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STRONG COMMUTATIVITY PRESERVING ADDITIVE MAPS ON INVERTIBLE TRIANGULAR MATRICES OVER $\mathbb{F}_2$

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Abstract. Let $T_n(\mathbb{F}_2)$ be the ring of $n\times n$ upper triangular matrices over the Galois field $\mathbb{F}_2$ of two elements. In this paper we characterize strong commutativity preserving additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on invertible matrices for $n=2$ and $n \geqslant 5$. This result completes a recent result obtained by Chooi et al. in [14] and yields a comprehensive structural characterization of strong commutativity preserving additive maps on rank k upper triangular matrices over division rings. Some irregular forms are included to exemplify the complexity in structure of strong commutativity preserving additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on invertible matrices for $n=3$ and 4.

1. Introduction and results

Let $\mathscr{R}$ be a ring and let $\mathscr{S}$ be a nonempty subset of $\mathscr{R}$. For any $x,y \in \mathscr{R}$, we denote by $[x,y] = xy - yx$ the commutator of x and y. A map $\psi : \mathscr{R} \to \mathscr{R}$ is called *commutativity preserving* on $\mathscr{S}$ if $[\psi(x),\psi(y)] = 0$ whenever $[x,y] = 0$ for all $x,y \in \mathscr{S}$. Watkins' pioneering result on commutativity preserving linear maps [33] inspired several analogous results for rings, matrix spaces and operator algebras; see [4, 5, 9, 12, 28, 32] and references therein. In particular, Bell and Mason [3] introduced the notion of strong commutativity preserving maps in 1992. A map $\psi : \mathscr{R} \to \mathscr{R}$ is said to be *strong commutativity preserving* on $\mathscr{S}$ if $[\psi(x),\psi(y)] = [x,y]$ for all $x,y \in \mathscr{S}$. A strong commutativity preserving map is commutativity preserving, but the converse is not true in general. Subsequently, Bell and Daif [2] proved that a semiprime ring $\mathscr{R}$ admitting a strong commutativity preserving derivation on $\mathscr{R}$ must necessarily be commutative. Following this, Brešar and Miers [8] characterized strong commutativity preserving additive maps $\psi : \mathscr{R} \to \mathscr{R}$ on semiprime rings $\mathscr{R}$ and proved that ψ is of the form

$$\psi(x) = \lambda x + \mu(x) \tag{1}$$

for all $x \in \mathscr{R}$, where λ is an element in the extended centroid $\mathscr{C}$ of $\mathscr{R}$ satisfying $\lambda^2 = 1$ and $\mu : \mathscr{R} \to \mathscr{C}$ is an additive map. The significance of this structural result has attracted considerable attention and there has been remarkable progress in the study

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of strong commutativity preserving maps on various rings and algebras, as evidenced in [1, 10, 11, 15, 18, 20–23, 25, 27, 30, 31, 34].

Let $n \geqslant 2$ be an integer and let $M_n(\mathbb{F})$ be the ring of $n \times n$ matrices over a field $\mathbb{F}$. Motivated by the elegant and astounding results obtained in addressing linear preserver problems [19, 29] on matrices, Franca [16, 17] initiated the study of functional identities [6, 7] on matrices, in particular, commuting additive maps on subsets of $M_n(\mathbb{F})$ that are not closed under addition such as invertible matrices, singular matrices and rank k matrices. In light of this, Liu [24] characterized strong commutativity preserving maps on the subset of invertible (respectively, singular) matrices of $M_n(\mathbb{D})$ over division rings $\mathbb{D}$ and showed that these maps conform to the standard form (1). In a later development, Liu et al. [26] extended this result to the set of all rank k matrices of $M_n(\mathbb{D})$ for some fixed integer $1 \leqslant k \leqslant n$.

Let $T_n(\mathbb{D})$ denote the ring of $n \times n$ upper triangular matrices over a division ring $\mathbb{D}$. Most recently, Chooi et al. [14] obtained a complete structural characterization of strong commutativity preserving additive maps $\psi : T_n(\mathbb{D}) \to T_n(\mathbb{D})$ on rank k matrices, where $1 \leqslant k \leqslant n$ is a fixed integer such that $k \neq n$ when $\mathbb{D}$ is the Galois field $\mathbb{F}_2$ of two elements. They showed that such additive maps ψ are of the standard form (1) when $\mathbb{D}$ is a noncommutative division ring, but there are some irregular nonstandard forms when $\mathbb{D}$ is a field. Inspired by these results, in this paper, we complete the study in [14] by addressing a characterization of strong commutativity preserving additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on invertible matrices, i.e., additive maps ψ satisfying $[\psi(A), \psi(B)] = [A, B]$ for all invertible matrices $A, B \in T_n(\mathbb{F}_2)$, in Theorem 1.1 for $n \geqslant 5$ and Theorem 1.2 for $n = 2$. It is worth pointing out that the structural result of these maps is quite different from the result in [14]. For instance, the following additive map

$$A \mapsto A + \sum_{i=1}^{n} a_{ii} X_i$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$, where $X_1, \ldots, X_n \in T_n(\mathbb{F}_2)$ are some fixed matrices such that $X_1 + \cdots + X_n = 0$, is strong commutativity preserving on invertible triangular matrices over $\mathbb{F}_2$. Furthermore, there are many irregular forms of strong commutativity preserving additive maps $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ on invertible matrices for both $n = 3$ and $n = 4$. We include a selection of irregular forms for $n = 3$ and $n = 4$ in Examples 2.2 and 2.3 below. Because of the complexity of these mappings, a full characterization of strong commutativity preserving maps on invertible matrices may be an intractable problem for $n = 3$ and $n = 4$.

Here and subsequently, $\mathbb{F}_2$ is the Galois field of two elements, I_n is the $n \times n$ identity matrix and $E_{ij} \in T_n(\mathbb{F}_2)$ denotes the standard matrix unit whose (i, j)th entry is one and zero elsewhere. One sees immediately that $E_{ij}E_{st} = \delta_{js}E_{it}$ for $E_{ij}, E_{st} \in T_n(\mathbb{F}_2)$, where δ_{js} is the Kronecker delta, and $[A, B] = [B, A]$ for all $A, B \in T_n(\mathbb{F}_2)$.

We are now ready to present the main results in the study.

THEOREM 1.1. *Let $n \geqslant 5$ be an integer. Then $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is a strong commutativity preserving additive map on invertible matrices if and only if there exist scalars $\alpha, \beta, \gamma \in \mathbb{F}_2$, additive maps $\mu, \eta : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ and matrices $X_1, \ldots, X_n \in$*

$T_n(\mathbb{F}_2)$ *satisfying* $X_1+\cdots+X_n=0$ *such that*

$$\psi(A)=A+\mu(A)I_n+\eta(A)E_{1n}+\sum_{i=1}^{n}a_{ii}X_i+\Phi_{\alpha,\beta,\gamma}(A)$$

for all $A=(a_{ij})\in T_n(\mathbb{F}_2)$, *where* $\Phi_{\alpha,\beta,\gamma}:T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ *is the additive map defined by*

$$\Phi_{\alpha,\beta,\gamma}(A)=(\gamma a_{12}+\alpha a_{n-1,n})E_{1,n-1}+(\beta a_{12}+\gamma a_{n-1,n})E_{2n}$$

for all $A=(a_{ij})\in T_n(\mathbb{F}_2)$.

THEOREM 1.2. *An additive map* $\psi:T_2(\mathbb{F}_2)\to T_2(\mathbb{F}_2)$ *is strong commutativity preserving on invertible matrices if and only if there exists an additive map* $\mu:T_2(\mathbb{F}_2)\to\mathbb{F}_2$ *and matrices* $X_1,X_2\in T_2(\mathbb{F}_2)$ *satisfying* $X_1+X_2=0$ *such that either*

$$\psi(A)=A+\mu(A)I_2+a_{11}X_1+a_{22}X_2+\begin{pmatrix}0&\alpha a_{11}+a_{12}\\0&\beta a_{11}+(\alpha+1)\gamma a_{12}\end{pmatrix}$$

for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$, *where* $\alpha,\beta,\gamma\in\mathbb{F}_2$ *are some fixed scalars, or*

$$\psi(A)=A+\mu(A)I_2+a_{11}X_1+a_{22}X_2+\begin{pmatrix}0&\alpha a_{11}\\0&\beta(\alpha a_{11}+a_{12})\end{pmatrix}$$

for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$, *where* $\alpha,\beta\in\mathbb{F}_2$ *are some fixed scalars.*

2. Examples

In this section, we address some examples of strong commutativity preserving additive maps $\psi:T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ on invertible matrices.

EXAMPLE 2.1. Let $n\geqslant 3$ be an integer. Let $\psi:T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ be the additive map defined by

$$\psi(A)=A+\mu(A)I_n+\eta(A)E_{1n}+\sum_{i=1}^{n}a_{ii}X_i+\Phi_{\alpha,\beta,\gamma}(A)$$

for all $A=(a_{ij})\in T_n(\mathbb{F}_2)$, where $\mu,\eta:T_n(\mathbb{F}_2)\to\mathbb{F}_2$ are additive maps, $X_1,\ldots,X_n\in T_n(\mathbb{F}_2)$ are matrices satisfying $X_1+\cdots+X_n=0$, and $\Phi_{\alpha,\beta,\gamma}:T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ is the additive map defined by

$$\Phi_{\alpha,\beta,\gamma}(A)=(\gamma a_{12}+\alpha a_{n-1,n})E_{1,n-1}+(\beta a_{12}+\gamma a_{n-1,n})E_{2n}$$

for all $A=(a_{ij})\in T_n(\mathbb{F}_2)$, where $\alpha,\beta,\gamma\in\mathbb{F}_2$ are fixed scalars in which $\alpha\beta=\gamma^2$ when $n=3$. We will show that ψ is a strong commutativity preserving additive map on invertible matrices. Let $A=(a_{ij})$, $B=(b_{ij})\in T_n(\mathbb{F}_2)$ be invertible. Then $a_{ii}=1=b_{ii}$ for $i=1,\ldots,n$, and thus $\sum_{i=1}^n a_{ii}X_i=0=\sum_{i=1}^n b_{ii}X_i$ and

$$\begin{aligned}[\psi(A),\psi(B)]=&[A+\mu(A)I_n+\eta(A)E_{1n}+\Phi_{\alpha,\beta,\gamma}(A),\\ &\quad B+\mu(B)I_n+\eta(B)E_{1n}+\Phi_{\alpha,\beta,\gamma}(B)]\\ =&[A,B]+[A,\Phi_{\alpha,\beta,\gamma}(B)]+[\Phi_{\alpha,\beta,\gamma}(A),B]+[\Phi_{\alpha,\beta,\gamma}(A),\Phi_{\alpha,\beta,\gamma}(B)].\end{aligned}$$

We note that $[\Phi_{\alpha,\beta,\gamma}(A),\Phi_{\alpha,\beta,\gamma}(B)]=(\alpha\beta+\gamma^2)(a_{12}b_{23}+b_{12}a_{23})E_{13}=0$ when $n=3$, and $[\Phi_{\alpha,\beta,\gamma}(A),\Phi_{\alpha,\beta,\gamma}(B)]=0$ when $n\geqslant 4$. Also, one easily sees that

$$[A,\Phi_{\alpha,\beta,\gamma}(B)]=(a_{12}(\gamma b_{n-1,n}+\beta b_{12})+(\gamma b_{12}+\alpha b_{n-1,n})a_{n-1,n})E_{1n},$$

$$[\Phi_{\alpha,\beta,\gamma}(A),B]=((\gamma a_{12}+\alpha a_{n-1,n})b_{n-1,n}+b_{12}(\gamma a_{n-1,n}+\beta a_{12}))E_{1n}.$$

It follows that $[\psi(A),\psi(B)]=[A,B]$ for all invertible matrices $A,B\in T_n(\mathbb{F}_2)$ as claimed.

We now list a selection of some irregular forms of strong commutativity preserving additive maps $\psi:T_4(\mathbb{F}_2)\to T_4(\mathbb{F}_2)$ on invertible matrices.

EXAMPLE 2.2. Let $\mu:T_4(\mathbb{F}_2)\to\mathbb{F}_2$ be an additive map. Let $X_1,X_2,X_3,X_4\in T_4(\mathbb{F}_2)$ be such that $\sum_{i=1}^4 X_i=0$. Consider the additive map $\psi:T_4(\mathbb{F}_2)\to T_4(\mathbb{F}_2)$ defined by

$$\psi(A)=A+\mu(A)I_4+\sum_{i=1}^4 a_{ii}X_i$$

for all $A=(a_{ij})\in T_4(\mathbb{F}_2)$. In view of Example 2.1, we see that ψ is a strong commutativity preserving additive map on invertible matrices. Let $\eta:T_4(\mathbb{F}_2)\to\mathbb{F}_2$ be an additive map and $\alpha,\beta,\gamma\in\mathbb{F}_2$. For each $i=1,2,3,4,5$, let $\psi_i:T_4(\mathbb{F}_2)\to T_4(\mathbb{F}_2)$ be the additive map defined by

$$\psi_1(A)=\psi(A)+\begin{pmatrix}0&0&\gamma a_{12}+\alpha a_{34}&\eta(A)\\0&0&0&\beta a_{12}+\gamma a_{34}\\0&0&0&0\\0&0&0&0\end{pmatrix}\text{ for all }A=(a_{ij})\in T_4(\mathbb{F}_2),$$

$$\psi_2(A)=\psi(A)+\begin{pmatrix}0&a_{13}&0&a_{14}\\0&0&0&0\\0&0&a_{23}&a_{24}\\0&0&0&a_{23}\end{pmatrix}\text{ for all }A=(a_{ij})\in T_4(\mathbb{F}_2),$$

$$\psi_3(A)=\psi_2(A)+\begin{pmatrix}0&a_{12}&a_{12}+a_{13}&0\\0&0&a_{23}&a_{24}+a_{34}\\0&0&0&a_{34}\\0&0&0&0\end{pmatrix}\text{ for all }A=(a_{ij})\in T_4(\mathbb{F}_2),$$

$$\psi_4(A)=\psi_2(A)+\begin{pmatrix}0&0&a_{12}+a_{13}&0\\0&0&a_{23}&a_{24}+a_{34}\\0&0&0&0\\0&0&0&0\end{pmatrix}\text{ for all }A=(a_{ij})\in T_4(\mathbb{F}_2),$$

$$\psi_5(A) = \psi_2(A) + \begin{pmatrix} 0 & a_{12} & a_{12} & 0 \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ for all } A = (a_{ij}) \in T_4(\mathbb{F}_2).$$

Firstly, by Example 2.1, ψ_1 is a strong commutativity preserving additive map on invertible matrices. Next, we claim that ψ_2 is strong commutativity preserving on invertible matrices. We set

$$\phi(X) = \begin{pmatrix} 0 & x_{13} & 0 & x_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & x_{23} & x_{24} \\ 0 & 0 & 0 & x_{23} \end{pmatrix} \quad \text{for all } X = (x_{ij}) \in T_4(\mathbb{F}_2).$$

Let $A = (a_{ij})$, $B = (b_{ij}) \in T_4(\mathbb{F}_2)$ be invertible. Then $a_{ii} = 1 = b_{ii}$, $i = 1, \ldots, 4$. We see that $[\phi(A), \phi(B)] = (a_{14}b_{23} + b_{14}a_{23})E_{14}$, and

$$[A, \phi(B)] = \begin{pmatrix} 0 & 0 & a_{13}b_{23} + b_{13}a_{23} & a_{13}b_{24} + b_{13}a_{24} + a_{14}b_{23} \\ 0 & 0 & a_{23}b_{23} & a_{23}b_{24} + a_{24}b_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[\phi(A), B] = \begin{pmatrix} 0 & 0 & a_{13}b_{23} + b_{13}a_{23} & a_{13}b_{24} + b_{13}a_{24} + b_{14}a_{23} \\ 0 & 0 & b_{23}a_{23} & b_{23}a_{24} + b_{24}a_{23} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It follows that $[\psi_2(A), \psi_2(B)] = [\psi(A) + \phi(A), \psi(B) + \phi(B)] = [A, B] + [A, \phi(B)] + [\phi(A), B] + [\phi(A), \phi(B)] = [A, B]$ for all invertible matrices $A, B \in T_4(\mathbb{F}_2)$.

We show that ψ_3 is a strong commutativity preserving additive map on invertible matrices. Set

$$\varphi(X) = \begin{pmatrix} 0 & x_{12} + x_{13} & x_{12} + x_{13} & x_{14} \\ 0 & 0 & x_{23} & x_{24} + x_{34} \\ 0 & 0 & x_{23} & x_{24} + x_{34} \\ 0 & 0 & 0 & x_{23} \end{pmatrix}$$

for all $X = (x_{ij}) \in T_4(\mathbb{F}_2)$. Let $A = (a_{ij})$, $B = (b_{ij}) \in T_4(\mathbb{F}_2)$ be invertible matrices. We see that $[\varphi(A), \varphi(B)] = (a_{14}b_{23} + b_{14}a_{23})E_{14}$, and

$$[A, \varphi(B)] = \begin{pmatrix} 0 & 0 & \begin{matrix}(a_{12} + a_{13})b_{23} \\ +(b_{12} + b_{13})a_{23}\end{matrix} & \begin{matrix}(a_{12} + a_{13})(b_{24} + b_{34}) + \\ (b_{12} + b_{13})(a_{24} + a_{34}) + a_{14}b_{23}\end{matrix} \\ 0 & 0 & a_{23}b_{23} & a_{23}(b_{24} + b_{34}) + b_{23}(a_{24} + a_{34}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$[\varphi(A), B] = \begin{pmatrix} 0 & 0 & \begin{matrix}(b_{12} + b_{13})a_{23} \\ +(a_{12} + a_{13})b_{23}\end{matrix} & \begin{matrix}(b_{12} + b_{13})(a_{24} + a_{34}) + \\ (a_{12} + a_{13})(b_{24} + b_{34}) + b_{14}a_{23}\end{matrix} \\ 0 & 0 & b_{23}a_{23} & b_{23}(a_{24} + a_{34}) + a_{23}(b_{24} + b_{34}) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $[\psi_3(A), \psi_3(B)] = [\psi(A) + \varphi(A), \psi(B) + \varphi(B)] = [A, B]$ for all invertible matrices $A, B \in T_4(\mathbb{F}_2)$. Likewise, it can be verified by direct computations that ψ_4 and ψ_5 are strong commutativity preserving maps on invertible matrices.

The following example provides a selection of some irregular forms of strong commutativity preserving additive maps $\psi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ on invertible matrices.

EXAMPLE 2.3. Let $\mu : T_3(\mathbb{F}_2) \to \mathbb{F}_2$ be an additive map and let $X_1, X_2, X_3 \in T_3(\mathbb{F}_2)$ be such that $X_1 + X_2 + X_3 = 0$. We define the additive map $\psi : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ by

$$\psi(A) = A + \mu(A)I_3 + \sum_{i=1}^{3} a_{ii}X_i$$

for all $A = (a_{ij}) \in T_3(\mathbb{F}_2)$. For each $i = 1, \ldots, 6$, let $\psi_i : T_3(\mathbb{F}_2) \to T_3(\mathbb{F}_2)$ be the additive map defined by

$$\psi_1(A) = \psi(A) + \begin{pmatrix} 0 & \gamma a_{12} + \alpha a_{23} & \eta(A) \\ 0 & 0 & \beta a_{12} + \gamma a_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{for all } A = (a_{ij}) \in T_3(\mathbb{F}_2),$$

where $\eta : T_3(\mathbb{F}_2) \to \mathbb{F}_2$ is an additive map and $\alpha, \beta, \gamma \in \mathbb{F}_2$ are scalars satisfying $\alpha\beta = \gamma^2$, and

$$\psi_2(A) = \psi(A) + \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & a_{23} \end{pmatrix} \quad \text{for all } A = (a_{ij}) \in T_3(\mathbb{F}_2),$$

$$\psi_3(A) = \psi(A) + \begin{pmatrix} 0 & 0 & a_{13} \\ 0 & a_{12} & 0 \\ 0 & 0 & a_{12} \end{pmatrix} \quad \text{for all } A = (a_{ij}) \in T_3(\mathbb{F}_2),$$

$$\psi_4(A) = \psi(A) + \begin{pmatrix} 0 & 0 & a_{13} + a_{23} \\ 0 & 0 & a_{23} \\ 0 & 0 & a_{12} \end{pmatrix} \quad \text{for all } A = (a_{ij}) \in T_3(\mathbb{F}_2),$$

$$\psi_5(A) = \psi(A) + \begin{pmatrix} 0 & a_{23} & a_{13} \\ 0 & a_{12} + a_{23} & 0 \\ 0 & 0 & a_{12} + a_{23} \end{pmatrix} \quad \text{for all } A = (a_{ij}) \in T_3(\mathbb{F}_2),$$

$$\psi_6(A) = \psi(A) + \begin{pmatrix} 0 & a_{12} + a_{13} & a_{13} + a_{23} \\ 0 & a_{13} & a_{23} \\ 0 & 0 & a_{12} \end{pmatrix} \quad \text{for all } A = (a_{ij}) \in T_3(\mathbb{F}_2).$$

Since $\alpha\beta = \gamma^2$, it follows from Example 2.1 that ψ_1 is a strong commutativity preserving map on invertible matrices. We now claim that ψ_2 is strong commutativity preserving on invertible matrices. We set

$$\phi(X) = \begin{pmatrix} 0 & 0 & x_{13} \\ 0 & 0 & 0 \\ 0 & 0 & x_{23} \end{pmatrix}$$

for every $X=(x_{ij})\in T_3(\mathbb{F}_2)$. Let $A=(a_{ij})$, $B=(b_{ij})\in T_3(\mathbb{F}_2)$ be invertible. We see that $[\psi(A),\psi(B)]=[A,B]$, $[\phi(A),\phi(B)]=(a_{13}b_{23}+b_{13}a_{23})E_{13}$,

$$[A,\phi(B)]=\begin{pmatrix}0&0&a_{13}b_{23}\\0&0&a_{23}b_{23}\\0&0&0\end{pmatrix},$$

$$[\phi(A),B]=\begin{pmatrix}0&0&b_{13}a_{23}\\0&0&b_{23}a_{23}\\0&0&0\end{pmatrix}.$$

Then $[\psi_2(A),\psi_2(B)]=[\psi(A)+\phi(A),\psi(B)+\phi(B)]=[A,B]+[A,\phi(B)]+[\phi(A),B]+[\phi(A),\phi(B)]=[A,B]$ for all invertible matrices $A,B\in T_3(\mathbb{F}_2)$.

Next, we show that ψ_6 is a strong commutativity preserving map on invertible matrices. Set

$$\varphi(X)=\begin{pmatrix}0&x_{12}+x_{13}&x_{13}+x_{23}\\0&x_{13}&x_{23}\\0&0&x_{12}\end{pmatrix}$$

for all $X=(x_{ij})\in T_3(\mathbb{F}_2)$. Let $A=(a_{ij})$, $B=(b_{ij})\in T_3(\mathbb{F}_2)$ be invertible. A direct verification gives

$$\begin{aligned}[\varphi(A),\varphi(B)]&=\begin{pmatrix}0&a_{12}b_{13}+b_{12}a_{13}&a_{13}(b_{12}+b_{23})+b_{13}(a_{12}+a_{23})\\0&0&a_{23}(b_{12}+b_{13})+b_{23}(a_{12}+a_{13})\\0&0&0\end{pmatrix}\\&=[A,\varphi(B)]+[\varphi(A),B].\end{aligned}$$

Thus $[\psi_6(A),\psi_6(B)]=[\psi(A)+\varphi(A),\psi(B)+\varphi(B)]=[A,B]$ for all invertible matrices $A,B\in T_3(\mathbb{F}_2)$. In the same manner we can verify that ψ_3, ψ_4 and ψ_5 are strong commutativity preserving maps on invertible matrices.

3. Proofs

We begin with the following observation.

LEMMA 3.1. *Let $n\geqslant 2$ be an integer. Then $\psi: T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ is an additive map such that $\psi(A)=A$ for all invertible matrices $A\in T_n(\mathbb{F}_2)$ if and only if there exist $X_1,\dots,X_n\in T_n(\mathbb{F}_2)$ satisfying $X_1+\cdots+X_n=0$ such that*

$$\psi(A)=A+\sum_{i=1}^{n}a_{ii}X_i$$

for all $A=(a_{ij})\in T_n(\mathbb{F}_2)$.

Proof. For the sufficiency, let $A=(a_{ij})$, $B=(b_{ij})\in T_n(\mathbb{F}_2)$. Then

$$\psi(A+B)=(A+B)+\sum_{i=1}^{n}(a_{ii}+b_{ii})X_i=\psi(A)+\psi(B).$$

Hence ψ is an additive map. Let $A=(a_{ij})\in T_n(\mathbb{F}_2)$ be invertible. Then $a_{ii}=1$ for all i, and thus $\psi(A)=A+\sum_{i=1}^n X_i=A$ as desired.

We consider the necessity. Let $\phi: T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ be the additive map defined by

$$\phi(A)=\psi(A)+A$$

for all $A\in T_n(\mathbb{F}_2)$. Let $B\in T_n(\mathbb{F}_2)$ be invertible. Then $\phi(B)=0$, and so $[\phi(B),B]=0$. We infer that ϕ is a commuting additive map on invertible matrices that vanishes on invertible matrices. By [13, Lemma 2.6], there exist $X_1,\dots,X_n\in T_n(\mathbb{F}_2)$ satisfying $X_1+\cdots+X_n=0$ such that $\phi(A)=\sum_{i=1}^n a_{ii}X_i$ for all $A=(a_{ij})\in T_n(\mathbb{F}_2)$, which completes the proof. □

We first characterize strong commutativity preserving additive maps $\psi: T_2(\mathbb{F}_2)\to T_2(\mathbb{F}_2)$ on invertible matrices.

Proof of Theorem 1.2. Note that if $H,K\in T_2(\mathbb{F}_2)$ are invertible, then $H=I_2+aE_{12}$ and $K=I_2+bE_{12}$ for some $a,b\in\mathbb{F}_2$. Thus $[H,K]=0$. We prove the sufficiency. Let $\mu: T_2(\mathbb{F}_2)\to\mathbb{F}_2$ be an additive map and let $X_1,X_2\in T_2(\mathbb{F}_2)$ be matrices satisfying $X_1+X_2=0$. We first consider the additive map $\psi: T_2(\mathbb{F}_2)\to T_2(\mathbb{F}_2)$ of the form

$$\psi(A)=A+\mu(A)I_2+a_{11}X_1+a_{22}X_2+\begin{pmatrix}0 & \alpha a_{11}+a_{12}\\ 0 & \beta a_{11}+(\alpha+1)\gamma a_{12}\end{pmatrix} \tag{2}$$

for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$, where $\alpha,\beta,\gamma\in\mathbb{F}_2$ are some fixed scalars. Let $A,B\in T_2(\mathbb{F}_2)$ be invertible. Then $A=I_2+aE_{12}$ and $B=I_2+bE_{12}$ for some $a,b\in\mathbb{F}_2$. It follows that $\psi(A)=A+\mu(A)I_2+(\alpha+a)E_{12}+(\beta+(\alpha+1)\gamma a)E_{22}$ by (2). Since $[xE_{12},X]=0$ for any $x\in\mathbb{F}_2$ and invertible matrix $X\in T_2(\mathbb{F}_2)$,

$$\begin{aligned}[\psi(A),\psi(B)] = {} & [A+(\beta+(\alpha+1)\gamma a)E_{22}, B+(\beta+(\alpha+1)\gamma b)E_{22}]\\ & +[(\beta+(\alpha+1)\gamma a)E_{22},(\alpha+b)E_{12}]\\ & +[(\alpha+a)E_{12},(\beta+(\alpha+1)\gamma b)E_{22}].\end{aligned}$$

Since $[A,B]=0$, $[(\beta+(\alpha+1)\gamma a)E_{22},(\alpha+b)E_{12}]+[(\alpha+a)E_{12},(\beta+(\alpha+1)\gamma b)E_{22}]=\beta(a+b)E_{12}$, $[A,(\beta+(\alpha+1)\gamma b)E_{22}]=a(\beta+(\alpha+1)\gamma b)E_{12}$ and $[(\beta+(\alpha+1)\gamma a)E_{22},B]=b(\beta+(\alpha+1)\gamma a)E_{12}$, it follows that

$$[\psi(A),\psi(B)]=\beta(a+b)E_{12}+a(\beta+(\alpha+1)\gamma b)E_{12}+b(\beta+(\alpha+1)\gamma a)E_{12}=0.$$

Then $[\psi(A),\psi(B)]=0=[A,B]$ for all invertible matrices $A\in T_2(\mathbb{F}_2)$ when ψ is of form (2).

We consider the additive map $\psi: T_2(\mathbb{F}_2)\to T_2(\mathbb{F}_2)$ of the form

$$\psi(A)=A+\mu(A)I_2+a_{11}X_1+a_{22}X_2+\begin{pmatrix}0 & \alpha a_{11}\\ 0 & \beta(\alpha a_{11}+a_{12})\end{pmatrix} \tag{3}$$

for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$, where $\alpha,\beta\in\mathbb{F}_2$ are some fixed scalars. Likewise, if $A=I_2+aE_{12}$ and $B=I_2+bE_{12}$ for some $a,b\in\mathbb{F}_2$, then

$$[\psi(A),\psi(B)]=[A+(\alpha\beta+\beta a)E_{22},B+(\alpha\beta+\beta b)E_{22}]+\alpha\beta(a+b)E_{12}.$$

Since $[A,(\alpha\beta+\beta b)E_{22}]=a(\alpha\beta+\beta b)E_{12}$ and $[(\alpha\beta+\beta a)E_{22},B]=b(\alpha\beta+\beta a)E_{12}$, we have

$$[\psi(A),\psi(B)]=a(\alpha\beta+\beta b)E_{12}+b(\alpha\beta+\beta a)E_{12}+\alpha\beta(a+b)E_{12}=0=[A,B].$$

Hence ψ is strong commutativity preserving on invertible matrices when ψ is of form (3).

For the necessity, we denote

$$\psi(I_2)=\begin{pmatrix} f_{11} & f_{12} \\ 0 & f_{22} \end{pmatrix}, \tag{4}$$

$$\psi(E_{12})=\begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix} \tag{5}$$

for some fixed scalars $f_{ij},g_{ij}\in\mathbb{F}_2$, $1\leqslant i\leqslant j\leqslant 2$. Since ψ is strong commutativity preserving on invertible matrices, it follows that $[\psi(I_2),\psi(I_2+E_{12})]=[I_2,I_2+E_{12}]=0$, which in turn gives $[\psi(I_2),\psi(E_{12})]=0$. By (4) and (5),

$$(f_{11}+f_{22})g_{12}=f_{12}(g_{11}+g_{22}). \tag{6}$$

We argue in the following two cases:

Case 1: $g_{12}=0$. We first consider $f_{12}=1$. Then $g_{11}=g_{22}$ by (6). Let $\beta_1=f_{11}+f_{22}\in\mathbb{F}_2$. By (4) and (5), $\psi(I_2)=f_{11}I_2+\beta_1E_{22}+E_{12}$ and $\psi(E_{12})=g_{11}I_2$. So, for $A=(a_{ij})\in\{I_2,E_{12}\}$,

$$\begin{aligned}\psi(A)&=a_{11}(f_{11}I_2+\beta_1E_{22}+E_{12})+a_{12}g_{11}I_2\\&=A+(a_{11}f_{11}+a_{12}g_{11}+a_{11})I_2+\begin{pmatrix} 0 & a_{11}+a_{12} \\ 0 & \beta_1a_{11} \end{pmatrix}.\end{aligned}$$

Let $\mu_1:T_2(\mathbb{F}_2)\to\mathbb{F}_2$ be the additive map defined by $\mu_1(A)=a_{11}f_{11}+a_{12}g_{11}+a_{11}$ for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$. Then, by the additivity of ψ, we have

$$\psi(A)=A+\mu_1(A)I_2+\begin{pmatrix} 0 & a_{11}+a_{12} \\ 0 & \beta_1a_{11} \end{pmatrix} \tag{7}$$

for all invertible matrices $A=(a_{ij})\in T_2(\mathbb{F}_2)$. Consider now $f_{12}=0$. Let $\gamma_1=g_{11}+g_{22}\in\mathbb{F}_2$. By (4) and (5), we get $\psi(E_{12})=g_{11}I_2+\gamma_1E_{22}$ and $\psi(I_2)=f_{11}I_2+\beta_1E_{22}$, with $\beta_1=f_{11}+f_{22}$. For $A=(a_{ij})\in\{I_2,E_{12}\}$,

$$\begin{aligned}\psi(A)&=a_{11}(f_{11}I_2+\beta_1E_{22})+a_{12}(g_{11}I_2+\gamma_1E_{22})\\&=A+(a_{11}f_{11}+a_{12}g_{11}+a_{11})I_2+\begin{pmatrix} 0 & a_{12} \\ 0 & \beta_1a_{11}+\gamma_1a_{12} \end{pmatrix}.\end{aligned}$$

By the additivity of ψ, we have

$$\psi(A)=A+\mu_1(A)I_2+\begin{pmatrix} 0 & a_{12} \\ 0 & \beta_1a_{11}+\gamma_1a_{12} \end{pmatrix} \tag{8}$$

for all invertible matrices $A=(a_{ij})\in T_2(\mathbb{F}_2)$. Together with (7) and (8), there exists $\alpha_1\in\mathbb{F}_2$ such that

$$\psi(A)=A+\mu_1(A)I_2+\begin{pmatrix}0 & \alpha_1a_{11}+a_{12}\\ 0 & \beta_1a_{11}+(\alpha_1+1)\gamma_1a_{12}\end{pmatrix}$$

for all invertible matrices $A=(a_{ij})\in T_2(\mathbb{F}_2)$. Let $\varphi: T_2(\mathbb{F}_2)\to T_2(\mathbb{F}_2)$ be the additive map defined by

$$\varphi(A)=\psi(A)+\mu_1(A)I_2+\begin{pmatrix}0 & \alpha_1a_{11}+a_{12}\\ 0 & \beta_1a_{11}+(\alpha_1+1)\gamma_1a_{12}\end{pmatrix} \tag{9}$$

for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$. Then $\varphi(A)=A$ for all invertible matrices $A\in T_2(\mathbb{F}_2)$. It follows from Lemma 3.1, there exist $X_1,X_2\in T_2(\mathbb{F}_2)$, with $X_1+X_2=0$, such that $\varphi(A)=A+a_{11}X_1+a_{22}X_2$ for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$. By (9), we thus obtain

$$\psi(A)=A+\mu_1(A)I_2+a_{11}X_1+a_{22}X_2+\begin{pmatrix}0 & \alpha_1a_{11}+a_{12}\\ 0 & \beta_1a_{11}+(\alpha_1+1)\gamma_1a_{12}\end{pmatrix}$$

for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$.

Case 2: $g_{12}=1$. Let $\alpha_2=f_{12}\in\mathbb{F}_2$ and $\beta_2=g_{11}+g_{22}\in\mathbb{F}_2$. Then $f_{11}=f_{22}+\alpha_2\beta_2$ by (6). In view of (4) and (5), $\psi(I_2)=(f_{22}+\alpha_2\beta_2)I_2+\alpha_2E_{12}+\alpha_2\beta_2E_{22}$ and $\psi(E_{12})=g_{11}I_2+E_{12}+\beta_2E_{22}$. For $A=(a_{ij})\in\{I_2,E_{12}\}$,

$$\begin{aligned}\psi(A)&=a_{11}((f_{22}+\alpha_2\beta_2)I_2+\alpha_2E_{12}+\alpha_2\beta_2E_{22})+a_{12}(g_{11}I_2+E_{12}+\beta_2E_{22})\\&=A+(a_{11}(1+f_{22}+\alpha_2\beta_2)+a_{12}g_{11})I_2+\begin{pmatrix}0 & \alpha_2a_{11}\\ 0 & \beta_2(\alpha_2a_{11}+a_{12})\end{pmatrix}.\end{aligned}$$

Let $\mu_2: T_2(\mathbb{F}_2)\to\mathbb{F}_2$ be the additive map defined by $\mu_2(A)=a_{11}(1+f_{22}+\alpha_2\beta_2)+a_{12}g_{11}$ for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$. By the additivity of ψ, we have

$$\psi(A)=A+\mu_2(A)I_2+\begin{pmatrix}0 & \alpha_2a_{11}\\ 0 & \beta_2(\alpha_2a_{11}+a_{12})\end{pmatrix}$$

for all invertible matrices $A=(a_{ij})\in T_2(\mathbb{F}_2)$. We now apply a similar argument as in Case 1 and use Lemma 3.1 to obtain matrices $Y_1,Y_2\in T_2(\mathbb{F}_2)$, with $Y_1+Y_2=0$, such that

$$\psi(A)=A+\mu_2(A)I_2+a_{11}Y_1+a_{22}Y_2+\begin{pmatrix}0 & \alpha_2a_{11}\\ 0 & \beta_2(\alpha_2a_{11}+a_{12})\end{pmatrix}$$

for all $A=(a_{ij})\in T_2(\mathbb{F}_2)$. This completes the proof. □

We move on to obtain a complete description of strong commutativity preserving additive maps $\psi: T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ on invertible matrices for $n\geqslant 5$. For any integer $n\geqslant 2$, it follows immediately from the Jacobi identity that if $\psi: T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ is an additive map satisfying $[\psi(A),\psi(B)]=[A,B]$ for all invertible matrices $A,B\in T_n(\mathbb{F}_2)$, then

$$[\psi(A),[B,C]]+[\psi(B),[C,A]]+[\psi(C),[A,B]]=0 \tag{10}$$

for all invertible matrices $A,B,C\in T_n(\mathbb{F}_2)$.

Our study starts with some preliminary results.

LEMMA 3.2. *Let $n \geqslant 3$ be an integer and let $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be a strong commutativity preserving additive map on invertible matrices. Then*

(i) $[\psi(I_n), \psi(E_{pq})] = 0$ *for all integers* $1 \leqslant p < q \leqslant n$.

(ii) $[\psi(I_n), E_{pq}] = 0$ *for all integers* $1 \leqslant p \leqslant n-2$ *and* $p+2 \leqslant q \leqslant n$.

(iii) $[\psi(E_{pq}), \psi(E_{st})] = \delta_{qs}E_{pt}$ *for all integers* $1 \leqslant p < q \leqslant n$ *and* $1 \leqslant s < t \leqslant n$ *with* $p \leqslant s$.

Proof. (i) Let $1 \leqslant p < q \leqslant n$ be integers. Since $[\psi(I_n), \psi(I_n)] = 0 = [\psi(I_n), \psi(I_n + E_{pq})]$, it follows that $[\psi(I_n), \psi(E_{pq})] = [\psi(I_n), \psi(I_n + I_n + E_{pq})] = [\psi(I_n), \psi(I_n)] + [\psi(I_n), \psi(I_n + E_{pq})] = 0$.

(ii) Let p and q be integers such that $1 \leqslant p \leqslant n-2$ and $p+2 \leqslant q \leqslant n$. Setting $A = I_n$, $B = I_n + E_{p,p+1}$ and $C = I_n + E_{p+1,q}$ in (10), we get

$$0 = [\psi(I_n), [I_n + E_{p,p+1}, I_n + E_{p+1,q}]] = [\psi(I_n), [E_{p,p+1}, E_{p+1,q}]] = [\psi(I_n), E_{pq}].$$

(iii) Consider integers $1 \leqslant p < q \leqslant n$ and $1 \leqslant s < t \leqslant n$ with $p \leqslant s$. By (i), we obtain $[\psi(I_n) + \psi(E_{pq}), \psi(I_n) + \psi(E_{st})] = [\psi(E_{pq}), \psi(E_{st})]$. Then

$$[\psi(E_{pq}), \psi(E_{st})] = [\psi(I_n + E_{pq}), \psi(I_n + E_{st})] = [I_n + E_{pq}, I_n + E_{st}] = \delta_{qs}E_{pt}.$$

This completes the proof. $\square$

LEMMA 3.3. *Let $n \geqslant 5$ be an integer and let $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be a strong commutativity preserving additive map on invertible matrices. Then the following hold.*

(i) *For each pair of integers* $1 \leqslant s < t \leqslant n$ *satisfying* $(s,t) \neq (1,2)$ *and* $s \neq 2$, $[\psi(E_{st}), E_{1k}] = 0$ *for all integers* $3 \leqslant k \leqslant n$ *with* $k \neq s$.

(ii) *For each pair of integers* $1 \leqslant s < t \leqslant n$ *satisfying* $(s,t) \neq (n-1,n)$ *and* $t \neq n-1$, $[\psi(E_{st}), E_{kn}] = 0$ *for all integers* $1 \leqslant k \leqslant n-2$ *with* $k \neq t$.

(iii) *For each integer* $3 \leqslant t \leqslant n$, $[\psi(E_{2t}), E_{1k}] + \delta_{3t}[\psi(E_{13}), E_{2k}] = 0$ *for all integers* $4 \leqslant k \leqslant n$.

(iv) *For each integer* $1 \leqslant s \leqslant n-2$, $[\psi(E_{s,n-1}), E_{kn}] + \delta_{s,n-2}[\psi(E_{n-2,n}), E_{k,n-1}] = 0$ *for all integers* $1 \leqslant k \leqslant n-3$.

(v) $[\psi(E_{st}), E_{tq}] + [\psi(E_{pq}), E_{sp}] = 0$ *for all integers* $1 \leqslant s < t < p < q \leqslant n$.

Proof. (i) Let $1 \leqslant s < t \leqslant n$ and $3 \leqslant k \leqslant n$ be integers such that $(s, t) \neq (1, 2)$, $s \neq 2$ and $k \neq s$. Setting $A = I_n + E_{st}$, $B = I_n + E_{12}$, $C = I_n + E_{2k}$ in (10) gives $[\psi(I_n + E_{st}), E_{1k}] = 0$. Since $[\psi(I_n), E_{1k}] = 0$ by Lemma 3.2 (ii), we have $[\psi(E_{st}), E_{1k}] = 0$.

(ii) Let $1 \leqslant s < t \leqslant n$ and $1 \leqslant k \leqslant n-2$ be integers such that $(s, t) \neq (n-1, n)$, $t \neq n-1$ and $k \neq t$. Taking $A = I_n + E_{st}$, $B = I_n + E_{k,n-1}$, $C = I_n + E_{n-1,n}$ in (10), we get $[\psi(I_n + E_{st}), E_{kn}] = 0$. By Lemma 3.2 (ii), $[\psi(I_n), E_{kn}] = 0$, and thus $[\psi(E_{st}), E_{kn}] = 0$.

(iii) Let $3 \leqslant t \leqslant n$ and $4 \leqslant k \leqslant n$ be integers. Set $A = I_n + E_{2t}$, $B = I_n + E_{13}$, $C = I_n + E_{3k}$ in (10). We obtain $[\psi(I_n + E_{2t}), E_{1k}] + [\psi(I_n + E_{13}), \delta_{3t} E_{2k}] = 0$. The desired result follows from Lemma 3.2 (ii).

(iv) Let $1 \leqslant s \leqslant n-2$ and $1 \leqslant k \leqslant n-3$ be integers. Setting $A = I_n + E_{s,n-1}$, $B = I_n + E_{k,n-2}$, $C = I_n + E_{n-2,n}$ in (10) yields $[\psi(I_n + E_{s,n-1}), E_{kn}] + [\psi(I_n + E_{n-2,n}), \delta_{s,n-2} E_{k,n-1}] = 0$. The result follows from Lemma 3.2 (ii).

(v) Let s,t,p,q be integers with $1 \leqslant s < t < p < q \leqslant n$. Taking $A = I_n + E_{st}$, $B = I_n + E_{tp}$, $C = I_n + E_{pq}$ in (10), we get $[\psi(E_{st}), E_{tq}] + [\psi(E_{pq}), E_{sp}] = 0$ by Lemma 3.2 (ii). $\square$

The following technical lemma gives a structural result of strong commutativity preserving additive maps on the identity matrix and strictly upper triangular matrices.

LEMMA 3.4. *Let $n \geqslant 5$ be an integer and let $\psi : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ be a strong commutativity preserving additive map on invertible matrices. Then there exist $\alpha, \beta, \gamma \in \mathbb{F}_2$ and additive maps $\mu, \eta : T_n(\mathbb{F}_2) \to \mathbb{F}_2$ such that*

$$\psi(A) = A + \mu(A) I_n + \eta(A) E_{1n} + \Phi_{\alpha,\beta,\gamma}(A)$$

for all strictly upper triangular matrices $A \in T_n(\mathbb{F}_2)$ and $A = I_n$, where $\Phi_{\alpha,\beta,\gamma} : T_n(\mathbb{F}_2) \to T_n(\mathbb{F}_2)$ is the additive map defined by

$$\Phi_{\alpha,\beta,\gamma}(A) = (\gamma a_{12} + \alpha a_{n-1,n}) E_{1,n-1} + (\beta a_{12} + \gamma a_{n-1,n}) E_{2n}$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$.

Proof. For each pair of integers $1 \leqslant s < t \leqslant n$, we denote

$$\psi(E_{st}) = \sum_{1 \leqslant i \leqslant j \leqslant n} \alpha_{ij}^{(st)} E_{ij}, \tag{11}$$

$$\psi(I_n) = \sum_{1 \leqslant i \leqslant j \leqslant n} \alpha_{ij} E_{ij} \tag{12}$$

for some fixed scalars $\alpha_{ij}^{(st)}, \alpha_{ij} \in \mathbb{F}_2$, $1 \leqslant i \leqslant j \leqslant n$. Our first claim is that

$$\psi(I_n) = \alpha_{11} I_n + \sum_{i=1}^{2} \sum_{j=n-1}^{n} \alpha_{ij} E_{ij}. \tag{13}$$

Let $3 \leqslant p \leqslant n$ be an integer. By Lemma 3.2 (ii), $[\psi(I_n), E_{1p}] = 0$. It follows from (12) that

$$\sum_{1 \leqslant i \leqslant j \leqslant n} \alpha_{ij} E_{ij} E_{1p} + \sum_{1 \leqslant i \leqslant j \leqslant n} \alpha_{ij} E_{1p} E_{ij} = 0.$$

This yields $(\alpha_{11} + \alpha_{pp}) E_{1p} + \sum_{j=p+1}^{n} \alpha_{pj} E_{1j} = 0$. It is understood that $\sum_{j=p+1}^{n} \alpha_{pj} E_{1j}$ vanishes when $p = n$. Then for every $3 \leqslant p \leqslant n$,

$$\alpha_{pp} = \alpha_{11}, \tag{14}$$

$$\alpha_{pj} = 0 \quad \text{for } j = p+1, \ldots, n. \tag{15}$$

Likewise, for each integer $1 \leqslant p \leqslant n-2$, we get $[\psi(I_n), E_{pn}] = 0$ by Lemma 3.2 (ii). By (12), we have $(\alpha_{pp} + \alpha_{nn})E_{pn} + \sum_{i=1}^{p-1} \alpha_{ip} E_{in} = 0$. Then for every $1 \leqslant p \leqslant n-2$,

$$\alpha_{pp} = \alpha_{nn}, \tag{16}$$

$$\alpha_{ip} = 0 \quad \text{for } i = 1, \ldots, p-1. \tag{17}$$

Substituting (14)–(17) into (12) yields $\psi(I_n) = \alpha_{11} I_n + \sum_{i=1}^{2} \sum_{j=n-1}^{n} \alpha_{ij} E_{ij}$ as claimed.

We now study the structure of $\psi(E_{st})$ for every $1 \leqslant s < t \leqslant n$. The argument will be divided into three cases.

Case I-1: $s = 1$. Let $3 \leqslant p, q \leqslant n$ be integers. By Lemma 3.3 (i), $[\psi(E_{1q}), E_{1p}] = 0$. We thus obtain from (11) that $(\alpha_{11}^{(1q)} + \alpha_{pp}^{(1q)})E_{1p} + \sum_{j=p+1}^{n} \alpha_{pj}^{(1q)} E_{1j} = 0$. Then for every $3 \leqslant p, q \leqslant n$,

$$\alpha_{pp}^{(1q)} = \alpha_{11}^{(1q)}, \tag{18}$$

$$\alpha_{pj}^{(1q)} = 0 \quad \text{for } j = p+1, \ldots, n. \tag{19}$$

Next consider integers $1 \leqslant p \leqslant n-2$ and $2 \leqslant q \leqslant n$, with $p \neq q$ and $q \neq n-1$. By Lemma 3.3 (ii), we see that $[\psi(E_{1q}), E_{pn}] = 0$. It follows from (11) that $(\alpha_{pp}^{(1q)} + \alpha_{nn}^{(1q)})E_{pn} + \sum_{i=1}^{p-1} \alpha_{ip}^{(1q)} E_{in} = 0$. Then for every $1 \leqslant p \leqslant n-2$ and $2 \leqslant q \leqslant n$, with $p \neq q$ and $q \neq n-1$,

$$\alpha_{pp}^{(1q)} = \alpha_{nn}^{(1q)}, \tag{20}$$

$$\alpha_{ip}^{(1q)} = 0 \quad \text{for } i = 1, \ldots, p-1. \tag{21}$$

Substituting (18)–(21) into (11), we obtain

$$\psi(E_{1n}) = \alpha_{11}^{(1n)} I_n + \sum_{i=1}^{2} \sum_{j=n-1}^{n} \alpha_{ij}^{(1n)} E_{ij}. \tag{22}$$

Let $3 \leqslant p < q \leqslant n$ be integers. By Lemma 3.3 (v), we get $[\psi(E_{12}), E_{2q}] + [\psi(E_{pq}), E_{1p}] = 0$. It follows from (11) that $\sum_{i=1}^{2} \alpha_{i2}^{(12)} E_{iq} + \sum_{j=q}^{n} \alpha_{qj}^{(12)} E_{2j} + \alpha_{11}^{(pq)} E_{1p} + \sum_{j=p}^{n} \alpha_{pj}^{(pq)} E_{1j} = 0$. Thus

$$(\alpha_{11}^{(pq)} + \alpha_{pp}^{(pq)})E_{1p} + (\alpha_{pq}^{(pq)} + \alpha_{12}^{(12)})E_{1q} + (\alpha_{qq}^{(12)} + \alpha_{22}^{(12)})E_{2q} + \sum_{j=q+1}^{n} \alpha_{qj}^{(12)} E_{2j} + \sum_{j=p+1, j\neq q}^{n} \alpha_{pj}^{(pq)} E_{1j} = 0.$$

Then for every $3 \leqslant p < q \leqslant n$,

$$\alpha_{pq}^{(pq)} = \alpha_{12}^{(12)}, \tag{23}$$

and for every $4 \leqslant p \leqslant n$,

$$\alpha_{pp}^{(12)} = \alpha_{22}^{(12)}, \tag{24}$$

$$\alpha_{pj}^{(12)} = 0 \quad \text{for } j = p+1,\dots,n. \tag{25}$$

Substituting (20), (21), (24) and (25) into (11), we get

$$\psi(E_{12}) = \alpha_{11}^{(12)} I_n + \alpha_{12}^{(12)} E_{12} + \sum_{i=1}^{3} \sum_{j=n-1}^{n} \alpha_{ij}^{(12)} E_{ij}. \tag{26}$$

Now consider integers $3 \leqslant t \leqslant n-2$ and $t < p < q \leqslant n$. Then $[\psi(E_{1t}), E_{tq}] + [\psi(E_{pq}), E_{1p}] = 0$ by Lemma 3.3 (v), and thus

$$(\alpha_{pq}^{(pq)} + \alpha_{1t}^{(1t)})E_{1q} + (\alpha_{11}^{(pq)} + \alpha_{pp}^{(pq)})E_{1p} + (\alpha_{tt}^{(1t)} + \alpha_{qq}^{(1t)})E_{tq} + \sum_{i=2}^{t-1} \alpha_{it}^{(1t)} E_{iq} + \sum_{j=q+1}^{n} \alpha_{qj}^{(1t)} E_{tj} + \sum_{j=p+1,\, j\neq q}^{n} \alpha_{pj}^{(pq)} E_{1j} = 0.$$

Then for every $3 \leqslant t \leqslant n-2$ and $t < p < q \leqslant n$,

$$\alpha_{it}^{(1t)} = 0 \quad \text{for } i = 2,\dots,t-1, \tag{27}$$

$$\alpha_{pq}^{(pq)} = \alpha_{1t}^{(1t)}. \tag{28}$$

Taking $(p,q) = (n-1,n)$ in (28), together with (23), we obtain

$$\alpha_{1t}^{(1t)} = \alpha_{n-1,n}^{(n-1,n)} = \alpha_{12}^{(12)} \quad \text{for } t = 3,\dots,n-2. \tag{29}$$

Substituting (18)–(21), (27) and (29) into (11), we get

$$\psi(E_{1t}) = \alpha_{11}^{(1t)} I_n + \alpha_{12}^{(12)} E_{1t} + \sum_{i=1}^{2} \sum_{j=n-1}^{n} \alpha_{ij}^{(1t)} E_{ij} \tag{30}$$

for $t = 3,\dots,n-2$. Finally, we claim that

$$\psi(E_{1,n-1}) = \alpha_{11}^{(1,n-1)} I_n + \sum_{i=1}^{2} \sum_{j=n-2}^{n} \alpha_{ij}^{(1,n-1)} E_{ij}. \tag{31}$$

Let $1 \leqslant p \leqslant n-3$ be an integer. Since $\delta_{1,n-2} = 0$, it follows from Lemma 3.3 (iv) that $[\psi(E_{1,n-1}), E_{pn}] = 0$. By (11), $(\alpha_{pp}^{(1,n-1)} + \alpha_{nn}^{(1,n-1)})E_{pn} + \sum_{i=1}^{p-1} \alpha_{ip}^{(1,n-1)} E_{in} = 0$. Then for every $1 \leqslant p \leqslant n-3$, we get

$$\alpha_{pp}^{(1,n-1)} = \alpha_{nn}^{(1,n-1)}, \tag{32}$$

$$\alpha_{ip}^{(1,n-1)} = 0 \quad \text{for } i = 1,\dots,p-1. \tag{33}$$

Substituting (18), (19), (32) and (33) into (11), we prove (31) as desired.

Case I-2: $s = 2$. Then $3 \leqslant t \leqslant n$. Consider integer $4 \leqslant p \leqslant n$. By Lemma 3.3 (iii), we see that $[\psi(E_{2t}), E_{1p}] + \delta_{3t}[\psi(E_{13}), E_{2p}] = 0$. Since $\alpha_{12}^{(13)} = 0$ by (21), it follows from (11) that

$$(\alpha_{11}^{(2t)} + \alpha_{pp}^{(2t)})E_{1p} + \delta_{3t}(\alpha_{22}^{(13)} + \alpha_{pp}^{(13)})E_{2p} + \sum_{j=p+1}^{n} \alpha_{pj}^{(2t)} E_{1j} + \delta_{3t} \sum_{j=p+1}^{n} \alpha_{pj}^{(13)} E_{2j} = 0.$$

Then for every $3 \leqslant t \leqslant n$ and $4 \leqslant p \leqslant n$,

$$\alpha_{pp}^{(2t)} = \alpha_{11}^{(2t)}, \tag{34}$$

$$\alpha_{pj}^{(2t)} = 0 \quad \text{for } j = p+1, \ldots, n. \tag{35}$$

We now consider integers $3 \leqslant t \leqslant n$ and $1 \leqslant p \leqslant n-2$, with $t \neq n-1$ and $p \neq t$. It follows from Lemma 3.3 (ii) that $[\psi(E_{2t}), E_{pn}] = 0$. By (11), $(\alpha_{pp}^{(2t)} + \alpha_{nn}^{(2t)})E_{pn} + \sum_{i=1}^{p-1} \alpha_{ip}^{(2t)} E_{in} = 0$. Then for every $3 \leqslant t \leqslant n$ and $1 \leqslant p \leqslant n-2$, with $t \neq n-1$ and $p \neq t$,

$$\alpha_{pp}^{(2t)} = \alpha_{nn}^{(2t)}, \tag{36}$$

$$\alpha_{ip}^{(2t)} = 0 \quad \text{for } i = 1, \ldots, p-1. \tag{37}$$

Next, consider integers $3 \leqslant t \leqslant n-2$ and $t < p < q \leqslant n$. Then $[\psi(E_{2t}), E_{tq}] + [\psi(E_{pq}), E_{2p}] = 0$ by Lemma 3.3 (v). We infer from (11) that

$$\begin{aligned}(\alpha_{tt}^{(2t)} + \alpha_{qq}^{(2t)})E_{tq} + (\alpha_{2t}^{(2t)} + \alpha_{pq}^{(pq)})E_{2q} + \alpha_{12}^{(pq)} E_{1p} + (\alpha_{22}^{(pq)} + \alpha_{pp}^{(pq)})E_{2p} \\ + \sum_{i=1, i\neq 2}^{t-1} \alpha_{it}^{(2t)} E_{iq} + \sum_{j=q+1}^{n} \alpha_{qj}^{(2t)} E_{tj} + \sum_{j=p+1, j\neq q}^{n} \alpha_{pj}^{(pq)} E_{2j} = 0.\end{aligned}$$

Then for every $3 \leqslant t \leqslant n-2$ and $t < p < q \leqslant n$,

$$\alpha_{pq}^{(pq)} = \alpha_{2t}^{(2t)}, \tag{38}$$

$$\alpha_{qq}^{(2t)} = \alpha_{tt}^{(2t)}, \tag{39}$$

$$\alpha_{it}^{(2t)} = 0 \quad \text{for } i = 1, 3, \ldots, t-1. \tag{40}$$

Taking $(p,q) = (n-1, n)$ in (38), together with (23), we get

$$\alpha_{2t}^{(2t)} = \alpha_{n-1,n}^{(n-1,n)} = \alpha_{12}^{(12)} \quad \text{for } t = 3, \ldots, n-2. \tag{41}$$

Substituting (34) – (37) and (39) – (41) into (11), we obtain

$$\psi(E_{2t}) = \alpha_{11}^{(2t)} I_n + \alpha_{12}^{(12)} E_{2t} + \sum_{i=1}^{3} \sum_{j=n-1}^{n} \alpha_{ij}^{(2t)} E_{ij} \tag{42}$$

for $t = 3, \ldots, n-2$.

We next consider $t = n-1$. Let $1 \leqslant p \leqslant n-3$ be an integer. Note that $\delta_{2,n-2} = 0$. Then $[\psi(E_{2,n-1}), E_{pn}] = 0$ by Lemma 3.3 (iv). By (11), $(\alpha_{pp}^{(2,n-1)} + \alpha_{nn}^{(2,n-1)})E_{pn} + \sum_{i=1}^{p-1} \alpha_{ip}^{(2,n-1)} E_{in} = 0$. Then for every $1 \leqslant p \leqslant n-3$,

$$\alpha_{pp}^{(2,n-1)} = \alpha_{nn}^{(2,n-1)}, \tag{43}$$

$$\alpha_{ip}^{(2,n-1)} = 0 \quad \text{for } i = 1, \ldots, p-1. \tag{44}$$

Next, setting $A = I_n + E_{2,n-1}$, $B = I_n + E_{12}$, $C = I_n + E_{23}$ in (10), together with Lemma 3.2 (ii), we obtain $[\psi(E_{2,n-1}), E_{13}] + [\psi(E_{23}), E_{1,n-1}] = 0$. It follows from (11) that

$$(\alpha_{11}^{(2,n-1)} + \alpha_{33}^{(2,n-1)})E_{13} + \left(\alpha_{11}^{(23)} + \alpha_{n-1,n-1}^{(23)} + \alpha_{3,n-1}^{(2,n-1)}\right)E_{1,n-1} + \left(\alpha_{n-1,n}^{(23)} + \alpha_{3n}^{(2,n-1)}\right)E_{1n} + \sum_{j=4}^{n-2} \alpha_{3j}^{(2,n-1)} E_{1j} = 0.$$

We thus obtain

$$\alpha_{33}^{(2,n-1)} = \alpha_{11}^{(2,n-1)}, \tag{45}$$

$$\alpha_{3j}^{(2,n-1)} = 0 \quad \text{for } j = 4, \ldots, n-2, \tag{46}$$

and $\alpha_{11}^{(23)} + \alpha_{n-1,n-1}^{(23)} + \alpha_{3,n-1}^{(2,n-1)} = 0$ and $\alpha_{n-1,n}^{(23)} + \alpha_{3n}^{(2,n-1)} = 0$. Since $\alpha_{11}^{(23)} = \alpha_{n-1,n-1}^{(23)}$ by (34), and $\alpha_{n-1,n}^{(23)} = 0$ by (35), it follows that

$$\alpha_{3,n-1}^{(2,n-1)} = 0 \quad \text{and} \quad \alpha_{3n}^{(2,n-1)} = 0. \tag{47}$$

Substituting (34), (35) and (43)–(47) into (11) yields

$$\psi(E_{2,n-1}) = \alpha_{11}^{(2,n-1)} I_n + \sum_{i=1}^{2} \sum_{j=n-2}^{n} \alpha_{ij}^{(2,n-1)} E_{ij}. \tag{48}$$

Now, letting $A = I_n + E_{2n}$, $B = I_n + E_{12}$, $C = I_n + E_{23}$ in (10), together with Lemma 3.2 (ii), we get $[\psi(E_{2n}), E_{13}] + [\psi(E_{23}), E_{1n}] = 0$. It follows from (11) that

$$\left(\alpha_{11}^{(2n)} + \alpha_{33}^{(2n)}\right)E_{13} + \left(\alpha_{11}^{(23)} + \alpha_{nn}^{(23)} + \alpha_{3n}^{(2n)}\right)E_{1n} + \sum_{j=4}^{n-1} \alpha_{3j}^{(2n)} E_{1j} = 0.$$

Then

$$\alpha_{3j}^{(2n)} = 0 \quad \text{for } j = 4, \ldots, n-1, \tag{49}$$

and $\alpha_{11}^{(23)} = \alpha_{nn}^{(23)} + \alpha_{3n}^{(2n)}$. Since $\alpha_{11}^{(23)} = \alpha_{nn}^{(23)}$ by (34), we get

$$\alpha_{3n}^{(2n)} = 0. \tag{50}$$

Substituting (34)–(37), (49) and (50) into (11), we obtain

$$\psi(E_{2n}) = \alpha_{11}^{(2n)} I_n + \sum_{i=1}^{2} \sum_{j=n-1}^{n} \alpha_{ij}^{(2n)} E_{ij}. \tag{51}$$

Case I-3: $3 \leqslant s \leqslant n-1$. Recall that $s < t \leqslant n$. Let p be an integer such that $3 \leqslant p \leqslant n$, with $p \neq s$. Then $[\psi(E_{st}), E_{1p}] = 0$ by Lemma 3.3 (i). From (11), we get $(\alpha_{11}^{(st)} + \alpha_{pp}^{(st)})E_{1p} + \sum_{j=p+1}^{n} \alpha_{pj}^{(st)} E_{1j} = 0$. Then for every $3 \leqslant s < t \leqslant n$ and $3 \leqslant p \leqslant n$, with $p \neq s$,

$$\alpha_{pp}^{(st)} = \alpha_{11}^{(st)}, \tag{52}$$

$$\alpha_{pj}^{(st)} = 0 \quad \text{for } j = p+1,\dots,n. \tag{53}$$

Next, consider integers p and q with $1 \leqslant p < q < s < t \leqslant n$. It follows from Lemma 3.3 (v) that $[\psi(E_{st}), E_{ps}] + [\psi(E_{pq}), E_{qt}] = 0$. By (11), we have

$$(\alpha_{pp}^{(st)} + \alpha_{ss}^{(st)})E_{ps} + (\alpha_{pq}^{(pq)} + \alpha_{st}^{(st)})E_{pt} + (\alpha_{qq}^{(pq)} + \alpha_{tt}^{(pq)})E_{qt} + \sum_{i=1}^{p-1} \alpha_{ip}^{(st)} E_{is} + \sum_{i=1, i\neq p}^{q-1} \alpha_{iq}^{(pq)} E_{it} + \sum_{j=s+1,\, j\neq t}^{n} \alpha_{sj}^{(st)} E_{pj} + \sum_{j=t+1}^{n} \alpha_{tj}^{(pq)} E_{qj} = 0.$$

Then for every $3 \leqslant s < t \leqslant n$ and $1 \leqslant p \leqslant s-2$,

$$\alpha_{pp}^{(st)} = \alpha_{ss}^{(st)}, \tag{54}$$

$$\alpha_{sj}^{(st)} = 0 \quad \text{for } j = s+1,\dots,n, \text{ with } j \neq t, \tag{55}$$

$$\alpha_{ip}^{(st)} = 0 \quad \text{for } i = 1,\dots,p-1, \tag{56}$$

and $\alpha_{pq}^{(pq)} + \alpha_{st}^{(st)} = 0$ for all $1 \leqslant p < q < s$. Taking $(p,q) = (1,2)$ gives

$$\alpha_{st}^{(st)} = \alpha_{12}^{(12)}. \tag{57}$$

Taking $(s,t) = (n-1,n)$ and substituting (52)–(54), (56) and (57) into (11), we obtain

$$\psi(E_{n-1,n}) = \alpha_{11}^{(n-1,n)} I_n + \alpha_{12}^{(12)} E_{n-1,n} + \sum_{i=1}^{2} \sum_{j=n-2}^{n} \alpha_{ij}^{(n-1,n)} E_{ij}. \tag{58}$$

We now consider $s < t \leqslant n$, with $t \neq n-1$ and $(s,t) \neq (n-1,n)$. By Lemma 3.3 (ii), we have $[\psi(E_{st}), E_{pn}] = 0$ for all $1 \leqslant p \leqslant n-2$, with $p \neq t$. By (11), $(\alpha_{pp}^{(st)} + \alpha_{nn}^{(st)})E_{pn} + \sum_{i=1}^{p-1} \alpha_{ip}^{(st)} E_{in} = 0$. Then for every $3 \leqslant s < t \leqslant n$ and $1 \leqslant p \leqslant n-2$, with $(s,t) \neq (n-1,n)$, $t \neq n-1$ and $p \neq t$,

$$\alpha_{pp}^{(st)} = \alpha_{nn}^{(st)}, \tag{59}$$

$$\alpha_{ip}^{(st)} = 0 \quad \text{for } i = 1,\dots,p-1. \tag{60}$$

In view of Lemma 3.3 (v) that $[\psi(E_{st}), E_{tq}] + [\psi(E_{pq}), E_{sp}] = 0$ for all $t < p < q \leqslant n$. By (11),

$$(\alpha_{tt}^{(st)} + \alpha_{qq}^{(st)})E_{tq} + (\alpha_{st}^{(st)} + \alpha_{pq}^{(pq)})E_{sq} + (\alpha_{ss}^{(pq)} + \alpha_{pp}^{(pq)})E_{sp} + \sum_{i=1, i\neq s}^{t-1} \alpha_{it}^{(st)} E_{iq} + \sum_{i=1}^{s-1} \alpha_{is}^{(pq)} E_{ip} + \sum_{j=q+1}^{n} \alpha_{qj}^{(st)} E_{tj} + \sum_{j=p+1,\, j\neq q}^{n} \alpha_{pj}^{(pq)} E_{sj} = 0.$$

Then for every $3 \leqslant s < t \leqslant n-2$,

$$\alpha_{it}^{(st)} = 0 \quad \text{for } i = 1, \dots, t-1, \text{ with } i \neq s. \tag{61}$$

Substituting (52), (53), (55), (57) and (59)–(61) into (11), we have

$$\psi(E_{st}) = \alpha_{11}^{(st)} I_n + \alpha_{12}^{(12)} E_{st} + \sum_{i=1}^{2} \sum_{j=n-1}^{n} \alpha_{ij}^{(st)} E_{ij} \tag{62}$$

for every $3 \leqslant s < t \leqslant n$, with $(s,t) \neq (n-1,n)$ and $t \neq n-1$.

Next consider $t = n-1$. Then $3 \leqslant s \leqslant n-2$. By Lemma 3.3 (iv), we obtain $[\psi(E_{s,n-1}), E_{pn}] + \delta_{s,n-2}[\psi(E_{n-2,n}), E_{p,n-1}] = 0$ for all $1 \leqslant p \leqslant n-3$. It follows from (11) that

$$\begin{aligned}\Big(\alpha_{pp}^{(s,n-1)} + \alpha_{nn}^{(s,n-1)} + \delta_{s,n-2}\alpha_{n-1,n}^{(n-2,n)}\Big) E_{pn} + \delta_{s,n-2}(\alpha_{n-1,n-1}^{(n-2,n)} + \alpha_{pp}^{(n-2,n)}) E_{p,n-1} \\ + \sum_{i=1}^{p-1} \alpha_{ip}^{(s,n-1)} E_{in} + \delta_{s,n-2} \sum_{i=1}^{p-1} \alpha_{ip}^{(n-2,n)} E_{i,n-1} = 0.\end{aligned}$$

Note that $\alpha_{n-1,n}^{(n-2,n)} = 0$ by (53). Then for every $3 \leqslant s \leqslant n-2$ and $1 \leqslant p \leqslant n-3$,

$$\alpha_{pp}^{(s,n-1)} = \alpha_{nn}^{(s,n-1)}, \tag{63}$$

$$\alpha_{ip}^{(s,n-1)} = 0 \quad \text{for } i = 1, \dots, p-1. \tag{64}$$

Substituting (52)–(55), (57), (63) and (64) into (11), we obtain

$$\psi(E_{s,n-1}) = \alpha_{11}^{(s,n-1)} I_n + \alpha_{12}^{(12)} E_{s,n-1} + \sum_{i=1}^{2} \sum_{j=n-2}^{n} \alpha_{ij}^{(s,n-1)} E_{ij} \tag{65}$$

for $s = 3, \dots, n-2$.

Denote $\lambda = \alpha_{12}^{(12)} \in \mathbb{F}_2$. We claim that

$$\lambda = 1. \tag{66}$$

To see this, let $3 \leqslant p \leqslant n-2$ be an integer. Then $[\psi(E_{12}), \psi(E_{2p})] = E_{1p}$ by Lemma 3.2 (iii). By (26) and (42), $\Big[\lambda E_{12} + \sum_{i=1}^{3} \sum_{j=n-1}^{n} \alpha_{ij}^{(12)} E_{ij}, \lambda E_{2p} + \sum_{i=1}^{3} \sum_{j=n-1}^{n} \alpha_{ij}^{(2p)} E_{ij}\Big] = E_{1p}$. Thus

$$(\lambda^2 + 1) E_{1p} + \lambda \Big(\alpha_{2,n-1}^{(2p)} E_{1,n-1} + \alpha_{2n}^{(2p)} E_{1n} + \delta_{3p}\alpha_{3,n-1}^{(12)} E_{2,n-1} + \delta_{3p}\alpha_{3n}^{(12)} E_{2n}\Big) = 0 \tag{67}$$

for all $3 \leqslant p \leqslant n-2$. Then $\lambda^2 = 1$, and hence $\lambda = 1$ as claimed.

We now further simplify the structure of $\psi(E_{12})$ and $\psi(E_{n-1,n})$ in (26) and (58), respectively. Let $3 \leqslant p \leqslant n-2$ be an integer. Lemma 3.2 (iii) gives $[\psi(E_{p,n-1}), \psi(E_{n-1,n})] = E_{pn}$. It follows from (58), (65) and (66) that

$$\alpha_{1,n-1}^{(p,n-1)} E_{1n} + \alpha_{2,n-1}^{(p,n-1)} E_{2n} + \delta_{p,n-2}\Big(\alpha_{1,n-2}^{(n-1,n)} E_{1,n-1} + \alpha_{2,n-2}^{(n-1,n)} E_{2,n-1}\Big) = 0.$$

By taking $p=n-2$, we have

$$\alpha_{1,n-2}^{(n-1,n)}=\alpha_{2,n-2}^{(n-1,n)}=0, \tag{68}$$

and for every $3\leqslant p\leqslant n-2$,

$$\alpha_{1,n-1}^{(p,n-1)}=\alpha_{2,n-1}^{(p,n-1)}=0. \tag{69}$$

Next, $[\psi(E_{12}),\psi(E_{n-1,n})]=0$ by Lemma 3.2 (iii). It follows from (26), (58), (66) and (68) that $\left[E_{12}+\sum_{i=1}^{3}\sum_{j=n-1}^{n}\alpha_{ij}^{(12)}E_{ij},\, E_{n-1,n}+\sum_{i=1}^{2}\sum_{j=n-1}^{n}\alpha_{ij}^{(n-1,n)}E_{ij}\right]=0$. Then

$$\alpha_{2,n-1}^{(n-1,n)}E_{1,n-1}+(\alpha_{2n}^{(n-1,n)}+\alpha_{1,n-1}^{(12)})E_{1n}+\alpha_{2,n-1}^{(12)}E_{2n}+\alpha_{3,n-1}^{(12)}E_{3n}=0.$$

Consequently, $\alpha_{2,n-1}^{(n-1,n)}=\alpha_{2,n-1}^{(12)}=\alpha_{3,n-1}^{(12)}=0$ and $\alpha_{2n}^{(n-1,n)}=\alpha_{1,n-1}^{(12)}$. Taking $p=3$ in (67) gives $\alpha_{3n}^{(12)}=0$. Combining these results with (26), (58), (66) and (68), we thus conclude that

$$\psi(E_{12})=\alpha_{11}^{(12)}I_n+E_{12}+\alpha_{1,n-1}^{(12)}E_{1,n-1}+\alpha_{1n}^{(12)}E_{1n}+\alpha_{2n}^{(12)}E_{2n}, \tag{70}$$

$$\psi(E_{n-1,n})=\alpha_{11}^{(n-1,n)}I_n+E_{n-1,n}+\alpha_{1,n-1}^{(n-1,n)}E_{1,n-1}+\alpha_{1n}^{(n-1,n)}E_{1n}+\alpha_{1,n-1}^{(12)}E_{2n}. \tag{71}$$

Denote $\mathscr{X}=\{(1,n-1),(1,n),(2,n-1),(2,n)\}$. We next show that for every $1\leqslant s<t\leqslant n$, with $(s,t)\notin\mathscr{X}\cup\{(1,2),(n-1,n)\}$,

$$\psi(E_{st})=\alpha_{11}^{(st)}I_n+E_{st}+\alpha_{1n}^{(st)}E_{1n}. \tag{72}$$

The proof will be divided into three cases.

Case II-1: $1\leqslant s<t\leqslant n$, with $(s,t)\notin\mathscr{X}\cup\{(1,2),(n-1,n)\}$, $s\neq 2$ and $t\neq n-1$. It follows from (30), (62) and (66) that

$$\psi(E_{st})=\alpha_{11}^{(st)}I_n+E_{st}+\sum_{i=1}^{2}\sum_{j=n-1}^{n}\alpha_{ij}^{(st)}E_{ij}. \tag{73}$$

By Lemma 3.2 (iii), we have $[\psi(E_{12}),\psi(E_{st})]=0$. Together with (70), we obtain

$$\left[E_{12}+\alpha_{1,n-1}^{(12)}E_{1,n-1}+\alpha_{1n}^{(12)}E_{1n}+\alpha_{2n}^{(12)}E_{2n},\, E_{st}+\sum_{i=1}^{2}\sum_{j=n-1}^{n}\alpha_{ij}^{(st)}E_{ij}\right]=0.$$

Since $s,t\notin\{2,n-1\}$, we have $\alpha_{2,n-1}^{(st)}E_{1,n-1}+\alpha_{2n}^{(st)}E_{1n}=0$. Then for every $1\leqslant s<t\leqslant n$, with $(s,t)\notin\mathscr{X}\cup\{(1,2),(n-1,n)\}$, $s\neq 2$ and $t\neq n-1$,

$$\alpha_{2,n-1}^{(st)}=\alpha_{2n}^{(st)}=0. \tag{74}$$

On the other hand, we have $[\psi(E_{st}),\psi(E_{n-1,n})]=0$ by Lemma 3.2(iii). Using a similar argument and applying (71), we obtain $\alpha_{1,n-1}^{(st)}E_{1n}+\alpha_{2,n-1}^{(st)}E_{2n}=0$. Then for every $1\leqslant s<t\leqslant n$, with $(s,t)\notin\mathscr{X}\cup\{(1,2),(n-1,n)\}$, $s\neq 2$ and $t\neq n-1$,

$$\alpha_{1,n-1}^{(st)}=0. \tag{75}$$

Substituting (74) and (75) into (73) yields $\psi(E_{st}) = \alpha_{11}^{(st)} I_n + E_{st} + \alpha_{1n}^{(st)} E_{1n}$ as claimed.

Case II-2: $s = 2$ and $3 \leqslant t \leqslant n-2$. By (42) and (66), we have

$$\psi(E_{2t}) = \alpha_{11}^{(2t)} I_n + E_{2t} + \sum_{i=1}^{3} \sum_{j=n-1}^{n} \alpha_{ij}^{(2t)} E_{ij} \tag{76}$$

By Lemma 3.2 (iii), we see that $[\psi(E_{13}), \psi(E_{2t})] = 0$. It follows from (30), (66) and (76) that $\alpha_{3,n-1}^{(2t)} E_{1,n-1} + \alpha_{3n}^{(2t)} E_{1n} = 0$. Then for every $3 \leqslant t \leqslant n-2$,

$$\alpha_{3,n-1}^{(2t)} = \alpha_{3n}^{(2t)} = 0. \tag{77}$$

Next, we see that $[\psi(E_{2t}), \psi(E_{n-1,n})] = 0$ by Lemma 3.2 (iii). Applying (71) and (76), we obtain $\alpha_{1,n-1}^{(2t)} E_{1n} + \alpha_{2,n-1}^{(2t)} E_{2n} + \alpha_{3,n-1}^{(2t)} E_{3n} = 0$. Then for every $3 \leqslant t \leqslant n-2$,

$$\alpha_{1,n-1}^{(2t)} = \alpha_{2,n-1}^{(2t)} = 0. \tag{78}$$

Moreover, in view of (67), we see that for every $3 \leqslant t \leqslant n-2$,

$$\alpha_{2n}^{(2t)} = 0. \tag{79}$$

Substituting (77), (78) and (79) into (76) yields $\psi(E_{2t}) = \alpha_{11}^{(2t)} I_n + E_{2t} + \alpha_{1n}^{(2t)} E_{1n}$ as desired.

Case II-3: $3 \leqslant s \leqslant n-2$ and $t = n-1$. By (65) and (66), we get

$$\psi(E_{s,n-1}) = \alpha_{11}^{(s,n-1)} I_n + E_{s,n-1} + \sum_{i=1}^{2} \sum_{j=n-2}^{n} \alpha_{ij}^{(s,n-1)} E_{ij}. \tag{80}$$

Note that Lemma 3.2 (iii) gives $[\psi(E_{s,n-1}), \psi(E_{n-2,n})] = 0$. Together with (62), (66) and (80), we obtain $\alpha_{1,n-2}^{(s,n-1)} E_{1n} + \alpha_{2,n-2}^{(s,n-1)} E_{2n} = 0$. Then for every $3 \leqslant s \leqslant n-2$,

$$\alpha_{1,n-2}^{(s,n-1)} = \alpha_{2,n-2}^{(s,n-1)} = 0. \tag{81}$$

Likewise, by Lemma 3.2 (iii), we get $[\psi(E_{12}), \psi(E_{s,n-1})] = 0$. It follows from (70), (80) and (81) that $\alpha_{2,n-1}^{(s,n-1)} E_{1,n-1} + \alpha_{2n}^{(s,n-1)} E_{1n} = 0$. Then for every $3 \leqslant s \leqslant n-2$,

$$\alpha_{2,n-1}^{(s,n-1)} = \alpha_{2n}^{(s,n-1)} = 0. \tag{82}$$

Moreover, in view of (69), we see that $\alpha_{1,n-1}^{(s,n-1)} = 0$ for $3 \leqslant s \leqslant n-2$. Combining this result with (80), (81) and (82) gives $\psi(E_{s,n-1}) = \alpha_{11}^{(s,n-1)} I_n + E_{s,n-1} + \alpha_{1n}^{(s,n-1)} E_{1n}$, which completes the proof of (72).

We continue to refine the structure of $\psi(I_n)$ in (13) as well as $\psi(E_{st})$, $(s,t) \in \mathscr{X}$, in (22), (31), (48) and (51). To do this, we first note that $[\psi(I_n), \psi(E_{12})] = 0$ by Lemma 3.2 (i). It follows from (13) and (70) that $\alpha_{2,n-1} E_{1,n-1} + \alpha_{2n} E_{1n} = 0$. Then

$$\alpha_{2,n-1} = \alpha_{2n} = 0. \tag{83}$$

By Lemma 3.2 (i), $[\psi(I_n),\psi(E_{n-1,n})]=0$. It follows from (13), (71) and (83) that $\alpha_{1,n-1}E_{1n}=0$, and so $\alpha_{1,n-1}=0$. Together with (13) and (83), we obtain

$$\psi(I_n)=\alpha_{11}I_n+\alpha_{1n}E_{1n}. \tag{84}$$

Next, by Lemma 3.2 (iii), we get $[\psi(E_{1i}),\psi(E_{12})]=0,\ i=n-1,n$. It follows from (22), (31) and (70) that $\alpha_{2,n-2}^{(1,n-1)}E_{1,n-2}+\alpha_{2,n-1}^{(1,n-1)}E_{1,n-1}+\alpha_{2n}^{(1,n-1)}E_{1n}=0$ and $\alpha_{2,n-1}^{(1n)}E_{1,n-1}+\alpha_{2n}^{(1n)}E_{1n}=0$. Consequently,

$$\alpha_{2,n-2}^{(1,n-1)}=\alpha_{2,n-1}^{(1,n-1)}=\alpha_{2n}^{(1,n-1)}=0, \tag{85}$$

$$\alpha_{2,n-1}^{(1n)}=\alpha_{2n}^{(1n)}=0. \tag{86}$$

By Lemma 3.2 (iii), we have $[\psi(E_{in}),\psi(E_{n-1,n})]=0,\ i=1,2$. It follows from (22), (51) and (71) that $\alpha_{1,n-1}^{(in)}E_{1n}+\alpha_{2,n-1}^{(in)}E_{2n}=0$ for $i=1,2$. Then

$$\alpha_{1,n-1}^{(sn)}=\alpha_{2,n-1}^{(sn)}=0 \quad \text{for } s=1,2. \tag{87}$$

One sees immediately from (22), (86) and (87) that

$$\psi(E_{1n})=\alpha_{11}^{(1n)}I_n+\alpha_{1n}^{(1n)}E_{1n}. \tag{88}$$

In view of Lemma 3.2 (iii), we see that $[\psi(E_{12}),\psi(E_{2i})]=E_{1i},\ i=n-1,n$. It follows from (48), (51) and (70) that $E_{1,n-1}+\sum_{j=n-2}^{n}\alpha_{2j}^{(2,n-1)}E_{1j}=0$ and $E_{1n}+\sum_{j=n-1}^{n}\alpha_{2j}^{(2n)}E_{1j}=0$. Then

$$\alpha_{2,n-1}^{(2,n-1)}=\alpha_{2n}^{(2n)}=1, \tag{89}$$

$$\alpha_{2,n-2}^{(2,n-1)}=\alpha_{2n}^{(2,n-1)}=0. \tag{90}$$

We infer from (51), (87) and (89) that

$$\psi(E_{2n})=\alpha_{11}^{(2n)}I_n+E_{2n}+\alpha_{1n}^{(2n)}E_{1n}. \tag{91}$$

Let $1\leqslant p\leqslant 2$ be an integer. Setting $A=I_n+E_{p,n-1}$, $B=I_n+E_{n-2,n-1}$, $C=I_n+E_{n-1,n}$ in (10), together with Lemma 3.2 (ii), we obtain $[\psi(E_{p,n-1}),E_{n-2,n}]+[\psi(E_{n-2,n-1}),E_{pn}]=0$. It follows from (31), (48) and (72) that $\alpha_{1,n-2}^{(p,n-1)}E_{1n}+\alpha_{2,n-2}^{(p,n-1)}E_{2n}=0$. Then

$$\alpha_{1,n-2}^{(s,n-1)}=\alpha_{2,n-2}^{(s,n-1)}=0 \quad \text{for } s=1,2. \tag{92}$$

In addition, we have $[\psi(E_{i,n-1}),\psi(E_{n-1,n})]=E_{in},\ i=1,2$, by Lemma 3.2 (iii). We then see from (31), (48) and (71) that $\alpha_{1,n-1}^{(i,n-1)}E_{1n}+\alpha_{2,n-1}^{(i,n-1)}E_{2n}=E_{in},\ i=1,2$, which in turn gives $\alpha_{1,n-1}^{(1,n-1)}=1$ and $\alpha_{1,n-1}^{(2,n-1)}=0$. Combining these results with (31), (48), (85), (89), (90) and (92), we conclude that

$$\psi(E_{s,n-1})=\alpha_{11}^{(s,n-1)}I_n+E_{s,n-1}+\alpha_{1n}^{(s,n-1)}E_{1n} \tag{93}$$

for $s=1,2$. We are done.

Let $\mu,\eta : T_n(\mathbb{F}_2)\to\mathbb{F}_2$ be the additive maps defined by

$$\mu(A)=a_{11}(\alpha_{11}+1)+\sum_{1\leqslant i<j\leqslant n} a_{ij}\alpha_{11}^{(ij)},$$

$$\eta(A)=a_{11}\alpha_{1n}+a_{1n}+\sum_{1\leqslant i<j\leqslant n} a_{ij}\alpha_{1n}^{(ij)}$$

for all $A=(a_{ij})\in T_n(\mathbb{F}_2)$. We denote $\gamma=\alpha_{1,n-1}^{(12)}\in\mathbb{F}_2$, $\alpha=\alpha_{1,n-1}^{(n-1,n)}\in\mathbb{F}_2$ and $\beta=\alpha_{2n}^{(12)}\in\mathbb{F}_2$. Let $\Phi_{\alpha,\beta,\gamma}: T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ be the additive map defined by

$$\Phi_{\alpha,\beta,\gamma}(A)=\big(\gamma a_{12}+\alpha a_{n-1,n}\big)E_{1,n-1}+\big(\beta a_{12}+\gamma a_{n-1,n}\big)E_{2n}$$

for all $A=(a_{ij})\in T_n(\mathbb{F}_2)$. Note that $\mu(I_n)=\alpha_{11}+1$, $\eta(I_n)=\alpha_{1n}$ and $\Phi_{\alpha,\beta,\gamma}(I_n)=0$. By (84),

$$\psi(I_n)=I_n+\mu(I_n)I_n+\eta(I_n)E_{1n}+\Phi_{\alpha,\beta,\gamma}(I_n).$$

Moreover, in view of (70)–(72), (88), (91) and (93), together with the additivity of ψ, we see that for every strictly upper triangular matrix $A=\sum_{1\leqslant i<j\leqslant n} a_{ij}E_{ij}\in T_n(\mathbb{F}_2)$,

$$\begin{aligned}
\psi(A) &= \sum_{1\leqslant i<j\leqslant n}\psi(a_{ij}E_{ij})\\
&= a_{12}\,\alpha_{1,n-1}^{(12)}E_{1,n-1}+a_{12}\,\alpha_{2n}^{(12)}E_{2n}+a_{n-1,n}\,\alpha_{1,n-1}^{(n-1,n)}E_{1,n-1}\\
&\quad +a_{n-1,n}\,\alpha_{1,n-1}^{(12)}E_{2n}+\sum_{1\leqslant i<j\leqslant n,\,(i,j)\neq(1,n)} a_{ij}E_{ij}+\sum_{1\leqslant i<j\leqslant n} a_{ij}\alpha_{11}^{(ij)}I_n\\
&\quad +\sum_{1\leqslant i<j\leqslant n} a_{ij}\alpha_{1n}^{(ij)}E_{1n}\\
&= \gamma a_{12}E_{1,n-1}+\beta a_{12}E_{2n}+\alpha a_{n-1,n}E_{1,n-1}+\gamma a_{n-1,n}E_{2n}\\
&\quad +\sum_{1\leqslant i<j\leqslant n} a_{ij}E_{ij}+\sum_{1\leqslant i<j\leqslant n} a_{ij}\alpha_{11}^{(ij)}I_n+\Big(a_{1n}+\sum_{1\leqslant i<j\leqslant n} a_{ij}\alpha_{1n}^{(ij)}\Big)E_{1n}\\
&= A+\mu(A)I_n+\eta(A)E_{1n}+\Phi_{\alpha,\beta,\gamma}(A).
\end{aligned}$$

Consequently, $\psi(A)=A+\mu(A)I_n+\eta(A)E_{1n}+\Phi_{\alpha,\beta,\gamma}(A)$ for every strictly upper triangular matrix $A\in T_n(\mathbb{F}_2)$ and $A=I_n$. This proves the lemma. □

We are in a position to prove the main result of this section.

Proof of Theorem 1.1. The sufficiency follows immediately from Example 2.1. We now prove the necessity. By Lemma 3.4, there exist $\alpha,\beta,\gamma\in\mathbb{F}_2$, additive maps $\mu,\eta : T_n(\mathbb{F}_2)\to\mathbb{F}_2$ such that

$$\psi(A)=A+\mu(A)I_n+\eta(A)E_{1n}+\Phi_{\alpha,\beta,\gamma}(A)$$

for all strictly upper triangular matrices $A\in T_n(\mathbb{F}_2)$ and $A=I_n$. Let $\varphi : T_n(\mathbb{F}_2)\to T_n(\mathbb{F}_2)$ be the additive map defined by

$$\varphi(A)=\psi(A)+\mu(A)I_n+\eta(A)E_{1n}+\Phi_{\alpha,\beta,\gamma}(A)$$

for all $A \in T_n(\mathbb{F}_2)$. It is easily verified that $\varphi(A) = A$ for all invertible matrices $A \in T_n(\mathbb{F}_2)$. By Lemma 3.1, there exist matrices $X_1, \ldots, X_n \in T_n(\mathbb{F}_2)$ satisfying $X_1 + \cdots + X_n = 0$ such that $\varphi(A) = A + \sum_{i=1}^n a_{ii}X_i$ for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$. We thus obtain

$$\psi(A) = A + \mu(A)I_n + \eta(A)E_{1n} + \sum_{i=1}^n a_{ii}X_i + \Phi_{\alpha,\beta,\gamma}(A)$$

for all $A = (a_{ij}) \in T_n(\mathbb{F}_2)$, which completes the proof. □

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