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OPERATOR RADII OF COMMUTING PRODUCTS

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Abstract. Let $\omega_\rho(X)$ represent the operator radius of a bounded linear operator X on a Hilbert space $\mathscr{H}$, where $0<\rho\leqslant 2$ and let X and Y be bounded commutative operators on $\mathscr{H}$. In this article, the long-standing constant $k=1.169$ in the inequality $\omega_2(XY)\leqslant 1.169\omega_1(X)\omega_2(Y)$ is improved and more accurate estimates for the ratio $\omega_\rho(XY)/(\omega_1(X)\cdot\omega_\rho(Y))$ are obtained.

1. Introduction

According to Sz.-Nagy and Foias [10], a bounded linear operator X on a Hilbert space $\mathscr{H}$ is said to admit a unitary ρ-dilation if there exists a unitary operator V on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ such that for every positive integer n the following relation holds:

$$X^n = \rho Q_{\mathscr{H}} \circ V^n\big|_{\mathscr{H}},$$

where $\rho>0$ and $Q_{\mathscr{H}}$ represents the projection from $\mathscr{K}$ onto $\mathscr{H}$. The class of all such operators X is denoted by $\mathscr{D}_\rho$. Later on, Halbrook [4] and Williams [12] defined ρ-operator radius $\omega_\rho(X)$ of an operator X as the Minkowski functional of the absorbing set $\mathscr{D}_\rho$. Specifically,

$$\omega_\rho(X) = \inf\left\{\mu>0 \mid \mu^{-1}X\in\mathscr{D}_\rho\right\}.$$

It has been shown that $\omega_\rho(X)$ generally acts as a quasi-norm and satisfies the triangle inequality only for $0<\rho\leqslant 2$. Additionally, for $0<\rho<2$ it also satisfies the relation:

$$\rho\,\omega_\rho(X) = (2-\rho)\,\omega_{2-\rho}(X). \tag{1.1}$$

The notion of ρ-radii brings togather several key measures in operator theory: notably $\omega_\rho(X)$ equals the operator norm $\|X\|$ when $\rho=1$, the numerical radius $\omega(X)$ when $\rho=2$, and converges to the spectral radius $\nu(X)$ as ρ approaches infinity. The power inequality

$$\omega_\rho(X^n)\leqslant(\omega_\rho(X))^n, \tag{1.2}$$

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holds for all $\rho > 0$ and integer $n \geqslant 1$. However, unlike the operator norm and spectral radius, the function $\omega_\rho(\cdot)$ is not always sub-multiplicative, not even for commutative operators. For example, for commutative bounded operators X and Y, we have:

$$\omega(XY) \leqslant 2 \cdot \omega(X)\omega(Y).$$

A conjecture has been proposed regarding this [4]. It suggests that for $\rho > 0$ and commutative bounded operators X and Y the following inequality should hold:

$$\omega_\rho(XY) \leqslant \rho \cdot \omega_\rho(X)\omega_\rho(Y), \tag{1.3}$$

or more generally:

$$\omega_\rho(XY) \leqslant \sigma \cdot \omega_\rho(X)\omega_\sigma(Y), \tag{1.4}$$

where $\sigma > 0$.

Conjecture (1.4) has been confirmed for $\sigma \geqslant 2$, as shown by Okubo and Ando [8]. However, the special case of (1.4) where $\sigma = 1$ and $\rho = 2$ has received a lot of attention. Several results support the conjecture in this case. For instance, it has been validated if either X or Y is an isometry and $XY = YX$.

Additionally, it also holds true if X and Y double commute, i.e. $XY = YX$ and $XY^* = Y^*X$. Okubo and Ando [8] established the following inequality for commutative operators X and Y,

$$\omega(XY) \leqslant 1.169 \cdot \omega(X)\|Y\|, \tag{1.5}$$

which further lends credence to the conjecture's validity. Nevertheless, in 1988, Muller [7] produced a counterexample with the aid of a computer, thereby disproving the conjecture. This counterexample was subsequently generalized by Davidson et al. [1], demonstrating that the conjecture (1.4) also does not hold for $\rho > 1$ and $\sigma = 1$.

In this research article, we shall prove that $\omega_\rho(\cdot)$ is a sub-multiplicative norm on class of commutative bounded operators when $0 < \rho \leqslant 1$. Concerning conjecture (1.4), we will prove that for $0 < \rho \leqslant 2$:

$$\omega_\rho(XY) \leqslant 1.059016995 \cdot \rho \cdot \omega_\rho(X) \cdot \omega_\rho(Y).$$

Furthermore, we will refine and generalize inequality (1.5) for $0 < \rho \leqslant 2$, as follows:

$$\omega_\rho(XY) \leqslant 1.1396023296 \cdot \omega_\rho(X) \cdot \|Y\|.$$

Regarding conjecture (1.4), we will establish that if $0 < \rho,\ \sigma \leqslant 2$, then for commutative bounded operators X and Y, the following inequality holds:

$$\omega_\rho(XY) \leqslant L_\sigma \cdot \omega_\rho(X) \cdot \omega_\sigma(Y),$$

where L_σ is a monotonically increasing function of σ on $(0,2]$ and satisfies $L_\sigma \leqslant 1.2991\sigma$ for $0 < \sigma \leqslant 2$.

2. Main results

LEMMA 2.1. *Suppose $\omega(X) \leqslant 1$, and let g be a unit vector in a Hilbert space $\mathscr{H}$. Define the sequence u_n as*

$$u_n = \|X^n g\|^2 + \|X^{*n} g\|^2 \quad \textit{for} \quad n = 1,2,3,\ldots,$$

with $u_0 = 1$. Then there exists a parameter sequence $\{g_n\}_{n\geqslant 0}$ such that u_n can be written as

$$u_n = 4^n(1-g_0)g_n \prod_{i=1}^{n-1}(g_i - g_i^2), \quad \textit{with} \quad 0 \leqslant g_i \leqslant 1 \quad \textit{for all} \quad n \geqslant 0.$$

Proof. The condition $\omega(X) \leqslant 1$ implies that

$$\mathrm{Re}\langle Xh, h\rangle \leqslant \langle h, h\rangle \quad \text{for all} \quad h \in \mathscr{H}. \tag{2.1}$$

Integrating inequality (2.1) over $0 \leqslant \theta \leqslant 2\pi$ with

$$h = \lambda_0 g + \sum_{j=1}^{n} \lambda_j \mathrm{Re}(X^j e^i j\theta)g, \quad \text{(where } \lambda_j\text{'s are complex numbers)}$$

and using the orthogonality of $\{e^i j\theta\}$, we obtain the inequality

$$\mathrm{Re}\left\{\sum_{j=1}^{n} \lambda_{j-1}\bar{\lambda}_j u_j\right\} \leqslant |\lambda_0|^2 + \sum_{j=1}^{n} |\lambda_j|^2 u_j. \tag{2.2}$$

By Wall-Witzel theorem [11, p. 67], the quadratic form (2.2) is positive if and only if there exists sequence $\{g_n\}_{n\geqslant 0}$ with $0 \leqslant g_n \leqslant 1$ such that for all $n \in \mathbb{N}$

$$u_n = 4u_{n-1}(1-g_{n-1})g_n,$$

from which lemma follows immediately. $\square$

Let X and Y be bounded operators on Hilbert space $\mathscr{H}$ such that $XY = YX$, $\|Y\| = \omega(X) = 1$, and let g be a unit vector. Define the sequences u_n and v_n as follows:

$$u_n = \|X^n g\|^2 + \|X^{*n} g\|^2 \quad \text{for} \quad n = 1,2,3,\ldots,$$

and

$$v_n = u_{2^{n-1}} \quad \text{for} \quad n = 1,2,3,\ldots$$

Using the Cauchy-Schwarz inequality,

$$\begin{aligned} 4|\mathrm{Re}\langle YXg, g\rangle|^2 &\leqslant |\langle YXg + X^*Y^*g, g\rangle|^2 \leqslant \|YXg + X^*Y^*g\|^2 \\ &= \|YXg\|^2 + \|X^*Y^*g\|^2 + 2\mathrm{Re}\langle (YX)^2 g, g\rangle, \end{aligned}$$

but since $XY = YX$ and Y is a contraction,

$$2\,|\mathrm{Re}\langle YXg,g\rangle| \leqslant \sqrt{\|Xg\|^2 + \|X^*g\|^2 + 2\,|\mathrm{Re}\langle Y^2X^2g,g\rangle|}. \tag{2.3}$$

Given that $\|Y\| = 1$, by applying the Cauchy-Schwarz inequality, we can also obtain:

$$2\left|\mathrm{Re}\langle Y^{2^n}X^{2^n}g,g\rangle\right| \leqslant 2\|Y^{2^{n-1}}X^{2^{n-1}}g\|\,\|Y^{*2^{n-1}}X^{*2^{n-1}}g\| \leqslant \|X^{2^{n-1}}g\|^2 + \|X^{*2^{n-1}}g\|^2 = v_n. \tag{2.4}$$

By iteratively applying equation (2.3) for Y^2X^2 and subsequent terms, and then using equation (2.4), it was demonstrated in [8] that

$$2\,|\mathrm{Re}\langle YXg,g\rangle| \leqslant \sqrt{v_1 + \sqrt{v_2 + \sqrt{v_3 + \cdots + \sqrt{2v_n}}}}.$$

Since this inequality remains valid even if X is replaced by $e^{i\theta}X$, the inequality transforms into:

$$2\omega(XY) \leqslant \sqrt{v_1 + \sqrt{v_2 + \sqrt{v_3 + \cdots + \sqrt{2v_n}}}}. \tag{2.5}$$

Consider the functions $f_n : {\mathbb{R}_+}^n \to \mathbb{R}_+$ and $h_n : {\mathbb{R}_+}^{n-1} \to \mathbb{R}_+$ defined as:

$$f_n(v_1, v_2, \ldots, v_n) = \frac{1}{2}\sqrt{v_1 + \sqrt{v_2 + \sqrt{v_3 + \cdots + \sqrt{2v_n}}}},$$

and

$$h_n(v_1, v_2, \ldots, v_{n-1}) = \frac{1}{2}\sqrt{v_1 + \sqrt{v_2 + \sqrt{v_3 + \cdots + \sqrt{v_{n-1}}}}}.$$

Lemma (2.1) serves as a critical tool for estimating the functions f_n and h_n. Since $\omega(X) \leqslant 1$ and g is a unit vector, by Lemma (2.1) there exists a parameter sequence $0 \leqslant g_i \leqslant 1$ such that for all $n \in \mathbb{N}$,

$$v_n = 4^{2^{n-1}}(1-g_0)g_{2^{n-1}}\prod_{i=1}^{2^{n-1}}(g_i - g_i^2). \tag{2.6}$$

Substituting v_n from equation (2.6) into the functions f_n or h_n and using the fact that the function ψ defined on the interval $[0,1]$ as:

$$\psi(x) = a_1(x-x^2) + \sqrt{a_2(x-x^2) + \sqrt{a_3(x-x^2) + \cdots + \sqrt{a_n(x-x^2)}}},$$

where $a_i \geqslant 0$, attains its supremum at $x = \frac{1}{2}$, it can be easily established that:

$$\sup_{0\leqslant g_i\leqslant 1} f_n(v_1, v_2, \ldots, v_n) = \sup_{0\leqslant g_i\leqslant 1} f_n(u_1, u_2, \ldots, u_n), \tag{2.7}$$

and

$$\sup_{0\leqslant g_i\leqslant 1} h_n(v_1,v_2,\ldots,v_{n-1}) = \sup_{0\leqslant g_i\leqslant 1} h_n(u_1,u_2,\ldots,u_{n-1}). \tag{2.8}$$

It follows from equations (2.5), (2.7), and (2.8) that for $n\in\mathbb{N}$, the following inequalities hold:

$$\omega(XY) \leqslant \sup_{0\leqslant g_i\leqslant 1} f_n(u_1,u_2,\ldots,u_n) \quad \text{if} \quad u_n\neq 0,$$

and

$$\omega(XY) \leqslant \sup_{0\leqslant g_i\leqslant 1} h_n(u_1,u_2,\ldots,u_{n-1}) \quad \text{if} \quad u_n = 0,$$

where u_i are represented as functions of g_i by Lemma (2.1).

The following table presents the supremum values of the functions $\sup_{0\leqslant g_i\leqslant 1} f_n(u_1, u_2,\ldots,u_n)$ and $\sup_{0\leqslant g_i\leqslant 1} h_n(u_1,u_2,\ldots,u_{n-1})$ for n from 2 to 10. These values were estimated using the Differential Evolution algorithm from the SciPy library in Python, ensuring high accuracy.

n	$\max f_n(u_1,u_2,\ldots,u_n)$	$\max h_n(u_1,u_2,\ldots,u_{n-1})$
2	1.1687708945	1.0000000000
3	1.1418639505	1.0986841135
4	1.1396023295	1.1257654090
5	1.1399188876	1.1348184352
6	1.1403263898	1.1382661270
7	1.1405831892	1.1396967930
8	1.1407256150	1.1403265609
9	1.1408015590	1.1406157621
10	1.1408415226	1.1407527874

Table 1: *Maximum values of $f_n(u_1,u_2,\ldots,u_n)$ and $h_n(u_1,u_2,\ldots,u_{n-1})$ for different values of n*

From this table, we observe that $\sup_{0\leqslant g_i\leqslant 1} f_n$ attains its minimum value of 1.1396023295 when $n=4$, and $\sup_{0\leqslant g_i\leqslant 1} h_4 = 1.1257654090$. Thus, whether u_4 is zero or non-zero,

$$\omega(XY) \leqslant 1.1396023296.$$

By definition, $\sup_{0\leqslant g_i\leqslant 1} h_n$ increases as n increases. Furthermore, $\sup_{0\leqslant g_i\leqslant 1} f_n \geqslant \sup_{0\leqslant g_i\leqslant 1} h_n$. Consequently, the observation that $\sup_{0\leqslant g_i\leqslant 1} g_7 = 1.1396967930$ is greater than 1.1396023296 implies that for all $n\geqslant 7$, we have

$$\sup_{0\leqslant g_i\leqslant 1} f_n \geqslant 1.1396967930.$$

Therefore, 1.1396023296 is the best possible upper bound in the above inequality which can be obtained using using equation (2.5). The discussion above can be summarized as the following theorem.

THEOREM 2.1. *Let X and Y be commutative bounded linear operators on a Hilbert space $\mathscr{H}$. Then, the following inequality holds:*

$$\omega(XY) \leqslant 1.1396023296 \cdot \omega(X)\|Y\|.$$

Okubo and Ando [8] demonstrated, under the hypothesis of Theorem (2.1), that for $0 < \rho < \infty$, the following inequality holds:

$$\omega_\rho(XY) \leqslant K_\rho \cdot \omega_\rho(X)\|Y\|, \tag{2.9}$$

where K_ρ is defined as follows:

$$K_\rho = \begin{cases} \inf\limits_{0<\lambda<1} \left\{ \frac{1}{\lambda(2-\lambda)} + \frac{(\rho-1)^2}{(2-\rho-\lambda)^2} \right\}^{1/2} & \text{for } 0 < \rho < 1, \\ \inf\limits_{0<\lambda<1} \left\{ \frac{1}{\lambda(2-\lambda)} + \frac{(\rho-1)^2}{(\rho-\lambda)^2} \right\}^{1/2} & \text{for } 1 \leqslant \rho < \infty. \end{cases}$$

Now we present a corollary to Theorem (2.1) which establishes an inequality analogous to (2.9). Prior to this, we recall a proposition by Kittaneh and Zamani, which will be essential for proving the corollary [6].

PROPOSITION 2.1. [6, Theorem 3.1 and Remark 3.2] *Suppose $0 < \rho \leqslant 2$ and let X be a bounded linear operator on a Hilbert space $\mathscr{H}$. Then:*

$$\omega_\rho(X) = \frac{2}{\rho}\omega\left(\begin{bmatrix} O & \sqrt{\rho(2-\rho)}X \\ O & |1-\rho|X \end{bmatrix}\right).$$

With this proposition in hand, we are now in a position to formally present the previously mentioned corollary.

COROLLARY 2.1. *Let X and Y be commutative bounded linear operators on a Hilbert space $\mathscr{H}$. For any $0 < \rho \leqslant 2$, the following inequality holds:*

$$\omega_\rho(XY) \leqslant 1.1396023296 \cdot \omega_\rho(X)\|Y\|.$$

Proof. Consider 2×2 operator matrices

$$X' = \begin{bmatrix} O & \sqrt{\rho(2-\rho)}X \\ O & |1-\rho|X \end{bmatrix} \quad \text{and} \quad Y' = \begin{bmatrix} Y & O \\ O & Y \end{bmatrix}.$$

Since X and Y commute, we have

$$X'Y' = \begin{bmatrix} O & \sqrt{\rho(2-\rho)}X \\ O & |1-\rho|X \end{bmatrix} \begin{bmatrix} Y & O \\ O & Y \end{bmatrix} = \begin{bmatrix} O & \sqrt{\rho(2-\rho)}XY \\ O & |1-\rho|XY \end{bmatrix} = \begin{bmatrix} O & \sqrt{\rho(2-\rho)}YX \\ O & |1-\rho|YX \end{bmatrix}. \tag{2.10}$$

Similarly,

$$Y'X' = \begin{bmatrix} Y & O \\ O & Y \end{bmatrix} \begin{bmatrix} O & \sqrt{\rho(2-\rho)}X \\ O & |1-\rho|X \end{bmatrix} = \begin{bmatrix} O & \sqrt{\rho(2-\rho)}YX \\ O & |1-\rho|YX \end{bmatrix}.$$

Thus, $X'Y' = Y'X'$. Applying Proposition (2.1) and then using Theorem (2.1) in subsequent steps, we have

$$\begin{aligned}
\omega_\rho(XY) &= \frac{2}{\rho}\omega\left(\begin{bmatrix} O & \sqrt{\rho(2-\rho)}XY \\ O & |1-\rho|XY \end{bmatrix}\right) \\
&= \frac{2}{\rho}\omega(X'Y'), \quad \text{(by (2.10))} \\
&\leqslant \frac{2(1.1396023296)}{\rho}\omega(X')\|Y'\|, \quad \text{(since } X'Y' = Y'X'\text{)} \\
&= 1.1396023296\, \omega_\rho(X)\|Y\|. \quad \square
\end{aligned}$$

REMARK 2.1. For fixed $\lambda \in (0,1)$, the function $(\rho-1)(2-\rho-\lambda)^{-1}$ decreases as ρ increases within the interval $0 < \rho \leqslant 1$. Consequently, K_ρ is a decreasing function on the interval $(0,1]$. In contrast, K_ρ is a monotonically increasing function on $[1,2]$. Since $K_1 = 1$ and $K_2 = \lim_{\rho\to 0^+} K_\rho = 1.2991$, it follows that Corollary (2.1) provides improved bounds over those given by inequality (2.9) for $\rho \in (0, 0.7577]$ and $\rho \in (1.24224, 2]$.

Next, we aim to refine yet another inequality established by Okubo and Ando [8]. They have shown that if X commutes with Y, then:

$$\omega_\rho(XY) \leqslant L'_\sigma \cdot \omega_\rho(X)\omega_\sigma(Y), \tag{2.11}$$

where L'_σ is defined as:

$$L'_\sigma = \begin{cases} \frac{(\sigma-1)+\sqrt{1+2\sigma-\sigma^2}}{2-\sigma} & \text{for } 0 < \sigma \leqslant 1, \\ \frac{1-\sigma+\sqrt{1+2\sigma-\sigma^2}}{2-\sigma} & \text{for } 1 \leqslant \sigma \leqslant 2, \\ \sigma & \text{for } 2 \leqslant \sigma < \infty. \end{cases}$$

This inequality verifies conjecture (1.4) for $\sigma \geqslant 2$ and $\rho > 0$. While analyzing their proof with Python for specific values of σ and ρ, we noticed that the bounds in inequality (2.11) could be further improved. Inspired by these observations, we have refined Okubo and Ando's original proof to obtain tighter bounds in inequality (2.11) for $0 < \rho$, $\sigma \leqslant 2$ in Theorem (2.2). Additionally, Theorem (2.3) provides even tighter bounds for the case when $\rho = \sigma$.

THEOREM 2.2. *Let X and Y be commutative bounded linear operators on Hlibert space $\mathscr{H}$, then for $0 < \sigma$, $\rho \leqslant 2$, the following inequality holds:*

$$\omega_\rho(XY) \leqslant L_\sigma \cdot \omega_\rho(X)\omega_\sigma(Y),$$

where L_σ is given by

$$L_\sigma = \begin{cases} \inf\limits_{0<\theta\leqslant 1} \frac{\theta(\sigma-1)+\sqrt{\theta^2(1-\sigma)^2+2\theta\sigma(2-\sigma)}}{\theta(2-\sigma)(2-\theta)} & \text{for } 0<\sigma\leqslant 1, \\ \inf\limits_{0<\theta\leqslant 1} \frac{\theta(1-\sigma)+\sqrt{\theta^2(1-\sigma)^2+2\theta\sigma(2-\sigma)}}{\theta(2-\sigma)(2-\theta)} & \text{for } 1\leqslant\sigma<2, \\ 2 & \text{for } \sigma=2. \end{cases}$$

Proof. Suppose $\omega(X)=\omega_\sigma(Y)=1$, with $\sigma\geqslant 1$. For $\beta>1$, using Theorem 1 in [8], it can be proved that a sufficient condition for $\omega(XY)\leqslant\beta$ is that the inequality

$$(2-\sigma)(2-\theta)^2\theta\beta^2+2(\sigma-1)\theta(2-\theta)\beta-2\sigma\geqslant 0, \tag{2.12}$$

has a solution θ satisfying equations,

$$0<1-\theta/2<1 \quad\text{and}\quad 1-2/\sigma<1/\beta(2-\theta)<1. \tag{2.13}$$

For $1\leqslant\sigma\leqslant 2$, equation (2.13) reduces to

$$0<\theta<2-1/\beta. \tag{2.14}$$

Since $\beta>1$, inequality (2.14) holds for all $0<\theta\leqslant 1$. So from inequality (2.12) it can be conclude that for $1\leqslant\sigma\leqslant 2$, a sufficient condition for $\omega(XY)\leqslant\beta$ is

$$\beta\geqslant\begin{cases} \inf\limits_{0<\theta\leqslant 1} \frac{\theta(\sigma-1)+\sqrt{\theta^2(1-\sigma)^2+2\theta\sigma(2-\sigma)}}{\theta(2-\sigma)(2-\theta)} & \text{for } 1\leqslant\sigma<2, \\ 2 & \text{for } \sigma=2. \end{cases}$$

This proves the assertion for $1\leqslant\sigma\leqslant 2$, when $\rho=2$. For $0<\sigma\leqslant 1$, the proof follows directly from the case $1\leqslant\sigma<2$, by using equation (1.1). Thus, the theorem is proved for $\rho=2$. To extend the proof to the general case where $0<\rho\leqslant 2$, we invoke Proposition 2.1 and proceed using analogous steps as outlined in the proof of Corollary 2.1. □

REMARK 2.2.

(i) Graphing calculators show that $L_\sigma\leqslant L'_\sigma$ for $0<\sigma\leqslant 2$ and $L_\sigma=L'_\sigma$, only when $\sigma=2$. Thus, Theorem (2.2) strictly improves the bound in inequality (2.11) for every $0<\sigma<2$.

(ii) The function L_σ is increasing on $(0,2]$ with $L_2=2$ and $\lim_{\sigma\to 0^+}L_\sigma=0$.

(iii) The ratio L_σ/σ is increasing on the interval $(0,1]$ and decreasing on $[1,2]$. Given that $L_1\approx 1.2991$, the following inequality holds for commutative operators X and Y:
$$\omega_\rho(XY)\leqslant 1.2991\cdot\sigma\cdot\omega_\rho(X)\cdot\omega_\sigma(Y),$$
where $0<\sigma,\rho\leqslant 2$.

THEOREM 2.3. *Let X and Y be commutative bounded linear operators on Hlibert space $\mathscr{H}$, then*

$$\omega_\rho(XY) \leqslant M_\rho \cdot \omega_\rho(X)\omega_\rho(Y), \quad (0<\rho<2)$$

where

$$M_\rho = \begin{cases} \inf\limits_{0<\theta<1} \frac{-(\rho-1)^2+\sqrt{(\rho-1)^4+\rho((2-\rho)(\rho-1)^2+(2-\rho)\theta^{-1}(2-\theta)^{-1}(2-\rho-\theta)^2)}}{(2-\rho)(2-\rho-\theta)} & \text{for } 0<\rho\leqslant 1, \\ \inf\limits_{0<\theta<1} \frac{-(\rho-1)^2+\sqrt{(\rho-1)^4+(2-\rho)(\rho(\rho-1)^2+\rho\theta^{-1}(2-\theta)^{-1}(\rho-\theta)^2)}}{(2-\rho)(\rho-\theta)} & \text{for } 1\leqslant\rho<2. \end{cases}$$

Proof. Suppose $\omega_\rho(X) = \omega_\rho(Y) = 1$, with $1<\rho<2$. For $\beta>1$, similar to Theorem 1.2, it can be deduced following the proof of Theorem 1 in [8] that a sufficient condition for $\omega(XY)\leqslant\beta$ is that the inequality

$$(2-\rho)(\rho-\theta)^2\beta^2+2(\rho-1)^2(\rho-\theta)\beta-\rho(\rho-1)^2-\rho\theta^{-1}(2-\theta)^{-1}(\rho-\theta)^2 \geqslant 0, \tag{2.15}$$

has a solution θ satisfying equations,

$$0<\theta<2, \quad \text{and} \quad 1-2/\rho<(\rho-1)/\beta(\rho-\theta)<1. \tag{2.16}$$

Since $\beta>1$, equation (2.16) holds for every $0<\theta<1$. From equation (2.15) it can be concluded that sufficient condition for $\omega_\rho(AB)\leqslant\beta$ is,

$$\beta \geqslant \inf_{0<\theta\leqslant 1} \frac{-(\rho-1)^2+\sqrt{(\rho-1)^4+(2-\rho)(\rho(\rho-1)^2+\rho\theta^{-1}(2-\theta)^{-1}(\rho-\theta)^2)}}{(2-\rho)(\rho-\theta)}.$$

This proves the theorem for $1<\rho<2$. Again for $0<\rho<1$, proof follows from case $1<\rho<2$ by using equation (1.1). □

Key observations regarding bound M_ρ in Theorem 2.3:

(i) The function M_ρ is monotonic increasing on the interval $0<\rho\leqslant 2$, with M_ρ bounded below by 0 and above by 2. Specifically, $M_1 = 1$. Consequently, for commutative operators X and Y, it follows that:

$$\omega_\rho(XY) \leqslant \omega(X)\cdot\omega(Y),$$

for all $0<\rho\leqslant 1$. Thus, $\omega_\rho(\cdot)$ is a sub-multiplicative norm on the set of commutative bounded operators for every $0<\rho\leqslant 1$.

(ii) The inequality $M_\rho \leqslant L_\rho$ holds for all $0<\rho\leqslant 2$. Therefore, Theorem 2.3 further improves the inequality in Theorem 2.2 for special case when $\rho=\sigma$.

(iii) The ratio M_ρ/ρ is bounded above by 1.059016995 on the interval $0<\rho\leqslant 2$. Hence, for commutative bounded operators X and Y, the following inequality holds for every $0<\rho\leqslant 2$:

$$\omega_\rho(XY) \leqslant 1.059016995\cdot\rho\cdot\omega_\rho(X)\cdot\omega_\rho(Y).$$

In the original proof of inequality (2.11) by Okubo and Ando, a key lemma is used. Specifically, for $\rho = 2$, this lemma can be stated as follows:

LEMMA 2.2. *If* $\omega(X) \leqslant 1$, *then for any unit vector* $h \in \mathscr{H}$ *and* $0 < \alpha < 1$, *the following inequality holds:*

$$\sum_{n=1}^{\infty} \alpha^{2n} \|X^n h\|^2 \leqslant \frac{\alpha}{1-\alpha}. \tag{2.17}$$

Assuming $\omega(X) \leqslant 1$ and unit vector $h \in \mathscr{H}$, following the same steps as in proof of Lemma (2.1), we can derive that for the sequence defined by

$$u'_n = \|X^n h\|^2,$$

there exists a parameter sequence g'_n such that

$$u'_n = 4^n (1 - g'_0) g'_n \prod_{i=1}^{n-1} (g'_i - g'^2_i), \tag{2.18}$$

where $0 \leqslant g'_i \leqslant 1$ for all $n \geqslant 0$.

Using this representation, we analyzed the tightness of the bound in inequality (2.17) through numerical methods with Python. Our findings suggest that the inequality can be improved as follows:

$$\sum_{n=1}^{\infty} \alpha^{2n} \|X^n h\|^2 \leqslant \begin{cases} 4\alpha^2 & \text{if } 0 < \alpha \leqslant \frac{1}{2}, \\ \frac{\alpha}{1-\alpha} & \text{if } \frac{1}{2} \leqslant \alpha < 1. \end{cases}$$

Following tables support our findings,

Table 2: *Test Table*

$0.00343225906 \leqslant \alpha \leqslant 0.5$	**Series Sum**	$4\alpha^2$	**As Expected?**
0.24051161	0.23138334	0.23138334	True
0.38118733	0.58121512	0.58121512	True
0.36190340	0.52389628	0.52389628	True
0.26749625	0.28621697	0.28621697	True
0.46393366	0.86093777	0.86093777	True
0.20486900	0.16788522	0.16788522	True
0.43967662	0.77326213	0.77326213	True
0.41576252	0.69143388	0.69143388	True
0.20849366	0.17387843	0.17387843	True
0.33823962	0.45762417	0.45762417	True
0.38794300	0.60199908	0.60199908	True
0.07194048	0.02070173	0.02070173	True
0.11395410	0.05194215	0.05194215	True
0.36482200	0.53238036	0.53238036	True
0.01652316	0.00109206	0.00109206	True
0.46767919	0.87489532	0.87489532	True
0.37831275	0.57248214	0.57248214	True
0.07017061	0.01969566	0.01969566	True

$0.00343225906 \leqslant \alpha \leqslant 0.5$	**Series Sum**	$4\alpha^2$	**As Expected?**
0.13061878	0.06824506	0.06824506	True
0.04614586	0.00851776	0.00851776	True
0.28573005	0.32656665	0.32656665	True
0.17095127	0.11689735	0.11689735	True
0.38294139	0.58657642	0.58657642	True
0.31046056	0.38554303	0.38554303	True
0.41336919	0.68349634	0.68349634	True
0.47583388	0.90567151	0.90567151	True
0.03457890	0.00478280	0.00478280	True
0.16143441	0.10424428	0.10424428	True
0.45171681	0.81619231	0.81619231	True
0.46336985	0.85884646	0.85884646	True
0.32248979	0.41599867	0.41599867	True
0.34969329	0.48914159	0.48914159	True
0.10387609	0.04316097	0.04316097,	True
0.10948238	0.04794557	0.04794557	True
0.19321618	0.14932997	0.14932997	True
0.38113089	0.58104304	0.58104304	True
0.32541982	0.42359223	0.42359223	True
0.36504165	0.53302163	0.53302163	True
0.26973637	0.29103083	0.29103083	True
0.15508306	0.09620302	0.09620302	True
0.24159587	0.23347425	0.23347425	True
0.46712215	0.87281242	0.87281242	True
0.48883043	0.95582076	0.95582076	True
0.22382601	0.20039233	0.20039233	True
0.27394054	0.30017369	0.30017369	True
0.39041474	0.60969468	0.60969468	True
0.46670269	0.87124561	0.87124561	True
0.34646233	0.48014459	0.48014459	True
0.29161768	0.34016349	0.34016349	True
0.22738305	0.20681221	0.20681221	True
0.44409390	0.78887759	0.78887759	True
0.26059595	0.27164099	0.27164099	True
0.15660941	0.09810603	0.09810603	True
0.31239568	0.39036424	0.39036424	True
0.08600828	0.02958970	0.02958970	True
0.32801411	0.43037303	0.43037303	True
0.13749697	0.07562166	0.07562166	True
0.49543824	0.98183621	0.98183621	True
0.44950687	0.80822570	0.80822570	True
0.37666229	0.56749794	0.56749794	True
0.25611900	0.26238777	0.26238777	True
0.34851828	0.48585996	0.48585996	True
0.16813088	0.11307198	0.11307198	True
0.06753800	0.01824552	0.01824552	True
0.26607187	0.28317696	0.28317696	True
0.00431775	0.00007457	0.00007457	True
0.27357349	0.29936982	0.29936982	True
0.19860958	0.15778307	0.15778307	True
0.44039045	0.77577501	0.77577501	True
0.17527064	0.12287920	0.12287920	True
0.43441363	0.75486080	0.75486080	True
0.03439848	0.00473302	0.00473302	True
0.01843283	0.00135908	0.00135908	True

$0.00343225906 \leqslant \alpha \leqslant 0.5$	**Series Sum**	$4\alpha^2$	**As Expected?**
0.38255163	0.58538301	0.58538301	True
0.01380817	0.00076266	0.00076266	True
0.38773836	0.60136414	0.60136414	True
0.28813292	0.33208232	0.33208232	True
0.28485917	0.32457898	0.32457898	True
0.03713988	0.00551748	0.00551748	True
0.45838368	0.84046241	0.84046241	True
0.31547522	0.39809845	0.39809845	True
0.43709564	0.76421040	0.76421040	True
0.48086671	0.92493117	0.92493117	True
0.31457569	0.39583146	0.39583146	True
0.05574448	0.01203773	0.01203773	True
0.38022120	0.57213541	0.57213541	True
0.38914565	0.59857457	0.59857457	True
0.35959738	0.50722736	0.50722736	True
0.14136481	0.07970272	0.07970272	True
0.11622696	0.05333858	0.05333858	True
0.49018446	0.97082557	0.97082557	True
0.33979629	0.45637082	0.45637082	True
0.31562763	0.39835637	0.39835637	True
0.46952740	0.88164335	0.88164335	True
0.15128138	0.09092746	0.09092746	True
0.43245744	0.73765126	0.73765126	True
0.14507583	0.08527364	0.08527364	True
0.11813130	0.05514687	0.05514687	True
0.09616832	0.03695685	0.03695685	True
0.14897922	0.08883151	0.08883151	True
0.15518318	0.09729289	0.09729289	True

Although for $0 \leqslant \alpha \leqslant 0.00343225959$, series sum comes out to be approximately $2.1\alpha^2$

Table 3: *Test Table*

$0 \leqslant \alpha \leqslant 0.00343225959$	**Series Sum**	**Threshold** $2.1\alpha^2$	$\leqslant$ **threshold?**
0.00087873625043655645	0.00000154435598817939	0.00000162157253544573	True
0.00004634800114383979	0.00000000429627442929	0.00000000451108814106	True
0.00120272030732632802	0.00000289307646024992	0.00000303772588907579	True
0.00194836683056748023	0.00000759252600432493	0.00000797187994355669	True
0.00316659334414559493	0.00002005643666263838	0.00002105735815509308	True
0.00157533077077603306	0.00000496334639208076	0.00000521150077844300	True
0.00137407648693596367	0.00000377617951365296	0.00000396498100309559	True
0.00209198607516713131	0.00000875315623024516	0.00000919045205125567	True
0.00201131857552461994	0.00000809109939968455	0.00000849534506572581	True
0.00187332860608887945	0.00000701894181381315	0.00000736965613942090	True
0.00070911840885987099	0.00000100569834128223	0.00000105598272734631	True
0.00025265898282280188	0.00000012767313135232	0.00000013405677936221	True
0.00334668908215602062	0.00002240291368022422	0.00002352068840650685	True
0.00313771720911532916	0.00001969228329855838	0.00002067506549719483	True
0.00300436562508790174	0.00001805389212520018	0.00001895504689934062	True
0.00284197366517258349	0.00001615480285684717	0.00001696131005842242	True
0.00251437289580099671	0.00001264486155326668	0.00001327634922419125	True
0.00197617610836324379	0.00000781081854334189	0.00000820107122365796	True
0.00058444494386253516	0.00000068315201816122	0.00000071730937405361	True
0.00264223586648743708	0.00001396369807061731	0.00001466096178572050	True

0.00343225906 $\leqslant \alpha \leqslant$ 0.5	**Series Sum**	**Threshold** $2.1\alpha^2$	$\leqslant$ **threshold?**
0.0024151668734934 6967	0.00001166667449055545	0.00001224936515632247	True
0.00203705204727470642	0.00000829947202916695	0.00000871412019094275	True
0.00235306722538189536	0.00001107440257043950	0.00001162754327104955	True
0.00146783940865705901	0.00000430911434342662	0.00000452456031217408	True
0.00296296914302385294	0.00001755975962033771	0.00001843629089927416	True
0.00081804436481177397	0.00000133839406124933	0.00000140531282388063	True
0.00028305185176536146	0.00000016023671441350	0.00000016824853665438	True
0.00204363340693891572	0.00000835318897140043	0.00000877051875410920	True
0.00267341458187440569	0.00001429521052429607	0.00001500900560581528	True
0.00089997648288242939	0.00000161991665154677	0.00000170091110645700	True
0.00339879187254646213	0.00002310597437660511	0.00002425875100506456	True
0.00329285690827449840	0.00002168792947452661	0.00002277010389857929	True
0.00043650187066946388	0.00000038106783880222	0.00000040012115450568	True
0.00179823954705546048	0.00000646751915591857	0.00000679069748404788	True
0.00013194314794904286	0.00000003481798918755	0.00000003655888801048	True
0.00015242172319302933	0.00000004646476448175	0.00000004878800157238	True
0.00055286566300816848	0.00000061132106952359	0.00000064188692680027	True
0.00087271243832118218	0.00000152325516015326	0.00000159941670000106	True
0.00104620162593921698	0.00000218907808026452	0.00000229852946844751	True
0.00014518818548808380	0.00000004215921929934	0.00000004426717933118	True
0.00180370541399931081	0.00000650689695857909	0.00000683204176302989	True
0.00076312720596965718	0.00000116472694327614	0.00000122296257823122	True
0.00304890843961255287	0.00001859324077632226	0.00001952126961359537	True
0.00182041840453854922	0.00000662804401280013	0.00000695923865192362	True
0.00201518096503078932	0.00000812220548807781	0.00000852800407582709	True
0.00136374393697783489	0.00000371960196899415	0.00000390557480385199	True
0.00119810229338223836	0.00000287090233184974	0.00000301444312135634	True
0.00282995559886670914	0.00001601845187631203	0.00001681816225226977	True
0.00145976185407219942	0.00000426181842273739	0.00000447489980826904	True
0.00116520543631045814	0.00000271541110433862	0.00000285117778849563	True
0.00116871352332871396	0.00000273178633054601	0.00000286837172918397	True
0.00023566164826458030	0.00000011107283109414	0.00000011662646617184	True
0.00173462805292285184	0.00000626287980582956	0.00000659018526078785	True
0.00259728379406664264	0.00001349451867273454	0.00001419613503537967	True
0.00024142097525221139	0.00000012227671162221	0.00000012720794872735	True
0.00063863694924764143	0.00000081877137449394	0.00000085994251720958	True
0.00154634330407432878	0.00000478572412846221	0.00000501743766247088	True
0.00328685577346705150	0.00002184303921787276	0.00002292333131035234	True
0.00170929117700879054	0.00000584037383435722	0.00000615969946421387	True
0.00016174008063085172	0.00000005229413248592	0.00000005524712150831	True
0.00107440628278891194	0.00000230567855976159	0.00000241177638084276	True
0.00207653561870861343	0.00000871376377567657	0.00000916113321737695	True
0.00072678737605774037	0.00000105893862252861	0.00000111284254270490	True
0.00004654939196808552	0.00000000432289237983	0.00000000453703760425	True
0.00324458335689237124	0.00002095345386160635	0.00002197991675143813	True
0.00217193041587425780	0.00000859001327558145	0.00000899231774328970	True
0.00322342466275262911	0.00002069538157584462	0.00002170545454485012	True
0.00019338607880201919	0.00000007455077362026	0.00000007867923486746	True
0.00010378324859109925	0.00000001959433979761	0.00000002048768291214	True
0.00135290441063351275	0.00000365537857930548	0.00000384219526419576	True
0.00201883138211603255	0.00000817142868009618	0.00000857807208822442	True
0.00065750317533504556	0.00000082598129025741	0.00000086821660708738	True
0.00038300589538207753	0.00000029211369092640	0.00000030627813147118	True
0.00309366167836692929	0.00001891184364768298	0.00001984947189756558	True
0.00101389756589758266	0.00000204532667315754	0.00000214796478333224	True

$0.00343225906 \leqslant \alpha \leqslant 0.5$	**Series Sum**	**Threshold** $2.1\alpha^2$	$\leqslant$ **threshold?**
0.0024347112952993892 3	0.000012429244363326 27	0.000013076185168616 33	True
0.0033957808303314287 6	0.000023062740874343 26	0.000024207319917014 25	True
0.0022457575224132607 3	0.000009973070436618 04	0.000010466276575501 85	True
0.0032852783581771592 6	0.000021818645348339 59	0.000022898704733650 73	True
0.0020194330906902235 3	0.000008163717799375 34	0.000008570877859586 96	True
0.0012959123460284240 9	0.000003485156669676 47	0.000003708946430271 46	True
0.0018066321171137853 4	0.000006603078058071 29	0.000006930169206322 09	True
0.0033614740195485904 2	0.000022620824679517 69	0.000023670501342651 53	True
0.0020970564738692162 5	0.000008813229112117 90	0.000009227467890208 95	True
0.0015200772604728311 9	0.000004607073184047 37	0.000004811425275253 24	True
0.0015662962763801300 5	0.000004897361077737 61	0.000005106468871736 73	True
0.0012872857862280927 6	0.000003533981124850 77	0.000003757791604256 09	True
0.0031533691746310607 9	0.000019764688583549 93	0.000020759430092867 80	True
0.0002582073735906058 1	0.000000133460509130 55	0.000000139564636644 52	True
0.0002863150577329586 1	0.000000163652788152 50	0.000000171279837507 28	True
0.0014925802719784525 7	0.000004449771226368 94	0.000004679409315922 35	True
0.0022096764234394736 5	0.000009738852628492 45	0.000010238263930716 46	True
0.0017378779537594480 9	0.000006284389606989 48	0.000006617529690882 72	True
0.0017806466200807272 4	0.000006346787019581 32	0.000006683801331237 22	True
0.0000262748150735156 8	0.000000001736860356 94	0.000000001830473072 18	True
0.0001041520706881471 4	0.000000019693178161 23	0.000000020585239059 48	True
0.0031154573647926369 1	0.000019278574576683 53	0.000020290224137018 84	True
0.0018636384429531958 2	0.000007013665815414 09	0.000007365075040969 95	True
0.0019154973064286360 5	0.000007323924253424 58	0.000007682186832938 96	True
0.0026213304765433292 3	0.000013805144434975 09	0.000014504930152842 88	True
0.0001325651815188487 2	0.000000035070542835 57	0.000000036957069279 95	True
0.0022188455617526225 3	0.000009782733171689 85	0.000010283306643382 64	True
0.0032480976424792080 5	0.000020997047488597 09	0.000022020551818688 20	True
0.0001609469543483398 5	0.000000051713631579 38	0.000000054576606087 52	True
0.0000343589300825696 7	0.000000001178771619 06	0.000000001238338556 65	True
0.0010866916688660216 3	0.000002358332859447 59	0.000002460063872689 99	True
0.0033435988332764536 9	0.000022469639248610 73	0.000023586631726278 16	True
0.0010146345737068257 6	0.000002051315913865 09	0.000002153507383646 17	True
0.0009899119558983498 9	0.000001979676651085 57	0.000002079277277146 85	True
0.0031159041731608584 8	0.000019282347460448 41	0.000020293637761598 42	True
0.0020319786411746462 5	0.000008285384684815 67	0.000008693129028649 18	True
0.0015036064214713364 8	0.000004528283261061 24	0.000004759822155949 09	True
0.0001174812732579538 1	0.000000027512743167 31	0.000000028715119758 42	True
0.0011892227042256429 7	0.000002690542357804 18	0.000002792862466494 68	True
0.0000550290896700139 8	0.000000003026071238 00	0.000000003178466758 72	True
0.0031431265080806725 3	0.000019605228418781 48	0.000020597377152643 69	True
0.0033423939546076055 7	0.000022466584054603 45	0.000023583209147124 20	True
0.0021552823051562772 1	0.000008678370862633 96	0.000009076420651929 29	True
0.0020193910995469270 0	0.000008160284632148 48	0.000008567565220815 54	True
0.0013632867869648182 0	0.000003721318911174 52	0.000003912556708826 09	True
0.0003860380284649416 7	0.000000300021357732 32	0.000000315628506070 91	True
0.0023968496719343790 9	0.000011604291616511 35	0.000012263866317974 85	True
0.0015951459422829349 6	0.000004797904317624 25	0.000005009879028587 88	True
0.0012750148985763731 5	0.000003489901499531 72	0.000003716388026565 04	True
0.0024926602998407284 6	0.000012279867621556 66	0.000012936559581785 09	True
0.0032367795092165400 7	0.000020568707459956 96	0.000021592856024684 14	True
0.0012216045796944313 2	0.000002907074624238 73	0.000003015736726779 93	True
0.0003130990502817414 5	0.000000182683411646 64	0.000000192232051106 13	True
0.0015290875743502807 7	0.000004746014302666 53	0.000004957545510146 67	True
0.0017117929193725975 6	0.000005866574058070 13	0.000006186572248516 99	True

$0.00343225906 \leqslant \alpha \leqslant 0.5$	**Series Sum**	**Threshold** $2.1\alpha^2$	$\leqslant$ **threshold?**
0.00212802433051948865	0.00000843978511585834	0.00000884570297937667	True
0.00007077117922975763	0.00000000501594412787	0.00000000525527578629	True
0.00024897274513327938	0.00000012219518664803	0.00000012711529455185	True
0.00337209048204228445	0.00002284699337774392	0.00002388314080421466	True
0.00116523740759197581	0.00000255585069293237	0.00000265967979974191	True
0.00320455991138778241	0.00002036832906513055	0.00002137048627388151	True
0.00249757041004892648	0.00001262124716633435	0.00001328318463627562	True
0.00060978188013682667	0.00000071091909651783	0.00000074770429687043	True
0.00156806958041711364	0.00000491446915112240	0.00000512447626478151	True
0.00316226137576048795	0.00001986609571106679	0.00002087258376439653	True
0.00004851028046326067	0.00000000527274537937	0.00000000553716008995	True
0.00207456386783961305	0.00000870850609900556	0.00000915575353312040	True
0.00312678215859728575	0.00001945638077137045	0.00002046437508404355	True
0.00115312721514670663	0.00000251854717160867	0.00000262153048543127	True
0.00011666234335784430	0.00000002733265135220	0.00000002853567938865	True
0.00077697492904601322	0.00000121783787485432	0.00000127050883231571	True
0.00114016589726995796	0.00000246836907502125	0.00000256927086881390	True
0.00324226359826158012	0.00002093627907374063	0.00002194926386893827	True
0.00147726251706815698	0.00000433714665853698	0.00000456734664670131	True
0.00107894286729982968	0.00000229073233064592	0.00000239571206084209	True
0.00043941584401445125	0.00000035338748589224	0.00000037187951375481	True
0.00315043964633714279	0.00001969974017041540	0.00002070254570893535	True
0.00033916128280522288	0.00000022141792437809	0.00000023149344536960	True
0.00001757030335162861	0.00000000020778080389	0.00000000021775153752	True
0.00018811256091176075	0.00000007180525704235	0.00000007579960545173	True
0.00112828397093486691	0.00000235080654992785	0.00000245064409148371	True
0.00056519595118770792	0.00000059049223894688	0.00000062182220504267	True
0.00174042071183406609	0.00000630016805441348	0.00000663440665459220	True
0.00323223218120527862	0.00002057085284756956	0.00002158445445255856	True
0.00012128472151683491	0.00000002862498941812	0.00000002984938672105	True
0.00179052488097846080	0.00000639991385806682	0.00000673845475777657	True
0.00019657495067136204	0.00000007802867937178	0.00000008224989009355	True
0.00333796002679302987	0.00002254514176306248	0.00002366321816390538	True
0.00121260353551751609	0.00000288876201515186	0.00000299462421270263	True
0.00313779511452367395	0.00001960653379656813	0.00002060545824841091	True
0.00029973749415427881	0.00000016593374129855	0.00000017386291834493	True
0.00070218403971528396	0.00000096408550189515	0.00000101573811642555	True
0.00241332251758496357	0.00001195266346400985	0.00001261015804669835	True
0.00122629645724732986	0.00000293009862094630	0.00000303773142531935	True

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REFERENCES

[1] KENNETH R. DAVIDSON AND JOHN A. R. HOLBROOK, *Numerical radii of zero-one matrices*, Michigan Math. J., no. 2, **35** (1988), 261–267.

[2] JOHN A. R. HOLBROOK, *Inequalities governing the operator radii associated with unitary ρ-dilations*, Michigan Math. J., **18** (1971), 149–159.

[3] JOHN A. R. HOLBROOK, *Multiplicative properties of the numerical radius in operator theory*, J. Reine Angew. Math., **237** (1969), 166–174.

[4] JOHN A. R. HOLBROOK, *On the power-bounded operators of Sz.-Nagy and Foiaş*, Acta Sci. Math. (Szeged), **29** (1968), 299–310.

[5] JOHN HOLBROOK AND JEAN-PIERRE SCHOCH, *Theory vs. experiment: multiplicative inequalities for the numerical radius of commuting matrices*, Oper. Theory Adv. Appl., **202** (2010), 273–284.

[6] FUAD KITTANEH AND ALI ZAMANI, *On the ρ-operator radii*, Linear Algebra Appl., **687** (2024), 132–156.

[7] VLADIMÍR MÜLLER, *The numerical radius of a commuting product*, Michigan Math. J., no. 2, **35** (1988), 255–260.

[8] K. OKUBO AND T. ANDO, *Operator radii of commuting products*, Proc. Amer. Math. Soc., **56** (1976), 203–210.

[9] BÉLA SZ.-NAGY, *Products of operators of classes C_ρ*, Rev. Roumaine Math. Pures Appl., **21** (1987), 13(1968), 897–899.

[10] BÉLA SZ.-NAGY AND CIPRIAN FOIAŞ, *On certain classes of power-bounded operators in Hilbert space*, Acta Sci. Math. (Szeged), **27** (1966), 17–25.

[11] H. S. WALL, *Analytic Theory of Continued Fractions*, D. Van Nostrand Co., Inc., New York, 1948, https://doi.org/10.1007/978-1-4612-0653-8.

[12] JAMES P. WILLIAMS, *Schwarz norms for operators*, Pacific J. Math., **24** (1968), 181–188.

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