that $T^n = 0$. We say that $T \in L(X)$ is quasinilpotent if $\lambda I - T$ is invertible for every nonzero $\lambda \in \mathbb{C}$. If $\lambda I - T$ is Fredholm for every nonzero $\lambda \in \mathbb{C}$, then $T \in L(X)$ is a Riesz operator. It is known that if a Riesz operator T commutes with some operator $S \in L(X)$, then ST is also Riesz.

If for $T \in L(X)$ there exists a pair of subspaces (M,N) such that $(M,N) \in \operatorname{Red}(T)$, T_M is Saphar and T_N is nilpotent, we say that T is of Saphar type [19]. If T_N is quasinilpotent or Riesz, we will say that T admits a generalized Saphar decomposition (GSD) [3] or a generalized Saphar-Riesz decomposition (GSRD) [17], respectively. In that case we write $T \in \operatorname{GSD}(M,N)$ or $T \in \operatorname{GSRD}(M,N)$, for short. We denote the Saphar type spectrum, the generalized Saphar spectrum and the generalized Saphar-Riesz spectrum of T by $\sigma_{St}(T)$, $\sigma_{gS}(T)$ and $\sigma_{gSR}(T)$, respectively.

Jacobson's lemma states that for operators $A,B \in L(X)$, $I - AB$ is invertible if and only if $I - BA$ is invertible. Equivalently, we can write this in terms of operator spectra:

$$\sigma(AB) \cup \{0\} = \sigma(BA) \cup \{0\}.$$

This lemma has been extended for other parts of the spectrum, some of which we gather in the following theorems.

THEOREM 1. *Let $A,B \in L(X)$. Then for each $H \in \{G_l, G_r, \Phi_l, \Phi_r, \Phi, W_l, W_r, W, B_l, B_r\}$ we have that*

$$\sigma_H(AB) \cup \{0\} = \sigma_H(BA) \cup \{0\}.$$

THEOREM 2. *Let $A,B \in L(X)$. Then $I - AB$ is Saphar if and only if $I - BA$ is Saphar.*

These and other results can be found, for example, in [1] and [18].

An operator $T \in L(X)$ is Drazin invertible [4] if there exists $S \in L(X)$ that satisfies

$$ST = TS, \quad STS = S \quad \text{and} \quad T - TST \quad \text{is nilpotent.}$$

Koliha [11] generalized this concept by replacing a request for a nilpotent operator with a quasinilpotent one, thus defining a generalized Drazin inverse of T.

Jacobson's lemma has been extended to these classes of operators in various settings (see for example [12, 16]). It also holds for numerous generalizations of the Drazin and the generalized Drazin invertible operators, some of which have been explored in [2, 9, 10].

Corach et al. [2] generalized the Jacobson' lemma to the operators AC and BA by showing that $I - AC$ is invertible if and only if $I - BA$ is invertible, where $A, B, C \in L(X)$ satisfy the condition $ABA = ACA$. Further generalizations ensued and Yan and Fang [13, 14] explored the spectral properties of operators AC and BD, where $A, B, C, D \in L(X)$ satisfy the conditions $ACD = DBD$ and $DBA = ACA$. On the other hand, Yan et al. [15] introduced the conditions $BAC = BDB$ and $CDB = CAC$ and in [9, 10, 15] were given extensions of Jacobson's lemma to several classes of operators.

In this paper we show the new extensions of Jacobson's lemma for the operators of Saphar type and the operators admitting the generalized Saphar decomposition and the generalized Saphar-Riesz decomposition. Section 2 contains some preliminary results that will be referred to in the later work. In Section 3 we prove that $I - AB$ is of Saphar type if and only if $I - BA$ is of Saphar type. As a corollary we get the extensions of Jacobson's lemma for all the classes of operators defined in [19]. Furthermore, we generalize some of these results to the operators AC and BD, under various conditions. In Section 4 we show the analogue of Jacobson's lemma for the operators AB and BA admitting generalized Saphar decomposition, which are defined in [5, 3]. Section 5 is dedicated to the observations of these problems for the operators satisfying generalized Saphar-Riesz decompositions and the left and the right generalized Drazin-Riesz invertible operators, all recently defined in [17].

2. Preliminaries

The following results are well known for operators $A, B, C, D \in L(X)$ satisfying conditions $BAC = BDB$ and $CDB = CAC$, and they can be found in [9] and [15]. Similarly, they also hold under conditions $ACD = DBD$ and $DBA = ACA$ as we state them here.

THEOREM 3. *Let $A, B, C, D \in L(X)$ satisfy $ACD = DBD$ and $DBA = ACA$. If $N(I - AC)$ is complemented in X and $P \in L(X)$ is the projection onto $N(I - AC)$, then $N(I - BD)$ is complemented in X and for an arbitrary $T \in L(X)$, the operator $Q \in L(X)$ defined by*

$$Q = BPT(I - BD) + BPD$$

is the projection onto $N(I - BD)$.

Proof. Follows similarly to the proof of [9, Theorem 3.3]. □

THEOREM 4. *Let $A,B,C,D \in L(X)$ satisfy $ACD = DBD$ and $DBA = ACA$. If $R(I-AC)$ is complemented in X and $P \in L(X)$ is a projection onto $R(I-AC)$, then $R(I-BD)$ is complemented in X and for an arbitrary $T \in L(X)$, an operator $Q \in L(X)$ defined by*

$$Q = I - BACPD - (I-BD)TPD$$

is a projection onto $R(I-BD)$.

Proof. Follows similarly to the proof of [9, Theorem 3.5]. □

THEOREM 5. *Let $A,B,C,D \in L(X)$ satisfy $ACD = DBD$ and $DBA = ACA$. Then for every $n \in \mathbb{N}$*

(i) *$N((I-AC)^n)$ is complemented in X if and only if $N((I-BD)^n)$ is complemented in X,*

(ii) *$R((I-AC)^n)$ is complemented in X if and only if $R((I-BD)^n)$ is complemented in X.*

Proof. By repeating the procedure in the proof of [14, Corollary 2.8] and using Theorems 3 and 4, we acquire the desired results. □

For $T \in L(X)$ and $n \in \mathbb{N}_0$ we define

$$c_n(T) = \dim R(T^n)/R(T^{n+1}) = \operatorname{codim}(R(T)+N(T^n)),$$
$$c'_n(T) = \dim N(T^{n+1})/N(T^n) = \dim(N(T)\cap R(T^n)).$$

The ascent and the descent of T are defined as $\operatorname{asc}(T) = \inf\{n \in \mathbb{N}_0 : c'_n(T) = 0\} = \inf\{n \in \mathbb{N}_0 : N(T^n) = N(T^{n+1})\}$ and $\operatorname{dsc}(T) = \inf\{n \in \mathbb{N}_0 : c_n(T) = 0\} = \inf\{n \in \mathbb{N}_0 : R(T^n) = R(T^{n+1})\}$. The essential ascent and the essential descent of T are defined as $a_e(T) = \inf\{n \in \mathbb{N}_0 : c'_n(T) < \infty\}$ and $d_e(T) = \inf\{n \in \mathbb{N}_0 : c_n(T) < \infty\}$.

THEOREM 6. *Let $A,B,C,D \in L(X)$ satisfy $ACD = DBD$ and $DBA = ACA$. Then*
(i) $\operatorname{asc}(I-AC) = \operatorname{asc}(I-BD)$,
(ii) $\operatorname{dsc}(I-AC) = \operatorname{dsc}(I-BD)$,
(iii) $a_e(I-AC) = a_e(I-BD)$,
(iii) $d_e(I-AC) = d_e(I-BD)$.

Proof. (i) Let $p = \operatorname{asc}(I-AC) < \infty$. Define the function

$$\phi : N((I-BD)^{p+1})/N((I-BD)^p) \to N((I-AC)^{p+1})/N((I-AC)^p)$$
$$\phi\left(x + N((I-BD)^p)\right) = Dx + N((I-AC)^p), \text{ for every } x \in N((I-BD)^{p+1}).$$

By [14, Lemma 2.1], this it is well defined.

The rest of the proof follows analogously to the proof of [15, Theorem 4.1(1)].

(ii) Let $q = \operatorname{dsc}(I-AC) < \infty$. Define the function

$$\psi : R((I-BD)^q)/R((I-BD)^{q+1}) \to R((I-AC)^q)/R((I-AC)^{q+1})$$
$$\psi\big(x + R((I-BD)^{q+1})\big) = Dx + R((I-AC)^{q+1}), \text{ for every } x \in R((I-BD)^q).$$

It is well defined by [14, Lemma 2.1].

The rest of the proof follows analogously to the proof of [15, Theorem 4.1(2)].

(iii) Follows from the definition of essential ascent and [14, Lemma 2.3].

(iv) Follows from the definition of essential descent and [14, Lemma 2.4]. □

3. Saphar type operators

THEOREM 7. *Let $A, B \in L(X)$. Then $I - AB$ is of Saphar type if and only if $I - BA$ is of Saphar type.*

Proof. If we put $T = I - AB$ and $S = I - BA$, it is easy to see that $BT = SB$ and $TA = AS$. Moreover, the first equality implies that for each $n \in \mathbb{N}$, $BT^n = S^nB$ stands.

Suppose that T is of Saphar type. According to [7, Theorem 4.4], there exists a projection $P \in L(X)$ commuting with T, such that $T + P$ is Saphar and TP is nilpotent. Therefore, $I - TP$ is invertible and we can define $Q = BP(I - TP)^{-1}A \in L(X)$. We will show that Q is the projection commuting with S, such that $S + Q$ is Saphar and SQ is nilpotent.

$$\begin{aligned}
Q^2 &= BP(I-TP)^{-1}ABP(I-TP)^{-1}A \\
&= BP(I-TP)^{-1}(I-T)P(I-TP)^{-1}A \\
&= BP(I-TP)^{-1}(I-TP)P(I-TP)^{-1}A \\
&= BP(I-TP)^{-1}PA = BP(I-TP)^{-1}A = Q.
\end{aligned}$$

Using the equations $BT = SB$ and $TA = AS$ we can simply see that S and Q commute. Indeed,

$$\begin{aligned}
SQ &= SBP(I-TP)^{-1}A = BTP(I-TP)^{-1}A \\
&= BP(I-TP)^{-1}TA = BP(I-TP)^{-1}AS = QS.
\end{aligned}$$

We have that

$$\begin{aligned}
S + Q &= I - BA + BP(I-TP)^{-1}A \\
&= I - B(I - P(I-TP)^{-1})A.
\end{aligned}$$

Since

$$\begin{aligned}
I - AB(I - P(I-TP)^{-1}) &= I - (I-T)(I - P(I-TP)^{-1}) \\
&= I - I + T + (I-T)P(I-TP)^{-1} \\
&= T + P
\end{aligned}$$

and $T+P$ is Saphar, from Theorem 2 we conclude that $S+Q$ is also Saphar.

Let $n\in\mathbb{N}$ be such that $(TP)^n=0$. Then $T^nP=0$ and

$$(SQ)^n=S^nQ=S^nBP(I-TP)^{-1}A=BT^nP(I-TP)^{-1}A=0,$$

so SQ is nilpotent. From [7, Theorem 4.4] we conclude that $S=I-BA$ is of Saphar type.

The converse holds by symmetry. $\square$

The degree of stable iteration of $T\in L(X)$ is defined as

$$\operatorname{dis}(T)=\inf\{n\in\mathbb{N}_0 : m\geqslant n,\ m\in\mathbb{N}\Longrightarrow R(T^n)\cap N(T)=R(T^m)\cap N(T)\}.$$

An operator $T\in L(X)$ is left Drazin invertible [8] if $p=\operatorname{asc}(T)<\infty$ and the subspace $R(T)+N(T^p)$ is complemented in X, while T is right Drazin invertible if $q=\operatorname{dsc}(T)<\infty$ and the subspace $N(T)\cap R(T^q)$ is complemented in X.

Živković-Zlatanović and Djordjević [19] introduced the concepts of the essentially left and right Drazin invertible operators and the left and right Weyl-Drazin invertible operators. An operator $T\in L(X)$ is said to be essentially left Drazin invertible if $a_e(T)<\infty$ and the subspace $R(T)+N(T^{\operatorname{dis}(T)})$ is complemented. If $d_e(T)<\infty$ and $N(T)\cap R(T^{\operatorname{dis}(T)})$ is a complemented subspace, than the operator T is essentially right Drazin invertible. Moreover, T is said to be left Weyl-Drazin invertible if it is essentially left Drazin invertible and

$$\dim(N(T)\cap R(T^{\operatorname{dis}(T)}))\leqslant\operatorname{codim}(R(T)+N(T^{\operatorname{dis}(T)})).$$

Analogously, T is right Weyl-Drazin invertible if it is essentially right Drazin invertible and

$$\operatorname{codim}(R(T)+N(T^{\operatorname{dis}(T)}))\leqslant\dim(N(T)\cap R(T^{\operatorname{dis}(T)})).$$

We denote by $\sigma_{lD}(T)$, $\sigma_{rD}(T)$, $\sigma_{lD}^e(T)$, $\sigma_{rD}^e(T)$, $\sigma_{DW_l}(T)$ and $\sigma_{DW_r}(T)$ the left Drazin spectrum, the right Drazin spectrum, the essentially left Drazin spectrum, the essentially right Drazin spectrum, the left Weyl-Drazin spectrum and the right Weyl-Drazin spectrum of T.

THEOREM 8. *Let $A,B\in L(X)$. Then*
(i) $\sigma_{St}(AB)\cup\{0\}=\sigma_{St}(BA)\cup\{0\}$,
(ii) *for each $\sigma_*\in\{\sigma_{lD},\sigma_{rD},\sigma_{lD}^e,\sigma_{rD}^e,\sigma_{DW_l},\sigma_{DW_r}\}$ we have that*

$$\sigma_*(AB)\cup\{0\}=\sigma_*(BA)\cup\{0\}.$$

Proof. (i) Follows from Theorem 7.
(ii) From part (i), [19, Corollary 5.3] and Theorem 1 we get that

$$\begin{aligned}\sigma_{lD}^e(AB)\cup\{0\}&=\Big(\sigma_{St}(AB)\cup\{0\}\Big)\cup\Big(\big(\operatorname{acc}\sigma_{\Phi_l}(AB)\big)\cup\{0\}\Big)\\&=\Big(\sigma_{St}(BA)\cup\{0\}\Big)\cup\Big(\big(\operatorname{acc}\sigma_{\Phi_l}(BA)\big)\cup\{0\}\Big)\\&=\sigma_{lD}^e(BA)\cup\{0\}.\end{aligned}$$

The rest of the proof follows analogously from [19, Corollaries 5.4 and 5.5]. $\square$

EXAMPLE 1. Let H be a complex Hilbert space and let $T = U|T|$ be the polar decomposition of $T \in L(H)$, where $|T| = (T^*T)^{\frac{1}{2}}$. The Althunge transform of T is given by $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. If we set $A = U|T|^{\frac{1}{2}}$ and $B = |T|^{\frac{1}{2}}$, then $AB = T$ and $BA = \tilde{T}$. Therefore,

$$\sigma_*(T) \cup \{0\} = \sigma_*(\tilde{T}) \cup \{0\},$$

for each $\sigma_* \in \{\sigma_{St}, \sigma_{lD}, \sigma_{rD}, \sigma_{lD}^e, \sigma_{rD}^e, \sigma_{DW_l}, \sigma_{DW_r}\}$.

We are able to generalize some of the results of Theorem 8 to the cases of operators AC and BD, where $A,B,C,D \in L(X)$ satisfy certain conditions. In the following we explore several such conditions.

THEOREM 9. *Let $A,B,C,D \in L(X)$ satisfy $ACD = DBD$ and $DBA = ACA$. Then for each $\sigma_* \in \{\sigma_{lD}, \sigma_{rD}, \sigma_{lD}^e, \sigma_{rD}^e\}$ we have that*

$$\sigma_*(AC) \cup \{0\} = \sigma_*(BD) \cup \{0\}.$$

Proof. Firstly, we show the equality $\sigma_{lD}(AC) \cup \{0\} = \sigma_{lD}(BD) \cup \{0\}$. It suffices to prove that $I - AC$ is left Drazin invertible if and only if $I - BD$ is left Drazin invertible. From [6, Theorem 2.5] we know that $I - AC$ is left Drazin invertible if and only if $\operatorname{asc}(I - AC) < \infty$ and the subspaces $N((I - AC)^n)$ and $R((I - AC)^n)$ are topologically complemented for each $n \geqslant \operatorname{asc}(I - AC)$. By Theorem 6, $\operatorname{asc}(I - AC) = \operatorname{asc}(I - BD)$. Therefore, Theorem 5 shows that the subspaces $N((I - AC)^n)$ and $R((I - AC)^n)$ are topologically complemented for each $n \geqslant \operatorname{asc}(I - AC)$ if and only if the subspaces $N((I - BD)^n)$ and $R((I - BD)^n)$ are topologically complemented for each $n \geqslant \operatorname{asc}(I - BD)$. Applying again [6, Theorem 2.5] we acquire the desired equivalence.

The proof for the cases $\sigma_* \in \{\sigma_{rD}, \sigma_{lD}^e, \sigma_{rD}^e\}$ follows analogously, by using [6, Theorems 3.2, 3.5 and 3.6] respectively. $\square$

COROLLARY 1. *Let $A,B,C \in L(X)$ satisfy $ABA = ACA$. Then for each $\sigma_* \in \{\sigma_{lD}, \sigma_{rD}, \sigma_{lD}^e, \sigma_{rD}^e\}$ we have that*

$$\sigma_*(AC) \cup \{0\} = \sigma_*(BA) \cup \{0\}.$$

THEOREM 10. *Let $A,B,C,D \in L(X)$ satisfy $BAC = BDB$ and $CDB = CAC$. Then for each $\sigma_* \in \{\sigma_{lD}, \sigma_{rD}, \sigma_{lD}^e, \sigma_{rD}^e\}$ we have that*

$$\sigma_*(AC) \cup \{0\} = \sigma_*(BD) \cup \{0\}.$$

Proof. The proof can be acquired by repeating the procedure from the previous theorem and applying the appropriate results from [9]. $\square$

It remains a question whether the equality $\sigma_*(AC) \cup \{0\} = \sigma_*(BD) \cup \{0\}$ holds for $\sigma_* \in \{\sigma_{St}, \sigma_{DW_l}, \sigma_{DW_r}\}$, under the previously observed sets of conditions.

4. Operators admitting a generalized Saphar decomposition

We will show that the analogue of Jacobson's lemma also holds for the classes of operators admitting generalized Saphar decomposition. However, it remains an open question wether these results can be extended to the operators AC and BD, where $A,B,C,D \in L(X)$ satisfy certain sets of conditions.

We will need the following theorem.

THEOREM 11. *Let $T \in L(X)$. Then T admits a* GSD *if and only if there exists a projection $P \in L(X)$, commuting with T, such that TP is quasinilpotent and $T+P$ is Saphar.*

Proof. Assume that T admits a GSD, and that M and N are closed, T-invariant subspaces such that $(M,N) \in \mathrm{Red}(T)$, where T_M is Saphar and T_N is quasinilpotent. Let $P \in L(X)$ be the projection for which $R(P) = N$ and $N(P) = M$. Since $(M,N) \in \mathrm{Red}(T)$ we have that T and P commute. It is obvious that $(M,N) \in \mathrm{Red}(TP)$ and $(M,N) \in \mathrm{Red}(T+P)$. Now from $TP = (TP)_M \oplus (TP)_N = 0 \oplus T_N$ we have that TP is quasinilpotent. Observe that $T+P = (T+P)_M \oplus (T+P)_N = T_M \oplus (T_N + I_N)$ and $T_N + I_N$ is invertible. From [19, Lemma 3.11] we conclude that $T+P$ is Saphar.

Conversely, let $P \in L(X)$ be the projection commuting with T, such that TP is quasinilpotent and $T+P$ is Saphar. If we set $M = N(P)$ and $N = R(P)$, having in mind the commutativity of T and P, we know that M and N are closed, T-invariant subspaces, such that $(M,N) \in \mathrm{Red}(T)$. Furthermore, $T_N = (TP)_N$ is quasinilpotent as a reduction of the quasinilpotent operator TP. Since $T+P$ is Saphar, again from [19, Lemma 3.11] it follows that $T_M = T_M + P_M = (T+P)_M$ is Saphar. Therefore, T admits a GSD. $\square$

THEOREM 12. *Let $A,B \in L(X)$. Then $I-AB$ admits a* GSD *if and only if $I-BA$ admits a* GSD.

Proof. Put $T = I-AB$ and $S = I-BA$ and suppose that T admits a GSD. According to Theorem 11, there exists a projection $P \in L(X)$, commuting with T, such that TP is quasinilpotent and $T+P$ is Saphar. Then $I-TP$ is invertible and it is easy to see that $BT = SB$ and $TA = AS$.

Set $Q = BP(I-TP)^{-1}A \in L(X)$. By repeating the part of the proof of Theorem 7 we acquire that Q is the projection commuting with S, such that $S+Q$ is Saphar. We show now that SQ is quasinilpotent.

Notice that

$$\begin{aligned} I-SQ &= I-SBP(I-TP)^{-1}A \\ &= I-BTP(I-TP)^{-1}A \\ &= I-BP(TP)(I-TP)^{-1}A \\ &= I-BP(I-(I-TP))(I-TP)^{-1}A \\ &= I-BP((I-TP)^{-1}-I)A. \end{aligned}$$

On the other hand,

$$\begin{aligned} I-A[BP((I-PT)^{-1}-I)] &= I-(I-T)P((I-PT)^{-1}-I) \\ &= I-(I-T)P(I-PT)^{-1}+(I-T)P \\ &= I-P+P-TP = I-TP \end{aligned}$$

and this operator is invertible since TP is quasinilpotent. From Jacobson's lemma we get that $I-SQ = I-[BP((I-PT)^{-1}-I)]A$ is also invertible, so we conclude that SQ is quasinilpotent. Thus, according to Theorem 11, we have shown that the operator $S = I-BA$ admits a GSD.

The converse holds by symmetry. $\square$

The quasinilpotent part of an operator $T \in L(X)$ is defined by

$$H_0(T) = \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

The analytical core of T, denoted by $K(T)$, is the set of all $x \in X$ for which there exist $\delta > 0$ and a sequence $(u_n)_n$ in X satisfying

$$Tu_1 = x, \;\; Tu_{n+1} = u_n \text{ and } \|u_n\| \leqslant c^n \|x\| \text{ for all } n \in \mathbb{N}.$$

In [5], the authors have defined the left generalized Drazin invertible operators as those operators $T \in L(X)$ for which $H_0(T)$ is closed and for which there exists a closed subspace M of X such that $(M, H_0(T)) \in \text{Red}(T)$ and that $T(M)$ is a complemented subspace of M. If $K(T)$ is closed and there exists a closed subspace N of X such that $N \subset H_0(T)$ and $(K(T), N) \in \text{Red}(T)$, and $K(T) \cap N(T)$ is complemented in $K(T)$, then T is called the right generalized Drazin invertible operator.

Dimitrijević and Živković-Zlatanović [3] generalized this concept by defining the essentially left and right generalized Drazin invertible operators and the left and right Weyl-g-Drazin invertible operators. An operator $T \in L(X)$ is essentially left generalized Drazin invertible if there exists $(M,N) \in \text{Red}(T)$ such that $N \subset H_0(T)$, $N(T) \cap M$ is finite-dimensional and $T(M)$ is complemented in M, while T is essentially right generalized Drazin invertible if there exists $(M,N) \in \text{Red}(T)$ such that $N \subset H_0(T)$, $M \supset K(T)$, $R(T) \cap M$ is of finite codimension in M and $N(T) \cap M$ is complemented in M. If T is both essentially left and right generalized Drazin invertible then we say that T is Fredholm-g-Drazin invertible.

We say that T is left Weyl-g-Drazin invertible if there exists $(M,N) \in \text{Red}(T)$ such that $N \subset H_0(T)$, $T(M)$ is complemented in M and $N(T) \cap M$ is of finite dimension no greater than the dimension of $M/T(M)$. The operator T is right Weyl-g-Drazin invertible if there exist closed subspaces $N \subset H_0(T)$ and $M \supset K(T)$ such that $(M,N) \in \text{Red}(T)$, $N(T) \cap M$ is complemented in M and $R(T) \cap M$ is of finite codimension in M, no greater then the dimension of $N(T) \cap M$. We say that T is Weyl-g-Drazin invertible if it is both left and right Weyl-g-Drazin invertible.

For $H \in \{G_l, G_r, \Phi_l, \Phi_r, \Phi, W_l, W_r, W\}$ we denote by $\sigma_{gDH}(T)$ the left generalized Drazin spectrum, the right generalized Drazin spectrum, the essentially left generalized

Drazin spectrum, the essentially right generalized Drazin spectrum, the Fredholm-g-Drazin spectrum, the left Weyl-g-Drazin spectrum, the right Weyl-g-Drazin spectrum and the Weyl-g-Drazin spectrum of T.

THEOREM 13. *Let* $A,B \in L(X)$. *Then*
(i) $\sigma_{gS}(AB)\cup\{0\} = \sigma_{gS}(BA)\cup\{0\}$,
(ii) *for each* $H \in \{G_l, G_r, \Phi_l, \Phi_r, \Phi, W_l, W_r, W\}$ *we have that*

$$\sigma_{gDH}(AB)\cup\{0\} = \sigma_{gDH}(BA)\cup\{0\}.$$

Proof. (i) Follows from Theorem 12.

(ii) From the proof of [3, Theorem 4.4] we have that $\sigma_{gDH}(T) = \sigma_{gS}(T)\cup \operatorname{acc}\sigma_H(T)$ for an arbitrary $T \in L(X)$ and for each $H \in \{G_l, G_r, \Phi_l, \Phi_r, \Phi, W_l, W_r, W\}$. From part (i) and Theorem 1 we get that

$$\begin{aligned} \sigma_{gDH}(AB)\cup\{0\} &= \Big(\sigma_{gS}(AB)\cup\{0\}\Big)\cup\Big(\big(\operatorname{acc}\sigma_H(AB)\big)\cup\{0\}\Big) \\ &= \Big(\sigma_{gS}(BA)\cup\{0\}\Big)\cup\Big(\big(\operatorname{acc}\sigma_H(BA)\big)\cup\{0\}\Big) \\ &= \sigma_{gDH}(BA)\cup\{0\}. \quad \square \end{aligned}$$

From Theorem 13 we can see that for the operators T and $\tilde{T}$ in Example 1 we can also conclude that

$$\sigma_{gS}(T)\cup\{0\} = \sigma_{gS}(\tilde{T})\cup\{0\} \quad \text{and} \quad \sigma_{gDH}(T)\cup\{0\} = \sigma_{gDH}(\tilde{T})\cup\{0\},$$

for each $H \in \{G_l, G_r, \Phi_l, \Phi_r, \Phi, W_l, W_r, W\}$.

5. Operators admitting a generalized Saphar-Riesz decomposition

The concept of generalized Saphar-Riesz decomposition was defined recently in [17], alongside the classes of left and right generalized Drazin-Riesz invertible operators. An operator $T \in L(X)$ is left generalized Drazin-Riesz invertible if there is $S \in L(X)$ such that

$$TST = ST^2, \;\; S^2T = S, \;\; T - TST \;\text{ is Riesz.}$$

Analogously, T is right generalized Drazin-Riesz invertible if there is $S \in L(X)$ such that

$$TST = T^2S, \;\; TS^2 = S, \;\; T - TST \;\text{ is Riesz.}$$

By $\sigma^l_{gDR}(T)$ and $\sigma^r_{gDR}(T)$ we denote the left generalized Drazin-Riesz spectrum and the right generalized Drazin-Riesz spectrum of T.

In this section we will show that $I - AB$ admits a GSRD if and only if $I - BA$ admits a GSRD. We will see that this extends to the left and right generalized Drazin-Riesz invertible operators, not just for the operators AB and BA, but also for the operators AC and BD under various conditions.

LEMMA 1. [1, Corollary 2.151] *Let* $A,B \in L(X)$. *Then* AB *is Riesz if and only if* BA *is Riesz.*

THEOREM 14. *Let* $A,B \in L(X)$. *Then* $I-AB$ *admits a* GSRD *if and only if* $I-BA$ *admits a* GSRD.

Proof. Put $T = I-AB$ and $S = I-BA$ and suppose that T admits GSRD. According to [17, Theorem 4.2], there exists a projection $P \in L(X)$ commuting with T, such that $T+P$ is of Saphar type and TP is Riesz. From the commutativity of T, P and AB it follows that $TP(I+AB)$ is Riesz. Therefore, we can find a suitable $\lambda \in \mathbb{C}\setminus\{0\}$ such that $\lambda I - TP(I+AB)$ is invertible and we have the representation

$$(\lambda I - TP(I+AB))^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1}(TP(I+AB))^n = P\sum_{n=0}^{\infty} \lambda^{-n-1}(I-(AB)^2)^n.$$

From

$$P\sum_{n=0}^{\infty}(\lambda I-(AB)^2)^n\lambda^{-n-1} = P\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(\lambda-1)^k(I-(AB)^2)^{n-k}\lambda^{-n-1}$$

we can see that $L = P\sum_{n=0}^{\infty}(\lambda I-(AB)^2)^n\lambda^{-n-1}$ is well defined bounded linear operator. Moreover, L commutes with P, T and AB and $PL = LP = L$. Also,

$$\begin{aligned}
L(AB)^2 &= P\sum_{n=0}^{\infty}(\lambda I-(AB)^2)^n\lambda^{-n-1}(AB)^2\\
&= P\sum_{n=0}^{\infty}(\lambda I-(AB)^2)^n\lambda^{-n-1}(\lambda I-(\lambda I-(AB)^2))\\
&= P\sum_{n=0}^{\infty}(\lambda I-(AB)^2)^n\lambda^{-n} - P\sum_{n=0}^{\infty}(\lambda I-(AB)^2)^{n+1}\lambda^{-n-1} = P.
\end{aligned}$$

Let $Q = BLABA$. Firstly, as $Q^2 = BLABABLABA = BPLABA = BLABA = Q$ we know that Q is the projection. It is easy to see that $BT = SB$ and $TA = AS$, so we get that

$$SQ = SBLABA = BTLABA = BLABTA = BLABAS = QS. \tag{1}$$

Observe that

$$S+Q = I-BA+BLABA = I-B[(I-LAB)A]$$

and that

$$I-[(I-LAB)A]B = I-AB+L(AB)^2 = T+P.$$

As $T+P$ is of Saphar type, from Theorem 7 we conclude that $S+Q$ is also of Saphar type.

We can see from (1) that $SQ = BLABTA$. Hence, $SQ = BLABTA = BLPABTA = BLABTPA$. From the commutativity of TP and $ABLAB$, and the fact that TP is Riesz we have that $ABLABTP$ is also Riesz. Now, Lemma 1 allows us to deduce that $SQ = BLABTPA$ is Riesz and from [17, Theorem 4.2] we acquire that $S = I - BA$ admits a GSRD.

The converse holds by symmetry. □

THEOREM 15. *Let $A, B \in L(X)$. Then*
(i) $\sigma_{gSR}(AB) \cup \{0\} = \sigma_{gSR}(BA) \cup \{0\}$,
(ii) $\sigma^l_{gDR}(AB) \cup \{0\} = \sigma^l_{gDR}(BA) \cup \{0\}$,
(iii) $\sigma^r_{gDR}(AB) \cup \{0\} = \sigma^r_{gDR}(BA) \cup \{0\}$.

Proof. (i) Follows from Theorem 14.
(ii) From part (i), [17, Corollary 6.3] and Theorem 1 we get that

$$\begin{aligned}\sigma^l_{gDR}(AB) \cup \{0\} &= \Big(\sigma_{gSR}(AB) \cup \{0\}\Big) \cup \Big(\big(\operatorname{acc} \sigma_{B_l}(AB)\big) \cup \{0\}\Big) \\ &= \Big(\sigma_{gSR}(BA) \cup \{0\}\Big) \cup \Big(\big(\operatorname{acc} \sigma_{B_l}(BA)\big) \cup \{0\}\Big) \\ &= \sigma^l_{gDR}(BA) \cup \{0\}.\end{aligned}$$

(iii) Follows analogously to part (ii). □

From Theorem 15 we can see that for the operators T and $\tilde{T}$ in Example 1 we can also conclude that

$$\sigma_*(T) \cup \{0\} = \sigma_*(\tilde{T}) \cup \{0\}$$

for each $\sigma_* \in \{\sigma_{gSR}, \sigma^l_{gDR}, \sigma^r_{gDR}\}$.

The following results generalize Theorem 15 parts (ii) and (iii) to the operators AC and BD under the conditions observed in the earlier sections. However, it remains to be seen wether the equality $\sigma_{gSR}(AC) \cup \{0\} = \sigma_{gSR}(BD) \cup \{0\}$ can be acquired under any set of conditions.

THEOREM 16. *Let $A, B, C, D \in L(X)$ satisfy $ACD = DBD$ and $DBA = ACA$.*

(i) $I - BD$ is left generalized Drazin-Riesz invertible if and only if $I - AC$ is left generalized Drazin-Riesz invertible.

(ii) $I - BD$ is right generalized Drazin-Riesz invertible if and only if $I - AC$ is right generalized Drazin-Riesz invertible.

Proof. (i) Suppose that $S' \in L(X)$ is left generalized Drazin-Riesz inverse of $S = I - BD$. From [17, Theorem 5.5] the following holds:

$$\begin{aligned}&SS'S = S'S^2 \\ &S'^2S = S' = S'SS' \\ &S - SS'S \quad \text{is Riesz.}\end{aligned}$$

Operator $S^\pi = I - S'S$ is a projection commuting with S and S', such that $S'S^\pi = S^\pi S' = 0$ and $SS^\pi = S^\pi S$ is Riesz. From the commutativity of S, S^π and BD, we get that $S^\pi S(I+BD)$ is Riesz. Similarly as in the proof of Theorem 14, we conclude that $L = S^\pi \sum_{n=0}^{\infty} (\lambda I - (BD)^2)^n \lambda^{-n-1}$ is a bounded operator.

Set $T = I - AC$ and let

$$T' = (I - DLBAC)(I + AC) + DS'BAC.$$

We will prove that T' satisfies the conditions

$$\begin{aligned} &TT'T = T'T^2 \\ &T'^2 T = T' \\ &T - TT'T \quad \text{is Riesz.} \end{aligned}$$

It is easily seen that from the conditions $ACD = DBD$ and $DBA = ACA$ we have

$$\begin{gathered} (I \pm AC)D = D(I \pm BD), \\ BAC(I \pm AC) = (I \pm BD)BAC. \end{gathered}$$

Observe that

$$\begin{aligned} L(BD)^2 &= S^\pi \sum_{n=0}^{\infty} (\lambda I - (BD)^2)^n \lambda^{-n-1} (BD)^2 \\ &= S^\pi \sum_{n=0}^{\infty} (\lambda I - (BD)^2)^n \lambda^{-n-1} (\lambda I - (\lambda I - (BD)^2)) \\ &= S^\pi \sum_{n=0}^{\infty} (\lambda I - (BD)^2)^n \lambda^{-n} - S^\pi \sum_{n=0}^{\infty} (\lambda I - (BD)^2)^{n+1} \lambda^{-n-1} \\ &= S^\pi + S^\pi \sum_{n=0}^{\infty} (\lambda I - (BD)^2)^{n+1} \lambda^{-n-1} - S^\pi \sum_{n=0}^{\infty} (\lambda I - (BD)^2)^{n+1} \lambda^{-n-1} \\ &= S^\pi. \end{aligned}$$

Therefore,

$$\begin{aligned} T'T &= I - (AC)^2 - DLBAC(I - (AC)^2) + DS'BAC(I - AC) \\ &= I - DBAC - DL(I - (BD)^2)BAC + DS'(I - BD)BAC \\ &= I - DBAC - DLBAC + DS^\pi BAC + D(I - S^\pi)BAC \\ &= I - DLBAC. \end{aligned}$$

Since L commutes with BD, we get the equality

$$\begin{aligned} TT'T &= (I - AC)(I - DLBAC) = I - AC - (I - AC)DLBAC \\ &= I - AC - D(I - BD)LBAC = I - AC - DL(I - BD)BAC \\ &= I - AC - DLBAC(I - AC) = (I - DLBAC)(I - AC) = (T'T)T \\ &= T'T^2. \end{aligned}$$

Having in mind that $S^{\pi}S' = S'S^{\pi} = 0$ and using the previous observations we acquire the equality

$$\begin{aligned} T'^2T &= T'(I-DLBAC) = T' - (I-DLBAC)(I+AC)DLBAC - DS'BACDLBAC \\ &= T' - (I+AC)DLBAC + (I+AC)DLBACDLBAC - DS'BACDLBAC \\ &= T' - (I+AC)DLBAC + (I+AC)DL(BD)^2LBAC - DS'(BD)^2LBAC \\ &= T' - (I+AC)DLBAC + (I+AC)DLBAC - DS'PBAC \\ &= T'. \end{aligned}$$

Observe that

$$T - TT'T = (I-AC)DLBAC = \big(DL(I-BD)\big)BAC.$$

On the other hand,

$$BAC\big(DL(I-BD)\big) = (BD)^2LS = S^{\pi}S$$

and since $S^{\pi}S$ is a Riesz operator we conclude that $T - TT'T$ is Riesz. Thus we have proved that $I-AC$ is left generalized Drazin-Riesz invertible, with T' as its left generalized Drazin-Riesz inverse.

Conversely, suppose that $T = I-AC$ is left generalized Drazin-Riesz invertible with T' as its left generalized Drazin-Riesz inverse. Analogously, we can prove that $T^{\pi} = I - T'T$ is a projection commuting with T, such that TT^{π} is Riesz, and that

$$S' = \left(I - BACT^{\pi}\sum_{n=0}^{\infty}(\lambda I-(AC)^2)^n\lambda^{-n-1}D\right)(I+BD) + BACT'D$$

is well defined and is the left generalized Drazin-Riesz inverse of $I-AC$.

(ii) Follows from [17, Theorem 5.6], analogously to the proof of part (i). □

COROLLARY 2. *Let $A,B,C \in L(X)$ satisfy $ABA = ACA$. Then $I-BA$ is left (right) generalized Drazin-Riesz invertible if and only if $I-AC$ is left (right) generalized Drazin-Riesz invertible.*

THEOREM 17. *Let $A,B,C,D \in L(X)$ satisfy $BAC = BDB$ and $CDB = CAC$.*

(i) $I-BD$ is left generalized Drazin-Riesz invertible if and only if $I-AC$ is left generalized Drazin-Riesz invertible.

(ii) $I-BD$ is right generalized Drazin-Riesz invertible if and only if $I-AC$ is right generalized Drazin-Riesz invertible.

Proof. (i) Using the same annotations as in the proof of Theorem 16, the proof follows analogously if we set

$$T' = \left(I - ACDS^{\pi}\sum_{n=0}^{\infty}(\lambda I-(BD)^2)^n\lambda^{-n-1}B\right)(I+AC) + ACDS'B$$

and

$$S' = \left(I - BT^{\pi}\sum_{n=0}^{\infty}(\lambda I-(AC)^2)^n\lambda^{-n-1}ACD\right)(I+BD) + BT'ACD. \quad \square$$

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