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SUFFICIENT CONDITIONS FOR FACTOR POSETS OF FRAMES IN $\mathbb{R}^n$ AND THEIR GRAPH ASSOCIATIONS

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Abstract. A *frame* in $\mathbb{R}^n$ is a possibly redundant set of vectors $\{f_i\}_{i\in I}$ that span $\mathbb{R}^n$. A *tight frame* in $\mathbb{R}^n$ is a generalization of an orthonormal basis. A *factor poset* P of a frame is the collection of subsets of I, ordered by inclusion, such that $J \subseteq I$ is in P if and only if $\{f_j\}_{j\in J}$ is a tight frame. In [8], the authors studied the conditions for a given poset of index sets to be the factor poset of a frame. They gave a complete characterization of this "inverse factor poset problem" for $\mathbb{R}^2$ and a necessary condition for solving this problem in $\mathbb{R}^n$. In this paper we give sufficient conditions on poset $P \subseteq 2^I$ to be a factor poset of a frame and discuss some combinatorial conditions that are necessary for $\mathbb{R}^n$. We also study how to associate tight frames to the vertices of a given graph G such that G becomes the *intersection graph* of the resulting frame. By establishing a connection between poset characteristics and graph theory, we generate new tight frames. Further we establish the connection between the independence number of a graph and the maximum number of mutually disjoint index sets of prime tight subframes. We also provide an estimation of the size of the factor poset of a frame when the corresponding graph is a complete t-partite graph.

1. Introduction

In recent years, new focus has been given to representation systems that are not a basis, but still admit stable decomposition and reconstruction algorithms. The fundamental concept in this context is a frame, which is a redundant set of vectors that span a Hilbert space.

A tight frame is a special case of a frame, which has a reconstruction formula similar to that of an orthonormal basis. Because of this simple formulation of reconstruction, tight frames are employed in a variety of applications such as sampling, signal processing, smoothing, denoising, compression, image processing, and in other areas [5, 7].

A tight frame that cannot be subdivided into smaller tight frames is called a prime tight frame. A key finding in [13] states that every tight frame F in $\mathbb{R}^n$ can be expressed as a finite collection of prime tight frames known as prime factors of F. To investigate the structure of prime factors, the authors of [8] introduced the concept of a factor poset, where the sets of vectors corresponding to certain sets of indices of the factor

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poset form tight frames. This approach led to the study of various structural properties of frames and novel methods for constructing tight frames.

The main objective of this article is to study sufficient conditions on any of indices to be a factor poset of a frame. Additionally, we present a method for constructing such a frame. We then discuss some combinatorial conditions that are necessary for a poset to be a factor poset of a frame in $\mathbb{R}^n$.

Finally, in the last section, we explore the combinatorial structure of frames using graph theory. In graph theory, an intersection graph represents the pattern of intersections of a family of sets. We take the family of sets to be the collection of index sets of prime tight subframes within a given frame. By establishing a connection between poset characteristics and graph theory, we generate new tight frames. Further we establish the connection between the independence number of a graph and the maximum number of mutually disjoint index sets of prime tight subframes. We also provide an estimation of the size of the factor poset of a frame when the corresponding graph is a complete t-partite graph.

2. Preliminaries

A set $\{f_i\}_{i=1}^k \subseteq \mathbb{R}^n$, for $k \geqslant n$ is a frame for $\mathbb{R}^n$ with frame bounds $0 < A \leqslant B < \infty$ if for all $f \in \mathbb{R}^n$,

$$A||f||^2 \leqslant \sum_{i=1}^{k} |\langle f, f_i \rangle|^2 \leqslant B||f||^2. \tag{1}$$

A frame $F = \{f_i\}_{i=1}^k$ will also be expressed as a $n \times k$ matrix F whose column vectors are f_i, $i = 1, \ldots, k$. A frame $\{f_i\}_{i=1}^k$ is said to be tight if $A = B$ in (1), [1,6,11]. In terms of F as a $n \times k$ matrix, F is a tight frame if and only if the row vectors of F form a pairwise orthogonal collection, all having the same norm [[11], Proposition 3.25]. For simplicity we consider only frames without zero vectors.

The main objective of this article is to study the combinatorial structure of frames by using partially-ordered sets (posets). This study was developed by Chan et al, [8]. We outline the most relevant definitions and theorems of this work and for further details refer to [8]. We now define the notion of a factor poset of a given frame.

Given a finite frame $F = \{f_i\}_{i \in I}$ in $\mathbb{R}^n$, with index set I, we define its factor poset $\mathbb{F}(F) \subseteq 2^I$ to be the collection of subsets J of I such that the vectors indexed by J form a tight frame

$$\mathbb{F}(F) := \left\{ J \subseteq I : \ \{f_j\}_{j \in J} \text{ is a tight frame for } \mathbb{R}^n \right\},$$

where $\mathbb{F}(F)$ is partially ordered by inclusion.

We assume $\emptyset \in \mathbb{F}(F)$. If $F_0 \subseteq F$ and F_0 is a tight frame for $\mathbb{R}^n$, we say F_0 is a tight subframe. In general a tight frame can be decomposed into tight subframes. We say a tight frame F is prime if no proper subset of F is a tight frame [13]. The index sets of prime tight subframes of a frame are the focus of study in the factor poset of a frame. We define the empty cover of F, $EC(F)$, to be the collection of index sets in

$\mathbb{F}(F)$ of prime tight subframes, that is,

$$EC(F) := \left\{J \in \mathbb{F}(F) : J \neq \emptyset \text{ and } \nexists J' \in \mathbb{F}(F) \text{ with } \emptyset \subsetneq J' \subsetneq J\right\}.$$

In the event that the frame F is clear from the context we use the notation $\mathbb{F}$ and EC for the factor poset of F and the empty cover of F, respectively.

3. Sufficient conditions for the inverse factor poset problem in $\mathbb{R}^n$

In [8], the authors studied the conditions for a given poset of index sets to be the factor poset of a frame. They gave a complete characterization of this "inverse factor poset problem" for $\mathbb{R}^2$ and a necessary condition for solving the inverse factor poset problem in $\mathbb{R}^n$. In order to state this characterization, given the index set $I = \{1,\dots,k\}$, we let $\{\mathbf{e}_i\}_{i=1}^k$ be the standard orthonormal basis for $\mathbb{R}^k$. For $J \subseteq I$, we define the *index vector of* J, denoted by $[J]$, as

$$[J] := \sum_{j \in J} \mathbf{e}_j.$$

Given a poset $P \subseteq 2^I$ ordered by set inclusion, we define the *index span of* P, denoted by $\mathscr{I}(P)$, as

$$\mathscr{I}(P) = span\{[J] : J \in P\}.$$

We say that a poset P is *span-closed* if

$$\{\,[J] : J \in P\} = \mathscr{I}(P) \cap [2^I] \quad \text{where} \quad [2^I] = \{\,[J] : J \in 2^I\,\}.$$

By abuse of notation, we use the same terminology of empty cover for the following collection $EC(P)$ where P is any poset,

$$EC(P) := \left\{J \in P : J \neq \emptyset \text{ and } \nexists J' \in P \text{ with } \emptyset \subsetneq J' \subsetneq J\right\}.$$

THEOREM 3.1. ([8]) *Let $I = \{1,\dots,k\}$ be some finite index set and $P \subseteq 2^I$ be a poset ordered by set inclusion and which contains no singletons. If there exists some frame $F = \{f_i\}_{i \in I} \subseteq \mathbb{R}^n \setminus \{\mathbf{0}\}$ with factor poset P then P is span-closed.*

The following theorem gives a complete solution to the inverse factor poset problem in $\mathbb{R}^2$.

THEOREM 3.2. ([8]) *Let $I = \{1,\dots,k\}$ be some finite index set and $P \subseteq 2^I$ be a poset ordered by set inclusion and which contains no singletons. Then, there exists some frame $F = \{f_i\}_{i \in I} \subseteq \mathbb{R}^2 \setminus \{\mathbf{0}\}$ with factor poset P if and only if P is span-closed.*

REMARK 3.1. We consider a poset

$$P = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3,4\}\}$$

with $EC(P) = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$. Note that $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 \in \mathscr{I}(P)$, but $\{1,2,3,4,\} \notin P$. Thus poset P is not a factor poset since it is not span-closed.

Although the span-closed property doesn't solve the inverse factor poset problem in higher dimensions, it gives us an insight into the combinatorial structure of index sets of frames. To explore more properties of frames, we define the diagram vector associated with a vector $f \in \mathbb{R}^n$, denoted $\tilde{f}$, by

$$\tilde{f} := \frac{1}{\sqrt{n-1}} \begin{bmatrix} f^2(1) - f^2(2) \\ \vdots \\ f^2(n-1) - f^2(n) \\ \sqrt{2n} f(1) f(2) \\ \vdots \\ \sqrt{2n} f(n-1) f(n) \end{bmatrix} \in \mathbb{R}^{n(n-1)},$$

where the difference of squares $f^2(i) - f^2(j)$ and the product $f(i)f(j)$ occur exactly once for $i < j$, $i = 1, 2, \cdots, n-1$. The notion of diagram vectors in $\mathbb{R}^n$ was developed for a characterization of scalable frames in [9]. For more details about scalable frames refer to [4, 12].

The diagram vectors give us the following characterization of a tight frame:

THEOREM 3.3. ([9, 10]) *Let $\{f_i\}_{i=1}^k$ be a set of vectors in $\mathbb{R}^n$, not all of which are zero. Then $\{f_i\}_{i=1}^k$ is a tight frame if and only if $\sum_{i=1}^k \tilde{f}_i = \mathbf{0}$.*

In Theorem 3.4 below we present a sufficient condition satisfied by the empty cover of a poset P so that P is the factor poset of a frame in $\mathbb{R}^n$. Moreover, we describe how to construct such a frame using Lemmas 3.5–3.8.

THEOREM 3.4. *Let P be a poset with its empty cover $EC = \{E_i\}_{i=1}^m$. If EC satisfies*

1. *$|E_i| \geqslant n$ for all $i = 1, \ldots, m$, and*
2. *for each i, E_i intersects at most one other empty cover element,*

then there exists an n-dimensional frame whose factor poset is P.

To construct the corresponding frame for Theorem 3.4, we begin with the following lemma.

LEMMA 3.5. *Let $F = [f_1 \ldots f_p]$ and $G = [g_{p+1} \ldots g_{p+q}]$ be tight frames in $\mathbb{R}^n$. Then there exists an n-dimensional frame whose factor poset has empty cover $EC(F) \uplus EC(G)$.*

Proof. We embed the q-dimensional row vectors of frame G into $\mathbb{R}^{q+1}$ and rotate them through an angle θ around the axis $\mathbf{e}_{q+1}$. We denote by

$$G(\theta) = [g_{p+1}(\theta) \ldots g_{p+q}(\theta)]$$

the frame whose row vectors are the resulting rotated vectors of G. Then $EC(G(\theta)) = EC(G)$. Note that for any angle θ,

$$\sum_{j \in \bigcup E \in S(G)} \tilde{g}_j(\theta) = \mathbf{0}, \quad \text{and} \quad \sum_{j \in \bigcup E \in S(F)} \tilde{f}_j = \mathbf{0}, \tag{2}$$

where $S(G)$ and $S(F)$ are subsets in $EC(G)$ and $EC(F)$, respectively, that are mutually disjoint. We claim that there exists an angle θ such that the empty cover of the frame $[F \quad G(\theta)]$ is $EC(F) \dot{\bigcup} EC(G)$. To do this let

$$\mathbb{J} = \{(J_1, J_2) : J_1 \subseteq \{1, \dots, p\}, J_1 \notin S(F),$$
$$J_2 \subseteq \{p+1, \dots, p+q\}, J_2 \notin S(G),$$
$$|J_1 \cup J_2| \geqslant n\}.$$

We claim that for some θ, if $(J_1, J_2) \in \mathbb{J}$ then $\{f_j\}_{j \in J_1} \cup \{g_j(\theta)\}_{j \in J_2}$ cannot form a tight frame. That is, it is enough to show

$$\sum_{j \in J_1} \tilde{f}_i + \sum_{j \in J_2} \tilde{g}_i(\theta) \neq \mathbf{0}.$$

We first note by Theorem 3.3, for any $(J_1, J_2) \in \mathbb{J}$, $\sum_{j \in J_1} \tilde{f}_i \neq \mathbf{0}$ and $\sum_{j \in J_2} \tilde{g}_i(\theta) \neq \mathbf{0}$. We now suppose that there do not exist such a θ as claimed. Then for any θ, there exists $(J_1, J_2) \in \mathbb{J}$ such that $\sum_{j \in J_1} \tilde{f}_i + \sum_{j \in J_2} \tilde{g}_i(\theta) = \mathbf{0}$. Since $\theta \in [0, 2\pi]$ and $\mathbb{J}$ is finite, there exists $(J_1', J_2') \in \mathbb{J}$ so that for any $\theta \in A, \sum_{j \in J_1'} \tilde{f}_i + \sum_{j \in J_2'} \tilde{g}_i(\theta) = \mathbf{0}$, where A is an infinite subset of $[0, 2\pi]$. However this is not possible since $\sum_{j \in J_1'} \tilde{f}_i$ is a fixed vector and $\sum_{j \in J_2'} \tilde{g}_i(\theta)$ varies depending on θ unless $\sum_{j \in J_2'} \tilde{g}_i(\theta) = \sum_{j \in J_1'} \tilde{f}_i = \mathbf{0}$. This proves our claim. $\square$

We use the following prime tight frame denoted by F_k consisting of k vectors in $\mathbb{R}^n$ for $k \geqslant n$, in Lemmas 3.6, 3.7, and 3.8 below:

$$F_k = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{\sqrt{k-n+1}} & \dots & \frac{1}{\sqrt{k-n+1}} \end{bmatrix}. \tag{3}$$

LEMMA 3.6. *Let P be a poset with disjoint finite empty cover, where each empty cover element has greater than or equal to n elements. Then there exists an n-dimensional frame whose factor poset is P.*

Proof. Let $EC = \{E_i\}_{i=1}^m$ where the sets are pairwise disjoint and $|E_i| \geqslant n$. Without loss of generality, let $E_1 = \{1, 2, \dots, |E_1|\}$, $E_2 = \{|E_1|+1, |E_1|+2, \dots, |E_1|+|E_2|\}$, $\dots$, $E_m = \{|E_1| + \dots + |E_{m-1}| + 1, \dots, |E_1| + \dots + |E_{m-1}| + |E_m|\}$. Using Lemma 3.5 with $F = F_{|E_1|}$ and $G = F_{|E_2|}$ displayed in (3) above, there exists an n-dimensional frame $F(1)$ whose empty cover is $\{E_1, E_2\}$. Next, apply Lemma 3.5 with frames $F(1)$ and $F_{|E_3|}$ to obtain a frame $F(2)$ in $\mathbb{R}^n$ whose empty cover is $\{E_1, E_2, E_3\}$. Continue the process which ends when we apply Lemma 3.5 to $F(m-1)$ and $F_{|E_m|}$ to find a frame in $\mathbb{R}^n$ whose empty cover is $\{E_i\}_{i=1}^m$. $\square$

The next two lemmas deal with the construction of a frame when the poset has two intersecting empty cover elements. For this construction, given a matrix A, we define $A(r_1 : r_2, c_1 : c_2)$ to be the submatrix of A consisting of rows labeled r_1 through r_2 (both inclusive) and columns labeled c_1 through c_2 (both inclusive). In order to illustrate the method, we consider the following poset with intersecting sets E_1 and E_2.

$$P = \{\emptyset, E_1 = \{1,2,3,4\}, E_2 = \{1,5,6\}\}.$$

We would like to construct the following 3-dimensional frame

$$F = \begin{bmatrix} M_{1\times 1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & M'_{2\times 3} & M''_{2\times 2} \end{bmatrix}_{3\times 6}.$$

We want to choose submatrices $M'_{2\times 3}$ and $M''_{2\times 2}$ so that

$$\begin{bmatrix} M_{1\times 1} & \mathbf{0} \\ \mathbf{0} & M'_{2\times 3} \end{bmatrix}_{3\times 4} \quad \text{and} \quad \begin{bmatrix} M_{1\times 1} & \mathbf{0} \\ \mathbf{0} & M''_{2\times 2} \end{bmatrix}_{3\times 3}$$

are the only two prime tight subframes of F. To this end, we consider F_4 and F_3 in $\mathbb{R}^3$ using F_k displayed in (3). We first consider the prime tight frame F_4 and denote by M and M' its submatrices defined as

$$M := F_4(1:1, 1:1), \quad M' := F_4(2:3, 2:4)$$

so that

$$F_4 = \left[\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right] = \begin{bmatrix} M_{1\times 1} & \mathbf{0} \\ \mathbf{0} & M'_{2\times 3} \end{bmatrix}.$$

To obtain $M''_{2\times 2}$ we consider the prime tight frame F_3 which is the 3×3 identity matrix. We rotate the 3-dimensional row vectors of F_3 through an angle θ around the axis $\mathbf{e}_1$ which fixes the first row of F_3. We denote by $F_3(\theta)$ the frame whose row vectors are the resulting rotated vectors of F_3, namely,

$$F_3(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

We denote by M and M'' its submatrices defined as

$$M := F_3(\theta)(1:1, 1:1), \quad M'' := F_3(\theta)(2:3, 2:3)$$

so that

$$F_3(\theta) = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{array}\right] = \begin{bmatrix} M_{1\times 1} & \mathbf{0} \\ \mathbf{0} & M''_{2\times 2} \end{bmatrix}.$$

Then, the required frame F is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\sin\theta & \cos\theta \end{bmatrix}.$$

In particular, when $\theta = \pi/4$ we get $\mathbb{F}(F) = P$ given above.

The proofs of the next two lemmas are similar to the method described in the above example.

LEMMA 3.7. *Let $P = \{\emptyset, E_1, E_2\}$ with $|E_1| \geqslant n$ and $|E_2| \geqslant n$. If $|E_i \cap E_j| = m < n$, then there exists an n-dimensional frame whose factor poset is P.*

Proof. Without loss of generality, let

$$E_1 = \{1, \dots, m, m+1, \dots, p\} \quad \text{and}$$

$$E_2 = \{1, \dots, m, p+1, \dots, p+q-m\}.$$

We then consider the prime tight frame F_p in (3) and denote by M and M' its submatrices defined as

$$M := F_p(1:m, 1:m), \quad M' := F_p(m+1:n, m+1:p).$$

Next, we take the prime tight frame F_q in (3) and a q dimensional rotation matrix around axis $\mathbf{e}_1$ with the angle θ to obtain $R_{\mathbf{e}_1}(\theta)$. Let $G_q = (R_{\mathbf{e}_1}(\theta)F_q^T)^T$. We then denote by M'' its submatrix defined as

$$M'' := G_q(m+1:n, m+1:q).$$

Let

$$F = \begin{bmatrix} M & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & M' & M'' \end{bmatrix}_{n\times(p+q-m)}.$$

In G_q we choose $\theta \in [0, 2\pi]$ so that no column vector of M'' is equal to a column vector of M'. We then have $EC(F) = \{E_1, E_2\}$. $\square$

Next we consider when $P = \{\emptyset, E_1, E_2\}$ and $|E_1 \cap E_2| = m \geqslant n$. In this case we need the conditions $E_1 \setminus E_2 \neq \emptyset$ and $E_2 \setminus E_1 \neq \emptyset$. Otherwise $E_2 \notin EC$ or $E_1 \notin EC$.

LEMMA 3.8. *Let $P = \{\emptyset, E_1, E_2\}$ with $|E_1| \geqslant n$ and $|E_2| \geqslant n$. If $|E_i \cap E_j| = m \geqslant n$, $E_1 \setminus E_2 \neq \emptyset$, and $E_2 \setminus E_1 \neq \emptyset$, then there exists an n-dimensional frame whose factor poset is P.*

Proof. Without loss of generality, let

$$E_1 = \{1, \dots, m, m+1, \dots, p\} \quad \text{and}$$

$$E_2 = \{1, \dots, m, p+1, \dots, p+q-m\}.$$

We consider the prime tight frame F_p and denote by M and M' its submatrices defined as

$$M := F_p(1:n-1, 1:m), \quad M' := F_p(n:n, m+1:p).$$

We then define a row matrix $M'' := [a_1, \dots, a_{q-m}]$ and consider a frame

$$F = \begin{bmatrix} M & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & M' & M'' \end{bmatrix}_{n \times (p+q-m)}.$$

Next, we take the prime tight frame F_{q-m} in (3) and a $q-m$ dimensional rotation matrix around axis $\mathbf{e}_1$ with an angle θ to obtain $R_{\mathbf{e}_1}(\theta)$. Let $G_q = (R_{\mathbf{e}_1}(\theta)F_{q-m}^T)^T$. Choose $\theta \in [0, 2\pi]$ so that no column vector of M'' is equal to a column vector of M'. Then we have $EC(F) = \{E_1, E_2\}$. $\square$

We are now ready to prove Theorem 3.4:

Proof. If an empty cover element E_k does not intersect any other elements, we associate the prime tight frame $F_{|E_k|}$ to E_k. Suppose E_i intersects one other E_j in the empty cover. If $|E_i \cap E_j| < n$, using Lemma 3.6, we construct a corresponding frame F_{ij}. If $|E_i \cap E_j| \geqslant n$, then we require $E_i \setminus E_j \neq \emptyset$ and $E_j \setminus E_i \neq \emptyset$. Thus, using Lemma 3.7, we construct a corresponding frame F_{ij}. Now we consider the collection of all F_{ij} and all $F_{|E_k|}$ described above, and list them as $F_1, \dots, F_M$. Then a finite number of applications of Lemma 3.5 gives an n-dimensional frame whose factor poset has empty cover $\{E_i\}_{i=1}^m$. $\square$

4. Necessary conditions for the inverse factor poset problem in $\mathbb{R}^n$

As seen from the previous section, it is challenging to construct frames whose factor poset matches a given poset. In this section we will consider some combinatorial conditions that are necessary for a poset to be a factor poset of a frame in $\mathbb{R}^n$.

If a factor poset of a frame F contains $\{1,2,5,6\}$ and $\{3,4,5,6\}$, then from Theorem 3.3 we see that

$$\tilde{f}_1 + \tilde{f}_2 + \tilde{f}_5 + \tilde{f}_6 = \tilde{f}_3 + \tilde{f}_4 + \tilde{f}_5 + \tilde{f}_6 = \mathbf{0}$$

which results in the equality

$$\tilde{f}_1 + \tilde{f}_2 = \tilde{f}_3 + \tilde{f}_4.$$

Thus $\{1,2\}$ and $\{3,4\}$ can be considered as copies of each other. As a result, for a given frame $F = \{f_i\}_{i \in I}$ with factor poset $\mathbb{F}$, if J and K are copies of each other, then $\sum_{i \in J} \tilde{f}_i = \sum_{i \in K} \tilde{f}_i$. This motivates the following definition.

DEFINITION 4.1. ([8]) For a given poset P and any A, B in P we say that $A \setminus B$ and $B \setminus A$ are copies of each other. We denote copies by $A \setminus B \sim B \setminus A$.

Therefore, if we remove a set of vectors from a tight frame and replace it with a copy we will not affect the tightness of that frame. We make the following observation about sets that are copies of each other.

PROPOSITION 4.1. *Let $F = \{f_i\}_{i=1}^k$ be a frame in $\mathbb{R}^n$. If $\{1,2,\ldots,m\} \sim \{m+1\}$ in $\mathbb{F}$ and $f_1, f_2, \ldots, f_m$ are pairwise orthogonal, then there exists i such that $1 \leqslant i \leqslant m$ and f_i is collinear to f_{m+1}.*

Proof. Since $\{1,\ldots,m\} \sim \{m+1\}$, we know that $\tilde{f}_1 + \ldots + \tilde{f}_m = \tilde{f}_{m+1}$. Since $f_1, f_2, \ldots, f_m$ are pairwise orthogonal, without loss of generality, we assume that $f_1 = a_1\mathbf{e}_1, f_2 = a_2\mathbf{e}_2, \ldots, f_m = a_m\mathbf{e}_m$ where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ are the standard basis vectors in $\mathbb{R}^n$. Using the diagram vector equation above, we see that $f_{m+1}(i)f_{m+i}(j) = 0$ for $i < j$. This implies $f_{m+1}(i) = 0$ for all but one i. Therefore, there exists one i such that $1 \leqslant i \leqslant m$ and f_{m+1} is collinear to f_i. $\square$

REMARK 4.1. As a consequence of the above proposition if $\{1\} \sim \{2\}$ in $\mathbb{F}$, then $f_1 = \pm f_2$.

The following "closure condition" was given in [8].

PROPOSITION 4.2. ([8]) *For a poset P to be a factor poset of a frame, if J and K are copies of each other, and there exists $A \in P$ such that $J \subseteq A$ and $K \cap A = \emptyset$, then $(A \setminus J) \cup K \in P$.*

We give below additional necessary conditions for a poset P to be a factor poset using the notion of copies of sets. We define C_A to be the collection of all copies of A, i.e.,

$$C_A := \{B : B \sim A\}.$$

PROPOSITION 4.3. *Let P be a poset of subsets of $I = \{1,\ldots,k\}$. Assume for each $i \in I$, $C_{\{i\}}$ consists of only singleton sets. Then, P is a factor poset of a frame $\{f_i\}_{i\in I}$ only if $|\cup\{C_{\{i\}} : i \in I\}| \geqslant n$.*

Proof. Assume that there is a frame $F = \{f_i\}_{i=1}^k$ in $\mathbb{R}^n$ whose factor poset is P. Since each $C_{\{i\}}$ for $i \in I$ is a collection of only singleton sets, we get $\{i\} \sim \{j\}$ for some $j \in I$. From Remark 4.1 we see that $f_i = \pm f_j$ and hence $span\{f_i\} = span\{f_j\}$. Suppose $N = |\cup\{C_{\{i\}} : i \in I\}| < n$. Then $dim(span\{f_j : \{j\} \in \cup_{i\in I} C_i\}) \leqslant N < n$. This contradicts the assumption that $\{f_i\}_{i\in I}$ is a frame in $\mathbb{R}^n$. Hence $|\cup\{C_{\{i\}} : i \in I\}| \geqslant n$. $\square$

The following corollary follows from Proposition 4.3.

COROLLARY 4.4. *Let P be a poset of subsets of $I = \{1,\ldots,k\}$. Assume for each $i \in I$, either $\{i\}$ is in every set of the empty cover or $C_{\{i\}}$ is a collection of only singleton sets. Then, P is a factor poset of a frame $\{f_i\}_{i\in I}$ in $\mathbb{R}^n$ only if $|\cap_{E\in EC} E| + |\cup\{C_{\{i\}} : i \in I\}| \geqslant n$.*

5. Frames and intersection graphs

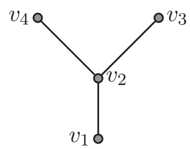
In graph theory, an intersection graph represents the pattern of intersections of a family of sets. In this section we take the family of sets to be the empty cover of the factor poset of a frame. If G is the intersection graph of the empty cover of the factor poset of a frame, then this allows us to construct tight frames corresponding to the vertices of the graph.

A *set system* is an ordered pair $(I,\mathbb{A})$ where $I=\{1,2,\ldots,n\}$ and $\mathbb{A}$ is a family of subsets of I. Given a set system we associate its *intersection graph* whose vertex set is $\mathbb{A}$ and two sets in $\mathbb{A}$ are considered adjacent in the graph if their intersection is nonempty [3].

DEFINITION 5.1. Let F be a frame and $\mathbb{F}(F)$ be its factor poset with empty cover $EC=\{E_i\}_{i=1}^m$. We say that G is the *intersection graph of the frame* F, if G is the intersection graph of the empty cover EC.

In this section we describe how to associate tight frames to the vertices of a given graph G such that G becomes the intersection graph of the resulting frame F.

As a first example we show that the following tree graph is the intersection graph of a frame by constructing a frame in $\mathbb{R}^2$.



To obtain this tree as the intersection graph of a frame F, we first associate to vertex v_1, a tight frame F_1 in $\mathbb{R}^2$ as follows:

$$F_1=\begin{bmatrix}\sqrt[4]{2} & \sqrt[4]{3} & 0\\ 0 & 0 & \sqrt{\sqrt{2}+\sqrt{3}}\end{bmatrix}=\begin{bmatrix}f_1 & f_2 & f_3\end{bmatrix}$$

where f_1, f_2, and f_3 are the column vectors of F_1. Then we associate to the vertices v_2,v_3, and v_4 the tight frames F_2,F_3, and F_4 as follows:

$$F_2=\begin{bmatrix}\sqrt[4]{2} & \sqrt[4]{5} & \sqrt[4]{7} & \sqrt[4]{11} & 0\\ 0 & 0 & 0 & 0 & c\end{bmatrix}=\begin{bmatrix}f_1 & f_4 & f_5 & f_6 & f_7\end{bmatrix}$$

where

$$c=\sqrt{\sqrt{2}+\sqrt{5}+\sqrt{7}+\sqrt{11}},$$

$$F_3=\begin{bmatrix}\sqrt[4]{5} & \sqrt[4]{13} & 0\\ 0 & 0 & \sqrt{\sqrt{5}+\sqrt{13}}\end{bmatrix}=\begin{bmatrix}f_4 & f_8 & f_9\end{bmatrix},$$

$$F_4=\begin{bmatrix}\sqrt[4]{7} & \sqrt[4]{17} & 0\\ 0 & 0 & \sqrt{\sqrt{7}+\sqrt{17}}\end{bmatrix}=\begin{bmatrix}f_5 & f_{10} & f_{11}\end{bmatrix},$$

where $f_i, i=1,2,\ldots,11$ are the column vectors. Then EC of the frame $F=\left[f_1 \ \ldots \ f_{11}\right]$ is $\{E_1=\{1,2,3\},E_2=\{4,8,9\},E_3=\{5,10,11\},E_4=\{1,4,5,6,7\}\}$. We note that the entries of F_i are specifically chosen so that $EC(F)=\{E_1,E_2,E_3,E_4\}$. Now we identify index sets of frame vectors to the corresponding vertices and write $v_1=\{1,2,3\}$, $v_2=\{1,4,5,6,7\}$, $v_3=\{4,8,9\}$, and $v_4=\{5,10,11\}$. And the factor poset of F is found by taking union of disjoint sets in $EC(F)$. That is, if S is any subcollection of EC whose elements are mutually disjoint, then $\bigcup_{E\in S}E\in\mathbb{F}(F)$. Since all possible subcollections of EC with disjoint elements are $\emptyset,\{E_1\},\{E_2\},\{E_3\},\{E_4\}$, $\{E_1,E_2\},\{E_1,E_3\},\{E_2,E_3\},\{E_1,E_2,E_3\}$, we have

$$\mathbb{F}(F)=\{\emptyset,\{1,2,3\},\{4,8,9\},\{5,10,11\},\{1,4,5,6,7\}$$

$$\{1,2,3,4,8,9\},\{1,2,3,5,10,11\},\{4,5,8,9,10,11\},\{1,2,3,4,5,8,9,10,11\}\}.$$

A general result based on the above example is proved in Proposition 5.3. It is also clear that the tree graph is the intersection graph of the frame F.

One may construct a frame in $\mathbb{R}^n$ by a simple modification of the above $\mathbb{R}^2$ frame F as follows. First, we extend all the existing vectors $a_1\mathbf{e}_1$ and $a_2\mathbf{e}_2$ in the $\mathbb{R}^2$ frame by adding 0 entries so that in the vectors $a_1\mathbf{e}_1$ and $a_2\mathbf{e}_2$ the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ now denote the standard basis vectors in $\mathbb{R}^n$. We then add additional vectors to the vertices of the graph of the form $a_2\mathbf{e}_3,a_2\mathbf{e}_4,\ldots,a_2\mathbf{e}_n$ where $\mathbf{e}_3,\mathbf{e}_4,\ldots,\mathbf{e}_n$ are the standard basis vectors in $\mathbb{R}^n$.

For example, the three dimensional frames associated to the vertices of the above tree graph are

$$F_1=\begin{bmatrix}\sqrt[4]{2} & \sqrt[4]{3} & 0 & 0\\ 0 & 0 & \sqrt{\sqrt{2}+\sqrt{3}} & 0\\ 0 & 0 & 0 & \sqrt{\sqrt{2}+\sqrt{3}}\end{bmatrix}=\left[f_1 \ f_2 \ f_3 \ f_4\right],$$

$$F_2=\begin{bmatrix}\sqrt[4]{2} & \sqrt[4]{5} & \sqrt[4]{7} & \sqrt[4]{11} & 0 & 0\\ 0 & 0 & 0 & 0 & c & 0\\ 0 & 0 & 0 & 0 & 0 & c\end{bmatrix}=\left[f_1 \ f_5 \ f_6 \ f_7 \ f_8 \ f_9\right]$$

where

$$c=\sqrt{\sqrt{2}+\sqrt{5}+\sqrt{7}+\sqrt{11}},$$

$$F_3=\begin{bmatrix}\sqrt[4]{5} & \sqrt[4]{13} & 0 & 0\\ 0 & 0 & \sqrt{\sqrt{5}+\sqrt{13}} & 0\\ 0 & 0 & 0 & \sqrt{\sqrt{5}+\sqrt{13}}\end{bmatrix}=\left[f_5 \ f_{10} \ f_{11} \ f_{12}\right],$$

$$F_4=\begin{bmatrix}\sqrt[4]{7} & \sqrt[4]{17} & 0 & 0\\ 0 & 0 & \sqrt{\sqrt{7}+\sqrt{17}} & 0\\ 0 & 0 & 0 & \sqrt{\sqrt{7}+\sqrt{17}}\end{bmatrix}=\left[f_6 \ f_{13} \ f_{14} \ f_{15}\right].$$

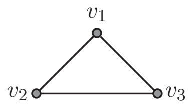
Then the corresponding EC is

$$\{\{1,2,3,4\},\{5,10,11,12\},\{6,13,14,15\},\{1,5,6,7,8,9\}\},$$

and the factor poset is

$$\mathbb{F}(F) = \{\emptyset, \{1,2,3,4\}, \{5,10,11,12\}, \{6,13,14,15\}, \{1,5,6,7,8,9\},$$
$$\{1,2,3,4,5,10,11,12\}, \{1,2,3,4,6,13,14,15\}, \{5,6,10,11,12,13,14,15\},$$
$$\{1,2,3,4,5,6,10,11,12,13,14,15\}\}.$$

As a second example we consider the following complete graph which has three vertices, v_1, v_2, v_3 and three edges.



We associate the following tight frames in $\mathbb{R}^2$ to each vertex of the graph.

$$F_1 = \begin{bmatrix} \sqrt[4]{2} & \sqrt[4]{3} & \sqrt[4]{5} & 0 \\ 0 & 0 & 0 & \sqrt{\sqrt{2}+\sqrt{3}+\sqrt{5}} \end{bmatrix} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 \end{bmatrix},$$

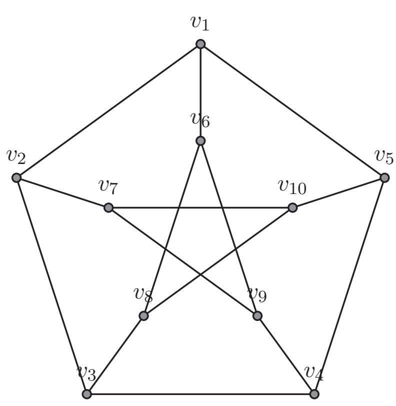
$$F_2 = \begin{bmatrix} \sqrt[4]{2} & \sqrt[4]{7} & \sqrt[4]{11} & 0 \\ 0 & 0 & 0 & \sqrt{\sqrt{2}+\sqrt{7}+\sqrt{11}} \end{bmatrix} = \begin{bmatrix} f_1 & f_5 & f_6 & f_7 \end{bmatrix},$$

$$F_3 = \begin{bmatrix} \sqrt[4]{3} & \sqrt[4]{7} & \sqrt[4]{13} & 0 \\ 0 & 0 & 0 & \sqrt{\sqrt{3}+\sqrt{7}+\sqrt{13}} \end{bmatrix} = \begin{bmatrix} f_2 & f_5 & f_8 & f_9 \end{bmatrix}.$$

Then for the frame $F = \{f_1, f_2, \ldots, f_9\}$ the empty cover EC is

$$\{\{1,2,3,4\}, \{1,5,6,7\}, \{2,5,8,9\}\}$$

and the factor poset $\mathbb{F}(F) = \{\emptyset, \{1,2,3,4\}, \{1,5,6,7\}, \{2,5,8,9\}\}$. For the final example, we consider the Petersen graph.



We associate the following frames in $\mathbb{R}^2$ to each vertex of the Petersen graph.

$$F_1 = \begin{bmatrix} \sqrt[4]{2} & \sqrt[4]{3} & \sqrt[4]{5} & \sqrt[4]{7} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{2}+\sqrt{3}+\sqrt{5}+\sqrt{7}} \end{bmatrix}$$
$$= \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \end{bmatrix},$$
$$F_2 = \begin{bmatrix} \sqrt[4]{2} & \sqrt[4]{11} & \sqrt[4]{13} & \sqrt[4]{17} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{2}+\sqrt{11}+\sqrt{13}+\sqrt{17}} \end{bmatrix}$$
$$= \begin{bmatrix} f_1 & f_6 & f_7 & f_8 & f_9 \end{bmatrix},$$
$$F_3 = \begin{bmatrix} \sqrt[4]{11} & \sqrt[4]{19} & \sqrt[4]{23} & \sqrt[4]{29} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{11}+\sqrt{19}+\sqrt{23}+\sqrt{29}} \end{bmatrix}$$
$$= \begin{bmatrix} f_6 & f_{10} & f_{11} & f_{12} & f_{13} \end{bmatrix},$$
$$F_4 = \begin{bmatrix} \sqrt[4]{19} & \sqrt[4]{31} & \sqrt[4]{37} & \sqrt[4]{41} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{19}+\sqrt{31}+\sqrt{37}+\sqrt{41}} \end{bmatrix}$$
$$= \begin{bmatrix} f_{10} & f_{14} & f_{15} & f_{16} & f_{17} \end{bmatrix},$$
$$F_5 = \begin{bmatrix} \sqrt[4]{3} & \sqrt[4]{31} & \sqrt[4]{43} & \sqrt[4]{47} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{3}+\sqrt{31}+\sqrt{43}+\sqrt{47}} \end{bmatrix}$$
$$= \begin{bmatrix} f_2 & f_{14} & f_{18} & f_{19} & f_{20} \end{bmatrix},$$
$$F_6 = \begin{bmatrix} \sqrt[4]{5} & \sqrt[4]{53} & \sqrt[4]{59} & \sqrt[4]{61} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{5}+\sqrt{53}+\sqrt{59}+\sqrt{61}} \end{bmatrix}$$
$$= \begin{bmatrix} f_3 & f_{21} & f_{22} & f_{23} & f_{24} \end{bmatrix},$$
$$F_7 = \begin{bmatrix} \sqrt[4]{13} & \sqrt[4]{67} & \sqrt[4]{71} & \sqrt[4]{73} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{13}+\sqrt{67}+\sqrt{71}+\sqrt{73}} \end{bmatrix}$$
$$= \begin{bmatrix} f_7 & f_{25} & f_{26} & f_{27} & f_{28} \end{bmatrix},$$
$$F_8 = \begin{bmatrix} \sqrt[4]{23} & \sqrt[4]{53} & \sqrt[4]{79} & \sqrt[4]{83} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{23}+\sqrt{53}+\sqrt{79}+\sqrt{83}} \end{bmatrix}$$
$$= \begin{bmatrix} f_{11} & f_{21} & f_{29} & f_{30} & f_{31} \end{bmatrix},$$
$$F_9 = \begin{bmatrix} \sqrt[4]{37} & \sqrt[4]{59} & \sqrt[4]{67} & \sqrt[4]{89} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{37}+\sqrt{59}+\sqrt{67}+\sqrt{89}} \end{bmatrix}$$
$$= \begin{bmatrix} f_{15} & f_{22} & f_{25} & f_{32} & f_{33} \end{bmatrix},$$

$$F_{10} = \begin{bmatrix} \sqrt[4]{43} & \sqrt[4]{71} & \sqrt[4]{79} & \sqrt[4]{97} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\sqrt{43}+\sqrt{71}+\sqrt{79}+\sqrt{97}} \end{bmatrix}$$

$$= \begin{bmatrix} f_{18} & f_{26} & f_{29} & f_{34} & f_{35} \end{bmatrix},$$

Then for the frame $F = \{f_1, f_2, \dots, f_{35}\}$ the empty cover EC is

$$\{\{1,2,3,4,5\},\{1,6,7,8,9\},\{6,10,11,12,13\},\{10,14,15,16,17\},$$

$$\{2,14,18,19,20\},\{3,21,22,23,24\},\{7,25,26,27,28\},$$

$$\{11,21,29,30,31\},\{15,22,25,32,33\},\{18,26,29,34,35\}\}.$$

We do not include $\mathbb{F}(F)$ for this example as it consists of 75 sets.

THEOREM 5.1. *Any simple graph G is an intersection graph of a frame F in $\mathbb{R}^n$.*

Proof. Since any graph G is a disjoint union of its connected components, we assume that G is a simple connected graph. Let $|G|$ denote the number of vertices in G. When $|G| = 1$, we associate to v_1 a tight frame. Suppose $|G| \geqslant 2$. We first show that G is an intersection graph of a frame in $\mathbb{R}^2$ and then extend the result to a frame in $\mathbb{R}^n$. First label the vertices of G in any order. Suppose $v_1, v_2, \dots, v_m$ are the labels of the vertices of G. Let $deg(v_i) = d_i$ and $N(v_i)$ be the set of vertices adjacent to v_i. Associate to the vertex v_1 the frame

$$F_1 = \begin{bmatrix} \sqrt{a_1^1} & \sqrt{a_2^1} & \dots & \sqrt{a_{d_1}^1} & \sqrt{b_0^1} & 0 \\ 0 & 0 & \dots & 0 & 0 & \sqrt{c_1} \end{bmatrix}$$

in $\mathbb{R}^2$ where $c_1 = a_1^1 + a_2^1 + \dots + a_{d_1}^1 + b_0^1$ and $a_i^1,\ b_0^1 > 0$ are different numbers for any i. For $i \geqslant 2$, we divide $N(v_i)$ into two disjoint subsets. One subset consists of vertices v_j where $1 \leqslant j \leqslant i-1$ such that v_i is adjacent to v_j. The other subset consists of vertices v_l where $i+1 \leqslant l \leqslant m$ such that v_i is adjacent to v_l. Denote by p_i and q_i the cardinality of these subsets so that $p_i + q_i = d_i = deg(v_i)$.

We now associate F_i to vertex v_i as follows:

$$F_i = \begin{bmatrix} \sqrt{a_1^i} & \sqrt{a_2^i} & \dots & \sqrt{a_{p_i}^i} & \sqrt{b_1^i} & \sqrt{b_2^i} & \dots & \sqrt{b_{q_i}^i} & \sqrt{b_0^i} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \sqrt{c_i} \end{bmatrix}$$

where the column vectors satisfy

1. $a_s^i > 0,\ b_t^i,\ b_0^i > 0$ for $1 \leqslant s \leqslant p_i,\ 1 \leqslant t \leqslant q_i$,

2. $c_i \neq \sum_{a \in A} a^2$,

$$A \subseteq \left\{ a : a \text{ is an entry of the first row of } F_j, 1 \leqslant j \leqslant i \right\}$$

except $c_i = a_1^i + a_2^i + \dots + a_{p_i}^i + b_1^i + b_2^i + \dots + b_{q_i}^i + b_0^i$,

3. $\sum_{b\in B_1} b^2 \neq \sum_{b\in B_2} b^2$ for $B_1 \neq B_2$,

$$B_1, B_2 \subseteq \left\{ b : b \text{ is an entry of the second row of } F_j, 1 \leqslant j \leqslant i \right\}$$

Here each vector $\begin{bmatrix} \sqrt{a_p^i} \\ 0 \end{bmatrix}$, $1 \leqslant p \leqslant p_i$, must be present in exactly one F_j for some $j < i$ that is associated with vertex v_j where v_j is adjacent to v_i. Also, we include a vector $\begin{bmatrix} \sqrt{b_q^i} \\ 0 \end{bmatrix}$, $1 \leqslant q \leqslant q_i$, in F_i for each vertex v_j with $j > i$ where v_j is adjacent to v_i.

Since $|G|$ is finite, the above process stops when a frame has been assigned to every vertex in G in the manner described above. Associating the set of indices E_i of the frame vectors F_i to the vertices v_i we see that G is the intersection graph of the empty cover $\{E_i\}_{i=1}^m$ and hence of the frame F which is the collection of distinct vectors in $\{F_i\}_{i=1}^m$.

For $\mathbb{R}^n$, we extend all the existing vectors $a_1\mathbf{e}_1$ and $a_2\mathbf{e}_2$ in the $\mathbb{R}^2$ frame by adding 0 entries to the vectors $a_1\mathbf{e}_1$ and $a_2\mathbf{e}_2$ such that $\mathbf{e}_1$ and $\mathbf{e}_2$ are now the standard basis vectors in $\mathbb{R}^n$. We add additional vectors to each set F_i associated to the vertices of the graph of the form $a_2\mathbf{e}_3, a_2\mathbf{e}_4, \ldots, a_2\mathbf{e}_n$ where $\mathbf{e}_3, \mathbf{e}_4, \ldots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{R}^n$. $\square$

In a graph any two nonadjacent vertices are said to be independent vertices. A set of mutually nonadjacent vertices is called an *independent set* of vertices. The size of a maximum independent set is called the *independence number* of the graph G and is denoted by $\alpha(G)$.

From the examples presented earlier we note that the tight frames associated with independent vertices are disjoint. Thus unions of frames associated to independent sets of the intersection graph appear in the factor poset. This result is proved below in Proposition 5.3.

PROPOSITION 5.2. ([2,8]) *For any factor poset $\mathbb{F}$ and any two elements $A, B \in \mathbb{F}$, we have*

$$A \cup B \in \mathbb{F} \iff A \cap B \in \mathbb{F} \iff A \setminus B \in \mathbb{F}.$$

The following proposition shows that any set in a factor poset can be decomposed into a disjoint union of sets in the empty cover. We give a different proof here for sake of completeness.

PROPOSITION 5.3. ([2]) *If F is a frame, then*

$$\mathbb{F} = \left\{ \dot{\bigcup_{E\in S}} E : S \text{ is any subcollection of } EC \right.$$

$$\left. \text{whose elements are mutually disjoint} \right\}.$$

Proof. Let

$$H = \left\{ \dot{\bigcup_{E \in S}} E \ : \ S \text{ is any subcollection of } EC \right.$$
$$\left. \text{whose elements are mutually disjoint} \right\}.$$

By Proposition 5.2, it is clear that $H \subseteq \mathbb{F}$.

We show that $\mathbb{F} \subseteq H$ using induction on the cardinality of nonempty sets $E \in \mathbb{F}$. Let $k_0 = \min\{|E| : E \in \mathbb{F}\}$. If $E \in \mathbb{F}$ and $|E| = k_0$, then by the definition of EC, $E \in EC$. Thus $E \in H$. Let $k > k_0$ and we assume that for all $A \in \mathbb{F}$, if $|A| < k$, then $A \in H$. Let $E \in \mathbb{F}$ such that $|E| = k$. If $E \in EC$, then $E \in H$. Suppose that $E \notin EC$. Then by the definition of EC, there exists $E' \in EC$ such that $E' \subseteq E$. Since $E' \cup E = E \in \mathbb{F}$, by Proposition 5.2, $E \setminus E' \in \mathbb{F}$. However, $|E \setminus E'| < k$, which implies that $E \setminus E' \in H$. Since $(E \setminus E') \cup E' = E$, $E \in H$. $\square$

We note that a set in $\mathbb{F}$ could be represented as a disjoint unioin of sets in EC in more than one way. For example, let

$$F = \begin{bmatrix} 3 & 0 & 4 & 0 & 0 & 5 \\ 0 & 3 & 0 & 4 & 5 & 0 \end{bmatrix}.$$

Then $EC = \{\{1,2\},\{3,4\},\{5,6\},\{1,3,5\},\{2,4,6\}\}$ and $\{1,\dots,6\} \in \mathbb{F}$. However, we have

$$\begin{aligned} \{1,2,3,4,5,6\} &= \{1,2\} \,\dot\cup\, \{3,4\} \,\dot\cup\, \{5,6\} \\ &= \{1,3,5\} \,\dot\cup\, \{2,4,6\}. \end{aligned}$$

The following two results follow from Proposition 5.3.

COROLLARY 5.4. *Suppose the m sets in the empty cover have sizes $l_1, l_2, \dots, l_m$. Then the size of any element of the factor poset must be of the form $\sum_{i=1}^m a_i l_i$ where $a_1, \dots, a_m \in \{0,1\}$.*

COROLLARY 5.5. *Let G be a simple graph with vertex set V. Let F be an associated frame in $\mathbb{R}^n$ so that G is the intersection graph of F. Then, by associating index sets of frame vectors to the vertices, we see that*

$$\mathbb{F} = \left\{ \dot{\bigcup_{E \in S}} E \ : \ S \textit{ is any subcollection of} \right.$$
$$\left. \textit{independent sets of vertices in } V \right\}.$$

Thus, we have that the independence number $\alpha(G)$ is equal to the maximum number of mutually disjoint subsets of EC.

Suppose G is a t-partite graph whose vertices are partitioned into t different independent sets. Let $A_1, A_2, \ldots, A_t$ denote the t independent sets and let $n_i := |A_i|$, for all $i \in \{1, \ldots, t\}$. The next result follows from Corollary 5.5.

COROLLARY 5.6. *If G is a complete t-partite graph, then the size of the factor poset of a frame with intersection graph G is given by*

$$\sum_{i=1}^{t} 2^{n_i} - (t-1).$$

REMARK 5.1. We note that the algorithm in the proof of Theorem 2.1 is not optimal. In many cases, we can assign a smaller size of F_i to each vertex v_i without including the vector $\begin{bmatrix} \sqrt{b_0^i} \\ 0 \end{bmatrix}$ in F_i. However, computational efficiency is not addressed in this article.

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