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ORDER BOUNDED STEVIĆ–SHARMA OPERATORS BETWEEN WEIGHTED DIRICHLET SPACES

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Abstract. The order bounded Stević-Sharma operators between weighted Dirichlet spaces are characterized, which generalizes the previous result obtained by Lin and his colleagues.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle and $H(\mathbb{D})$ be the set of all analytic functions on $\mathbb{D}$. Let dm be the normalized Lebesgue measure on $\mathbb{T}$ and dA be the normalized Lebesgue area measure on $\mathbb{D}$.

At the beginning of the paper, we give several definitions of the common function spaces. For $0 < p < \infty$, the Hardy space H^p consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p} = \Big(\sup_{0 \leqslant r < 1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) \Big)^{1/p} < \infty.$$

For $f \in H^p$, it follows from [2] that f has non-tangential limit a.e. $\mathbb{T}$. See [2] for more information of the space.

For $-1 < \alpha$ and $0 < p < \infty$, the weighted Bergman space A^p_α consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} = \Big(\int_{\mathbb{D}} |f(z)|^p dA_\alpha(z) \Big)^{1/p} < \infty,$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$. If $\alpha = 0$, it is the classical Bergman space, usually denoted by A^p. One can see [3, 9, 36] for more information about the weighted Bergman space.

The weighted Dirichlet space D^p_α consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{D^p_\alpha} = \Big(|f(0)|^p + \int_{\mathbb{D}} \big|f'(z)\big|^p dA_\alpha(z) \Big)^{1/p} < \infty.$$

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There are many studies about this space, see, for example, [6, 23, 24, 32, 34]. From the definitions, we see that $f \in D^p_\alpha$ if and only if $f' \in A^p_\alpha$. Moreover, $\|f'\|_{A^p_\alpha} \leqslant \|f\|_{D^p_\alpha}$.

Through some previous studies, we can find some relations between these spaces. For instance, it follows from [4] that if $p < \alpha+1$, then $D^p_\alpha = A^p_{\alpha-p}$. From the relations between Hardy spaces and weighted Dirichlet spaces (see [4, 23, 24]), we have that if $0 < p \leqslant 2$, then $D^p_{p-1} \subset H^p$; if $2 \leqslant p < \infty$, then $H^p \subset D^p_{p-1}$. It is well known that the Hardy space H^p is contained in the Bergman space A^{2p}. This is also true for the space D^p_{p-1}, that is, $D^p_{p-1} \subset H^p \subset A^{2p}$ for $0 < p \leqslant 2$, and $H^p \subset D^p_{p-1} \subset A^{2p}$ for $2 \leqslant p < \infty$. Therefore, if $0 < p \leqslant 2$ and $f \in D^p_{p-1}$, then $f \in H^p$, and so f has non-tangential limit a.e. $\mathbb{T}$. From this, we have that if $0 < p \leqslant 2$ and $f \in D^p_{p-1}$, then $f(re^{i\theta}) = O(1)$ as $r \to 1^-$ for a.e. $e^{i\theta} \in \mathbb{T}$.

Next, we introduce the definition of order bounded operators and the related studies. Let (X, ρ) be a metric space of analytic functions defined on Ω, $(\Omega, \mathscr{A}, \mu)$ a measure space and

$$L^p(\Omega, \mathscr{A}, \mu) = \left\{ f \mid f : \Omega \to \mathbb{C} \text{ is measurable and } \int_\Omega |f|^p d\mu < \infty \right\}.$$

An operator $T : X \to L^p(\Omega, \mathscr{A}, \mu)$ is said to be order bounded if there exists a non-negative function $g \in L^p(\Omega, \mathscr{A}, \mu)$ such that for all $f \in X$ with $\rho(f, 0) \leqslant 1$, it holds that

$$|T(f)(x)| \leqslant g(x) \text{ a.e. } [\mu].$$

This definition in the case when X is a Banach space of analytic functions on Ω was introduced by Hunziker and Jarchow in [12]. From the following facts, we see that order boundedness is an interesting class of properties. Kwapień in [14] and Schwartz in [25] proved that if X is a Banach space, μ is any measure, $1 \leqslant p < \infty$ and $T : X \to L^p(\mu)$ is order bounded, then T is p-integral. Ueki in [31] proved that every order bounded weighted composition operator between weighted Bergman spaces is bounded. Maybe motivated by these interesting studies, Gao et al. in [5] studied and obtained a sufficient condition for order boundedness of a weighted composition operator between Dirichlet spaces. Sharma in [26] proved that their condition is also necessary for order boundedness of a weighted composition operator between Dirichlet spaces. Recently, Lin et al. in [21] have characterized the order bounded weighted composition operators on weighted Dirichlet spaces and derivative Hardy spaces.

It is well known that in addition to weighted composition operators and composition operators, there are many other concrete operators such as Stević-Sharma operators. So, a natural problem is to characterize the order bounded Stević-Sharma operators. This is exactly the issue to be considered in this paper.

Let φ be an analytic self-map of $\mathbb{D}$. Associated with φ is the composition operator C_φ, which is defined by $C_\varphi f = f \circ \varphi$ for $f \in H(\mathbb{D})$. Let $u \in H(\mathbb{D})$. Associated with u is the multiplication operator M_u is defined by $M_u f = u \cdot f$ for $f \in H(\mathbb{D})$. As a product of M_u and C_φ, the weighted composition operator $W_{u,\varphi}$ is defined by $W_{u,\varphi} = M_u \cdot C_\varphi$, that is, $W_{u,\varphi} f = u \cdot f \circ \varphi$ for $f \in H(\mathbb{D})$.

Let D be the classical differentiation operator on $H(\mathbb{D})$, that is, $Df = f'$. The authors in [30] introduced the following operator, called Stević-Sharma operator,

$$T_{u_1,u_2,\varphi} = M_{u_1}C_\varphi + M_{u_2}C_\varphi D.$$

From a direct observation, we have that $W_{u,\varphi} = T_{u,0,\varphi}$. Now, we explain the reasons why this operator was introduced. Before studying the operator, the following six operators were studied many times (see, for example, [18, 19, 27]).

$$M_uC_\varphi D, \quad C_\varphi M_u D, \quad C_\varphi DM_u, \quad M_uDC_\varphi, \quad DM_uC_\varphi, \quad DC_\varphi M_u. \tag{1.1}$$

Other products containing differentiation operators can also be found in [12, 20, 28, 33] and the related references therein. However, it is easy to see that if one studies the operators in (1.1) one by one, it will require a great deal of time and effort. From the following relations, as we expected, Stević-Sharma operator can overcome this malpractice. More precisely, it holds that

$$M_uC_\varphi D = T_{0,u,\varphi}, \quad C_\varphi M_u D = T_{0,u\circ\varphi,\varphi}, \quad C_\varphi DM_u = T_{u'\circ\varphi,u\circ\varphi,\varphi},$$
$$M_uDC_\varphi = T_{0,u\varphi',\varphi}, \quad DM_uC_\varphi = T_{u',u\varphi',\varphi}, \quad DC_\varphi M_u = T_{(u'\circ\varphi)\varphi',(u\circ\varphi)\varphi',\varphi}.$$

Another reason to be interested in Stević-Sharma operators is that it needs to obtain further methods and techniques for studying its properties. For some later and continuous studies of this operator, see, for example, [7, 8, 24, 29].

In this paper, we characterize the order bounded Stević-Sharma operator between weighted Dirichlet spaces. As some applications, we can obtain the corresponding results for the operators in (1.1). Let us recall that the operator $T_{u_1,u_2,\varphi} : D^p_\alpha \to D^q_\beta$ is order bounded if and only if there exists a nonnegative function $g \in L^q(dA_\beta)$ such that for all $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \leqslant 1$, it holds that

$$\left|(T_{u_1,u_2,\varphi}f)'(z)\right| \leqslant g(z) \text{ a.e. } [dA_\beta].$$

We hope that the study can attract more attention for the order boundedness of other concrete operators. For example, one can continuously try to consider such a property for Toeplitz operators, Hankel operators and so on.

As usual, in this paper the positive numbers are denoted by C, and they may vary in different situations. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is a positive number C such that $a \leqslant Cb$ (resp. $a \geqslant Cb$). If $a \lesssim b$ and $b \gtrsim a$, we write $a \asymp b$.

2. Auxiliary lemmas

First, from the direct calculations, we have the next lemma.

LEMMA 2.1. *For each $f \in H(\mathbb{D})$, it holds that*

$$\left(T_{u_1,u_2,\varphi}f\right)'(z) = u_1'(z)f(\varphi(z)) + \left[u_1(z)\varphi'(z) + u_2'(z)\right]f'(\varphi(z)) + u_2(z)\varphi'(z)f''(\varphi(z)).$$

Let δ_z be the point evaluation functional at $z \in \mathbb{D}$ on D^p_α, that is, $\delta_z f = f(z)$ for each $f \in D^p_\alpha$. If δ_z is viewed as an operator on D^p_α, then for the norm of δ_z we have the following result (see [21]).

LEMMA 2.2. *Let $\alpha > -1$, $0 < p < \infty$ and $z \in \mathbb{D}$. Then the following statements hold.*

(a) If $p < \alpha + 2$, then $\|\delta_z\| \asymp \frac{1}{(1-|z|^2)^{\frac{\alpha+2}{p}-1}}$.

(b) If $p = \alpha + 2$, then $\|\delta_z\| \asymp \frac{1}{(\log \frac{2}{1-|z|^2})^{\frac{1-p}{p}}}$.

(c) If $p > \alpha + 2$, then $\|\delta_z\| \asymp 1$.

Let $j \in \mathbb{N}$. For a fixed $z \in \mathbb{D}$, denote $\delta_z^{(j)} f = f^{(j)}(z)$ as the jth derivative point evaluation functional at z on D^p_α. By using the similar point evaluation functional on A^p_α, we have the following result.

LEMMA 2.3. *Let $\alpha > -1$, $0 < p < \infty$ and $z \in \mathbb{D}$. Then*

$$\|\delta_z^{(j)}\| \lesssim \frac{1}{(1-|z|^2)^{\frac{\alpha+2}{p}+j-1}}.$$

Proof. Let $f \in D^p_\alpha$. From the definition, we have $f' \in A^p_\alpha$. For the functions in A^p_α, it follows from [19] that there exists a positive constant C independent of $f \in D^p_\alpha$ and $z \in \mathbb{D}$ such that

$$|f^{(j)}(z)| = |(f')^{(j-1)}(z)| \leqslant C \frac{\|f'\|_{A^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}+j-1}}. \tag{2.1}$$

Since

$$\|f'\|_{A^p_\alpha} \leqslant \|f\|_{D^p_\alpha},$$

it follows from (2.1) that

$$\|\delta_z^{(j)}\| \lesssim \frac{1}{(1-|z|^2)^{\frac{\alpha+2}{p}+j-1}}.$$

The proof is end. □

By Lemma 2.3, there exists a positive constant C independent of $f \in D^p_\alpha$ and $z \in \mathbb{D}$ such that

$$|f^{(j)}(z)| \leqslant C \frac{\|f\|_{D^p_\alpha}}{(1-|z|^2)^{\frac{\alpha+2}{p}+j-1}},$$

which shows that $\delta_z^{(j)}$ is bounded on D^p_α. In particular, by Riesz's representation theorem in Hilbert space theory, there exists a unique function $K_z^{[j]}$ in D^2_α such that

$$f^{(j)}(z) = \langle f, K_z^{[j]} \rangle$$

for all $f \in D^2_\alpha$. $K^{[0]}_z$ is usually called the reproducing kernel function in D^2_α.

In order to construct some functions in D^p_α, we need the following lemma (see [36]).

LEMMA 2.4. *Assume that $z \in \mathbb{D}$, $\alpha > -1$ and $c \geqslant 0$. Let*

$$J_{\alpha,c}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-\bar{z}w|^{\alpha+2+c}} dA(w).$$

Then we have the following asymptotic properties.

(a) If $c > 0$, then

$$J_{\alpha,c}(z) \asymp \frac{1}{(1-|z|^2)^c}.$$

(b) If $c = 0$, then

$$J_{\alpha,c}(z) \asymp \log \frac{1}{1-|z|^2}.$$

By using Lemma 2.4, we obtain some test functions in D^p_α as follows.

LEMMA 2.5. *Let $\alpha > -1$ and $0 < p < \infty$. For each $w \in \mathbb{D}$ and $j \in \mathbb{N}$, the following function belongs to D^p_α*

$$k_{w,j}(z) = \frac{(1-|w|^2)^{\frac{\alpha+2}{p}+j}}{(1-\overline{w}z)^{\frac{2\alpha+4}{p}+j-1}}.$$

Moreover,

$$\sup_{w\in\mathbb{D}} \|k_{w,j}\|_{D^p_\alpha} \lesssim 1. \tag{2.2}$$

Proof. From the calculations, it follows that

$$|k_{w,j}(0)| = (1-|w|^2)^{\frac{\alpha+2}{p}+j} \leqslant 1$$

and

$$k'_{w,j}(z) = c_{\alpha,j}\overline{w}\frac{(1-|w|^2)^{\frac{\alpha+2}{p}+j}}{(1-\overline{w}z)^{\frac{2\alpha+4}{p}+j}},$$

where $c_{\alpha,j} = \frac{2\alpha+4}{p} + j - 1$. Then, by Lemma 2.4 (a), there exists a positive constant C independent of w such that

$$\begin{aligned}
\|k_{w,j}\|^p_{D^p_\alpha} &= |k_{w,j}(0)|^p + \int_{\mathbb{D}} |k'_{w,j}(z)|^p (1-|z|^2)^\alpha dA(z) \\
&\leqslant 1 + c_{\alpha,j}(1-|w|^2)^{\alpha+2+pj} \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{|1-\overline{w}z|^{2\alpha+4+pj}} dA(z) \\
&= 1 + c_{\alpha,j}C.
\end{aligned}$$

From this, (2.2) holds, and then the desired result follows. $\square$

For the case $p = \alpha + 2$, we also can give some special functions in D^p_α.

LEMMA 2.6. *Let $\alpha > -1$ and $p = \alpha + 2$. For each $w \in \mathbb{D}$ and $j \in \mathbb{N}$, the following function belongs to D_α^p*

$$l_{w,j}(z) = \frac{(\log \frac{2}{1-\overline{w}z})^j}{(\log \frac{2}{1-|w|^2})^{\frac{1}{p}+j-1}}.$$

Moreover,

$$\sup_{w\in\mathbb{D}} \|l_{w,j}\|_{D_\alpha^p} \lesssim 1. \tag{2.3}$$

Proof. First we have $|1-\overline{w}z| < 2$ for all $z \in \mathbb{D}$. Then, by the definition of complex logarithmic function,

$$\begin{aligned}\left|\log \frac{2}{1-\overline{w}z}\right| &= \left|\log\left|\frac{2}{1-\overline{w}z}\right| + i\arg\left(\frac{2}{1-\overline{w}z}\right)\right| \\ &\leqslant \log\left|\frac{2}{1-\overline{w}z}\right| + \pi \\ &\leqslant \log \frac{4}{1-|w|^2} + \pi\end{aligned}$$

for all $z \in \mathbb{D}$. From this and the calculations, it follows that

$$\frac{\left|\log \frac{2}{1-\overline{w}z}\right|}{\log \frac{2}{1-|w|^2}} \leqslant \frac{\log \frac{4}{1-|w|^2} + \pi}{\log \frac{2}{1-|w|^2}} \leqslant \frac{\log 4 + \pi}{\log 2} \tag{2.4}$$

for all $z \in \mathbb{D}$.

On the other hand, we have

$$\left|l_{w,j}(0)\right| = \frac{(\log 2)^j}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}+j-1}} \leqslant (\log 2)^{1-\frac{1}{p}}$$

and

$$l'_{w,j}(z) = j\frac{\left(\log \frac{2}{1-\overline{w}z}\right)^{j-1}}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}+j-1}} \frac{\overline{w}}{1-\overline{w}z}.$$

Thus, by (2.4) and Lemma 2.4 (b), there exists a positive constant C independent of w such that

$$\begin{aligned}\|l_{w,j}\|_{D_\alpha^p}^p &= \left|l_{w,j}(0)\right|^p + \int_{\mathbb{D}} \left|l'_{w,j}(z)\right|^p \left(1-|z|^2\right)^\alpha dA(z) \\ &\leqslant (\log 2)^{p-1} + \left(\frac{\log 4 + \pi}{\log 2}\right)^{p(j-1)} \frac{1}{\log \frac{2}{1-|w|^2}} \int_{\mathbb{D}} \frac{\left(1-|z|^2\right)^\alpha}{|1-\overline{w}z|^{\alpha+2}} dA(z) \\ &= (\log 2)^{p-1} + \left(\frac{\log 4 + \pi}{\log 2}\right)^{p(j-1)} C.\end{aligned}$$

From this, it follows that (2.3) holds, and then the desired result follows. □

Next, we will use the functions $k_{w,j}$ and $l_{w,j}$ to construct new test functions. Such an approach has actually been formed and implemented in our papers (see [15–17] for early papers; see [10, 11] for recent papers).

LEMMA 2.7. *Let $\alpha > -1$, $0 < p < \infty$ and $w \in \mathbb{D}$. If $p < \alpha + 2$, then for each $k \in \{0,1,2\}$, there exist constants $c_{k,1}$, $c_{k,2}$ and $c_{k,3}$ such that the function*

$$f_{w,k}(z) = c_{k,1}k_{w,1}(z) + c_{k,2}k_{w,2}(z) + c_{k,3}k_{w,3}(z)$$

satisfies

$$f_{w,k}^{(k)}(w) = \frac{\overline{w}^k}{(1-|w|^2)^{\frac{\alpha+2}{p}+k-1}} \quad \text{and} \quad f_{w,k}^{(j)}(w) = 0 \tag{2.5}$$

for $j \in \{0,1,2\} \backslash \{k\}$. Moreover,

$$\sup_{w\in\mathbb{D}} \|f_{w,k}\|_{D_\alpha^p} \lesssim 1.$$

Proof. First, assume that $k = 0$. For the sake of convenience, we write $a = (2\alpha + 4)/p$. By a direct calculation, (2.5) is equivalent to the system

$$\begin{cases} c_{0,1} + c_{0,2} + c_{0,3} = 1 \\ ac_{0,1} + (a+1)c_{0,2} + (a+2)c_{0,3} = 0 \\ a(a+1)c_{0,1} + (a+1)(a+2)c_{0,2} + (a+2)(a+3)c_{0,3} = 0. \end{cases} \tag{2.6}$$

It is not difficult to see that there exist constants $c_{0,1}$, $c_{0,2}$ and $c_{0,3}$ such that the system (2.6) holds. For the rest of cases, the result can similarly proved. So, we omit the details. □

LEMMA 2.8. *Let $\alpha > -1$, $p = \alpha + 2$ and $w \in \mathbb{D}$. Then the function*

$$g_w(z) = 3l_{w,1}(z) - 3l_{w,2}(z) + l_{w,3}(z)$$

satisfies

$$g_w(w) = \frac{1}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}-1}} \quad \text{and} \quad g_w'(w) = g_w''(w) = 0. \tag{2.7}$$

Moreover,

$$\sup_{w\in\mathbb{D}} \|g_w\|_{D_\alpha^p} \lesssim 1. \tag{2.8}$$

Proof. From the calculations, we have

$$l'_{w,j}(w) = \frac{j}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}}} \frac{\overline{w}}{1-|w|^2}$$

for $j \in \{1,2,3\}$,

$$l''_{w,1}(w) = \frac{1}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}}} \frac{\overline{w}^2}{(1-|w|^2)^2},$$

$$l''_{w,2}(w) = \frac{2}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}+1}} \frac{\overline{w}^2}{(1-|w|^2)^2} + \frac{2}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}}} \frac{\overline{w}^2}{(1-|w|^2)^2},$$

and

$$l''_{w,3}(w) = \frac{6}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}+1}} \frac{\overline{w}^2}{(1-|w|^2)^2} + \frac{3}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}}} \frac{\overline{w}^2}{(1-|w|^2)^2}.$$

Then, it holds that

$$g_w(w) = \frac{1}{\left(\log \frac{2}{1-|w|^2}\right)^{\frac{1}{p}-1}},$$

$$g'_w(w) = 3l'_{w,1}(w) - 3l'_{w,2}(w) + l'_{w,3}(w) = 0,$$

and

$$g'_w(w) = 3l''_{w,1}(w) - 3l''_{w,2}(w) + l''_{w,3}(w) = 0.$$

Thus, (2.7) holds, and (2.8) follows from Lemma 2.6. □

3. Order bounded Stević-Sharma operators

For brevity, on $\mathbb{D}$ define the functions required as follows:

$$g_1(z) = \frac{|u'_1(z)|^q}{\left(1-|\varphi(z)|^2\right)^{\frac{q(\alpha+2)}{p}-q}} + \frac{|u_1(z)\varphi'(z) + u'_2(z)|^q}{\left(1-|\varphi(z)|^2\right)^{\frac{q(\alpha+2)}{p}}} + \frac{|u_2(z)\varphi'(z)|^q}{\left(1-|\varphi(z)|^2\right)^{\frac{q(\alpha+2)}{p}+q}},$$

$$g_2(z) = \frac{|u'_1(z)|^q}{\left(\log \frac{2}{1-|\varphi(z)|^2}\right)^{\frac{q(1-p)}{p}}} + \frac{|u_1(z)\varphi'(z) + u'_2(z)|^q}{(1-|\varphi(z)|^2)^q} + \frac{|u_2(z)\varphi'(z)|^q}{(1-|\varphi(z)|^2)^{2q}},$$

and

$$g_3(z) = \frac{|u_1(z)\varphi'(z) + u'_2(z)|^q}{\left(1-|\varphi(z)|^2\right)^{\frac{q(\alpha+2)}{p}}} + \frac{|u_2(z)\varphi'(z)|^q}{\left(1-|\varphi(z)|^2\right)^{\frac{q(\alpha+2)}{p}+q}}.$$

Objectively speaking, the literature on boundedness and compactness of Stević-Sharma operators between weighted Dirichlet spaces is largely absent. We have spent a lot of time to characterize the boundedness and compactness of such operators and also read many related references (for example, [1, 13, 22, 30, 35]), but we encounter some difficulties that are difficult to overcome in a short period of time. We have compared the difference between such operators on weighted Bergman spaces (see [30]) and on weighted Dirichlet spaces, and we find the difference being that the latter has an extra term that we have no way of dealing with. However, we have obtained the following sufficient condition. It is a notable point that the result does not depend on the choice of p and q.

PROPOSITION 3.1. *Let $-1<\alpha,\ \beta,\ 0<p,q<\infty,\ u_1,u_2\in H(\mathbb{D})$ and φ be an analytic self-map of $\mathbb{D}$. Then the following statements hold.*

(a) If $p<\alpha+2$ and $\int_{\mathbb{D}} g_1(z)dA_\beta(z)<\infty$, then $T_{u_1,u_2,\varphi}:D^p_\alpha\to D^q_\beta$ is bounded.

(b) If $p=\alpha+2$ and $\int_{\mathbb{D}} g_2(z)dA_\beta(z)<\infty$, then $T_{u_1,u_2,\varphi}:D^p_\alpha\to D^q_\beta$ is bounded.

(c) If $p>\alpha+2$ and $u_1\in D^q_\beta$ and $\int_{\mathbb{D}} g_3(z)dA_\beta(z)<\infty$, then $T_{u_1,u_2,\varphi}:D^p_\alpha\to D^q_\beta$ is bounded.

Proof. (a) For each $f\in D^p_\alpha$ and $z\in\mathbb{D}$, by Lemma 2.1-Lemma 2.3, we have

$$
\begin{aligned}
&\left|\left(T_{u_1,u_2,\varphi}f\right)'(z)\right|\\
&=\left|u_1'(z)f(\varphi(z))+\left[u_1(z)\varphi'(z)+u_2'(z)\right]f'(\varphi(z))+u_2(z)\varphi'(z)f''(\varphi(z))\right|\\
&\leqslant\left|u_1'(z)\right|\left|f(\varphi(z))\right|+\left|u_1(z)\varphi'(z)+u_2'(z)\right|\left|f'(\varphi(z))\right|+\left|u_2(z)\varphi'(z)\right|\left|f''(\varphi(z))\right|\\
&\lesssim\frac{|u_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}}+\frac{|u_1(z)\varphi'(z)+u_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}}+\frac{|u_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+1}},
\end{aligned}
$$

and so

$$
\|T_{u_1,u_2,\varphi}f\|^q_{D^q_\beta}\lesssim\int_{\mathbb{D}} g_1(z)dA_\beta(z)\|f\|^q_{D^p_\alpha}.
$$

From this, it follows that $T_{u_1,u_2,\varphi}:D^p_\alpha\to D^q_\beta$ is bounded.

Similar to the proof of (a), (b) and (c) can be proved, and then we omit them. □

For the order boundedness, we have the characterization as follows.

THEOREM 3.1. *Let $-1<\alpha,\ \beta,\ 0<p,q<\infty,\ u_1,u_2\in H(\mathbb{D})$ and φ an analytic self-map of $\mathbb{D}$. Then the following statements hold.*

(a) If $p<\alpha+2$, then $T_{u_1,u_2,\varphi}:D^p_\alpha\to D^q_\beta$ is order bounded if and only if

$$
\int_{\mathbb{D}} g_1(z)dA_\beta(z)<\infty.
$$

(b) If $p=\alpha+2$, then $T_{u_1,u_2,\varphi}:D^p_\alpha\to D^q_\beta$ is order bounded if and only if

$$
\int_{\mathbb{D}} g_2(z)dA_\beta(z)<\infty.
$$

(c) If $p > \alpha + 2$, then $T_{u_1,u_2,\varphi} : D^p_\alpha \to D^q_\beta$ is order bounded if and only if $u_1 \in D^q_\beta$ and

$$\int_{\mathbb{D}} g_3(z) dA_\beta(z) < \infty.$$

Proof. (a) Assume that $\int_{\mathbb{D}} g_1(z) dA_\beta(z) < \infty$. Let $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \leqslant 1$. Then by Lemma 2.1, Lemma 2.2 and Lemma 2.3, we have

$$\begin{aligned}
&\left|\left(T_{u_1,u_2,\varphi} f\right)'(z)\right| \\
&= \left|u_1'(z) f(\varphi(z)) + \left[u_1(z)\varphi'(z) + u_2'(z)\right] f'(\varphi(z)) + u_2(z)\varphi'(z) f''(\varphi(z))\right| \\
&\leqslant \left|u_1'(z)\right| \left|f(\varphi(z))\right| + \left|u_1(z)\varphi'(z) + u_2'(z)\right| \left|f'(\varphi(z))\right| + \left|u_2(z)\varphi'(z)\right| \left|f''(\varphi(z))\right| \\
&\lesssim \frac{|u_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} + \frac{|u_1(z)\varphi'(z) + u_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} + \frac{|u_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+1}}. \qquad (3.1)
\end{aligned}$$

Since $\int_{\mathbb{D}} g_1(z) dA_\beta(z) < \infty$, the function

$$h(z) = \frac{|u_1'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}-1}} + \frac{|u_1(z)\varphi'(z) + u_2'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}}} + \frac{|u_2(z)\varphi'(z)|}{(1-|\varphi(z)|^2)^{\frac{\alpha+2}{p}+1}}$$

belongs to $L^q\left(dA_\beta\right)$, and by (3.1), $\left|\left(T_{u_1,u_2,\varphi} f\right)'(z)\right| \lesssim h(z)$. Consequently, we deduce that $T_{u_1,u_2,\varphi} : D^p_\alpha \to D^q_\beta$ is order bounded.

Conversely, assume that $T_{u_1,u_2,\varphi} : D^p_\alpha \to D^q_\beta$ is order bounded. Then there exists a nonnegative function $h \in L^q\left(dA_\beta\right)$ such that for $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \leqslant 1$, it holds that

$$\left|\left(T_{u_1,u_2,\varphi} f\right)'(z)\right| \leqslant h(z) \ \text{ a.e. } \left[dA_\beta\right].$$

For $w \in \mathbb{D}$, it follows from Lemma 2.7 that there exists a function $f_{\varphi(w),0} \in D^p_\alpha$ with

$$\sup_{w \in \mathbb{D}} \|f_{\varphi(w),0}\|_{D^p_\alpha} \lesssim 1$$

such that

$$f_{\varphi(w),0}(\varphi(w)) = \frac{1}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}-1}} \quad \text{and} \quad f'_{\varphi(w),0}(\varphi(w)) = f''_{\varphi(w),0}(\varphi(w)) = 0.$$

Therefore,

$$\begin{aligned}
h(w) \gtrsim \left|\left(T_{u_1,u_2,\varphi} f_{\varphi(w),0}\right)'(w)\right| &= \left|u_1'(w) f_{\varphi(w),0}(\varphi(w))\right| \\
&= \frac{|u_1'(w)|}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}-1}} \ \text{ a.e. } [dA_\beta],
\end{aligned}$$

which implies that

$$\int_{\mathbb{D}} \frac{|u_1'(w)|^q}{(1-|\varphi(w)|^2)^{\frac{q(\alpha+2)}{p}-q}} dA_\beta(w) < \infty.$$

Take the test function $f_{\varphi(w),1}$. From Lemma 2.7, it follows that

$$\sup_{w\in\mathbb{D}} \left\|f_{\varphi(w),1}\right\|_{D^p_\alpha} \lesssim 1,$$

$$f'_{\varphi(w),1}(\varphi(w)) = \frac{\overline{\varphi(w)}}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \quad \text{and} \quad f_{\varphi(w),1}(\varphi(w)) = f''_{\varphi(w),1}(\varphi(w)) = 0.$$

Thus, we have

$$\begin{aligned} h(w) \gtrsim \left| \left(T_{u_1,u_2,\varphi} f_{\varphi(w),1}\right)'(w)\right| &= \left|\left[u_1(w)\varphi'(w)+u_2'(w)\right] f'_{\varphi(w),1}(\varphi(w))\right| \\ &= \frac{|u_1(w)\varphi'(w)+u_2'(w)|\,|\varphi(w)|}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \text{ a.e. } [dA_\beta]. \end{aligned}$$

For $|\varphi(w)|>1/2$, it holds that

$$\frac{|u_1(w)\varphi'(w)+u_2'(w)|}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \lesssim h(w) \text{ a.e. } [dA_\beta].$$

For $|\varphi(w)|\leqslant 1/2$, it holds that

$$\frac{1}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \lesssim 1.$$

Now, taking the constant function 1 and the function $f(z)=z$ in D^p_α, we obtain

$$\left|\left(T_{u_1,u_2,\varphi}1\right)'(z)\right| = \left|u_1'(z)\right| \lesssim h(z) \text{ a.e. } \left[dA_\beta\right]$$

and

$$\left|\left(T_{u_1,u_2,\varphi}f\right)'(z)\right| = \left|u_1'(z)\varphi(z)+u_1(z)\varphi'(z)+u_2'(z)\right| \lesssim h(z) \text{ a.e. } \left[dA_\beta\right].$$

Therefore,

$$\left|u_1(z)\varphi'(z)+u_2'(z)\right| \lesssim h(z) \text{ a.e. } \left[dA_\beta\right].$$

So, for $|\varphi(w)|\leqslant 1/2$, it holds that

$$\frac{|u_1(w)\varphi'(w)+u_2'(w)|}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \lesssim h(w) \text{ a.e. } \left[dA_\beta\right].$$

Consequently, for all $w\in\mathbb{D}$, we obtain

$$\frac{|u_1(w)\varphi'(w)+u_2'(w)|}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}}} \lesssim h(w) \text{ a.e. } \left[dA_\beta\right],$$

which shows that

$$\int_{\mathbb{D}} \frac{|u_1(w)\varphi'(w)+u_2'(w)|^q}{(1-|\varphi(w)|^2)^{\frac{2(\alpha+2)}{p}}} dA_\beta(w) < \infty.$$

Consider the function $f_{\varphi(w),2}$. It follows from Lemma 2.7 that $\sup_{w\in\mathbb{D}}\left\|f_{\varphi(w),2}\right\|_{D^p_\alpha} \lesssim 1$,

$$f''_{\varphi(w),2}(\varphi(w)) = \frac{\overline{\varphi(w)}^2}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+1}} \quad \text{and} \quad f_{\varphi(w),2}(\varphi(w)) = f'_{\varphi(w),2}(\varphi(w)) = 0.$$

From the order boundedness of $T_{u_1,u_2,\varphi}: D^p_\alpha \to D^q_\beta$, we have

$$h(w) \gtrsim \left|\left(T_{u_1,u_2,\varphi} f_{\varphi(w),2}\right)'(w)\right| = \frac{|u_2(w)\varphi'(w)|\,|\varphi(w)|^2}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+1}} \text{ a.e. } [dA_\beta].$$

From this, we see that for $|\varphi(w)| > 1/2$, it holds that

$$\frac{|u_2(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+1}} \lesssim h(w) \text{ a.e. } [dA_\beta].$$

Next, we consider the case $|\varphi(w)| \leqslant 1/2$. For this case, we have

$$\frac{1}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+1}} \lesssim 1.$$

Let $h(z) = z^2 \in D^p_\alpha$. From the order boundedness of $T_{u_1,u_2,\varphi}: D^p_\alpha \to D^q_\beta$, we have

$$\begin{aligned} h(z) &\gtrsim \left|\left(T_{u_1,u_2,\varphi} h\right)'(z)\right| \\ &= \left|u_1'(z)\varphi^2(z) + 2\left[u_1(z)\varphi'(z) + u_2'(z)\right]\varphi(z) + 2u_2(z)\varphi'(z)\right| \text{ a.e. } [dA_\beta]. \end{aligned}$$

Since we have obtained the relations

$$\left|u_1'(z)\right| \lesssim h(z), \quad \left|u_1(z)\varphi'(z) + u_2'(z)\right| \lesssim h(z) \text{ a.e. } [dA_\beta],$$

and the boundedness of $\varphi(z)$, we have

$$\left|u_2(z)\varphi'(z)\right| \lesssim h(z) \text{ a.e. } [dA_\beta].$$

Therefore, we obtain that for $|\varphi(w)| \leqslant 1/2$, it holds that

$$\frac{|u_2(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+1}} \lesssim h(w) \text{ a.e. } [dA_\beta].$$

In conclusion, for $w \in \mathbb{D}$,

$$\frac{|u_2(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+1}} \lesssim h(w) \text{ a.e. } [dA_\beta],$$

which gives

$$\int_{\mathbb{D}} \frac{|u_2(w)\varphi'(w)|}{(1-|\varphi(w)|^2)^{\frac{\alpha+2}{p}+1}} dA_\beta < \infty.$$

Accordingly, this completes the proof of (a).

(b) Assume that $\int_{\mathbb{D}} g_2(z)dA_\beta(z) < \infty$ holds. Let $f \in D^p_\alpha$ with $\|f\|_{D^p_\alpha} \leqslant 1$. Then by Lemma 2.1, Lemma 2.2, and Lemma 2.3, we have

$$\begin{aligned}
&\left|\left(T_{u_1,u_2,\varphi}f\right)'(z)\right| \\
&= \left|u_1'(z)f(\varphi(z)) + \left[u_1(z)\varphi'(z)+u_2'(z)\right]f'(\varphi(z)) + u_2(z)\varphi'(z)f''(\varphi(z))\right| \\
&\leqslant \left|u_1'(z)\right|\left|f(\varphi(z))\right| + \left|u_1(z)\varphi'(z)+u_2'(z)\right|\left|f'(\varphi(z))\right| + \left|u_2(z)\varphi'(z)\right|\left|f''(\varphi(z))\right| \\
&\lesssim \frac{|u_1'(z)|}{\left(\log\frac{2}{1-|\varphi(z)|^2}\right)^{\frac{1-p}{p}}} + \frac{|u_1(z)\varphi'(z)+u_2'(z)|}{1-|\varphi(z)|^2} + \frac{|u_2(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2}.
\end{aligned} \tag{3.2}$$

Let

$$h(z) = \frac{|u_1'(z)|}{\left(\log\frac{2}{1-|\varphi(z)|^2}\right)^{\frac{1-p}{p}}} + \frac{|u_1(z)\varphi'(z)+u_2'(z)|}{1-|\varphi(z)|^2} + \frac{|u_2(z)\varphi'(z)|}{\left(1-|\varphi(z)|^2\right)^2}.$$

Since $\int_{\mathbb{D}} g_2(z)dA_\beta(z) < \infty$, it shows that $h \in L^q\left(dA_\beta\right)$, and by (3.2) $\left|\left(T_{u_1,u_2,\varphi}f\right)'(z)\right| \lesssim h(z)$. This shows that $T_{u_1,u_2,\varphi} : D^p_\alpha \to D^q_\beta$ is order bounded.

Conversely, assume that $T_{u_1,u_2,\varphi} : D^p_\alpha \to D^q_\beta$ is order bounded. Then, there exists a nonnegative function $g \in L^q\left(dA_\beta\right)$ such that for $g \in D^p_\alpha$ with $\|g\|_{D^p_\alpha} \leqslant 1$, it holds that

$$\left|\left(T_{u_1,u_2,\varphi}g\right)'(z)\right| \leqslant g(z) \text{ a.e. } \left[dA_\beta\right].$$

Take the function $g_{\varphi(w)}$ defined in Lemma 2.8. By applying the order boundedness to this function, we obtain

$$g(w) \gtrsim \left|\left(T_{u_1,u_2,\varphi}g_{\varphi(w)}\right)'(w)\right| = \frac{|u_1'(w)|}{\left(\log\frac{2}{1-|\varphi(w)|^2}\right)^{\frac{1-p}{p}}} \text{ a.e. } \left[dA_\beta\right],$$

which implies that

$$\int_{\mathbb{D}} \frac{|u_1'(w)|^q}{\left(\log\frac{2}{1-|\varphi(w)|^2}\right)^{\frac{q(1-p)}{p}}} dA_\beta(w) < \infty.$$

The remaining proofs of (b) are similar to that of (a) with some modifications. So, we omit them.

(c) By Lemma 2.1, Lemma 2.2, and Lemma 2.3 and using the function in Lemma 2.7, the proof is also similar to that of (a). So, we omit it. □

REMARK 3.1. From Proposition 3.1 and Theorem 3.1, it follows that if $T_{u_1,u_2,\varphi} : D^p_\alpha \to D^q_\beta$ is order bounded, then it is bounded. Since $W_{u,\varphi} = T_{u,0,\varphi}$, Theorem 3.1 also generalizes the result obtained by Lin et al. in [21].

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