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COMPACT WEIGHTED COMPOSITION–DIFFERENTIATION OPERATORS OF ORDER n ON THE HARDY SPACE

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Abstract. The weighted composition–differentiation operator of order n is denoted by $D_{\psi,\varphi,n}$. In this paper, we investigate some basic properties of compact weighted composition–differentiation operators of order n on the Hardy space. Moreover, we obtain the upper estimate on the norm of the operator $D_{\psi,\varphi,n}$, in the case that $\|\varphi\|_\infty < 1$.

1. Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ which are analytic on $\mathbb{D}$. The space H^∞ is the set of bounded analytic functions f on $\mathbb{D}$ with $\|f\|_\infty = \sup\{|f(z)| : z \in \mathbb{D}\}$. The *Hardy space* H^2 is the Hilbert space of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2 = \left(\sup_{0<r<1} \int_{\partial\mathbb{D}} \left|f(r\zeta)\right|^2 dm(\zeta)\right)^{1/2} = \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta < \infty,$$

where m is the normalized arc-length Lebesgue measure. Note that there is a unitary isomorphism between this space and the Hilbert space $\{x \in L^2(\mathbb{T}, d\theta) : \hat{x}(n) = 0,\ n < 0\}$, where $\hat{x}$ is the Fourier coefficient of x. The unitary map is explicitly realized via the boundary values of $f \in H^2$, namely, $f \mapsto f^*$, where f^* is the boundary value of f which always exists and belongs to the $L^2(\mathbb{T}, d\theta)$, see [3, Theorem 2.2]. The inner product of H^2 is defined as follows

$$\langle f, g\rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} d\theta.$$

Let H be a Hilbert space of analytic functions. For each $w \in \mathbb{D}$, the linear functional for evaluation at w is denoted by e_w, that is $e_w(f) = f(w)$ for $f \in H$. If e_w is a bounded linear functional, then by Riesz Representation Theorem, there exists a unique function K_w in H that $\langle f, K_w\rangle = f(w)$. The functions K_w are called the *reproducing kernels* and a functional Hilbert space which the linear functionals e_w are bounded is

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called the *reproducing kernel Hilbert space* (see [3, p. 3]). The Hardy space is the reproducing kernel Hilbert space with reproducing kernels $K_w(z) = (1-\overline{w}z)^{-1}$ and note that $\|K_w\|^2 = K_w(w) = \frac{1}{1-|w|^2}$ (see [3, Theorem 2.10] and [3, Corollary 2.11]). For every $w \in \mathbb{D}$ and non-negative integer n, the evaluation of the n-th derivative of functions in H^2 at the point w is a bounded linear functional, so there is a unique function $K_w^{[n]}$ in H^2 such that $\langle f, K_w^{[n]}\rangle = f^{(n)}(w)$, where $f^{(n)}$ is the n-th derivative of f. For each $z \in \mathbb{D}$, the function $K_w^{[n]}$ is given by $K_w^{[n]}(z) = \frac{n! z^n}{(1-\overline{w}z)^{n+1}}$ with

$$\|K_w^{[n]}\|^2 = \sum_{j=n}^{\infty} (|w|^2)^{j-n} \left(\frac{j!}{(j-n)!}\right)^2.$$

Note that in the case of $n = 0$, the function $K_w^{[0]}$ is the reproducing kernel K_w (see [3, Theorem 2.16]).

For φ an analytic self-map of $\mathbb{D}$, the *composition operator* C_φ is defined for analytic functions f on $\mathbb{D}$ by $C_\varphi(f) = f \circ \varphi$. Every composition operator C_φ is bounded on H^2 (see [3, Corollary 3.7]).

For φ an analytic self-map of $\mathbb{D}$, let D_φ be the *composition–differentiation operator* so that $D_\varphi(f) = f' \circ \varphi$ for any $f \in H^2$. The study of operators D_φ was initially addressed by Hibschweiler, Portnoy, and Ohno (see [8] and [11]) and has been noticed by many researchers ([4], [5], [6], [10], and [14]). Ohno [11] characterized bounded and compact operators D_φ on H^2. For each positive integer n, we write $D_{\varphi,n}$ to denote the operator on H^2 given by the rule $D_{\varphi,n}(f) = f^{(n)} \circ \varphi$. For an analytic function $\psi\colon \mathbb{D} \to \mathbb{C}$, the *weighted composition–differentiation operator of order* n on H^2 is defined by the rule

$$D_{\psi,\varphi,n}(f) = \psi \cdot \left(f^{(n)} \circ \varphi\right).$$

To simplify notation, we write $D_{\psi,\varphi}$ to denote $D_{\psi,\varphi,1}$ and we call it a *weighted composition–differentiation operator*. In [4], the first author investigated the spectrum of $D_{\psi,\varphi,n}$. She characterized the spectrum of some compact operators $D_{\psi,\varphi,n}$. Moreover, under some conditions, she found the set containing the point spectrum of $D_{\psi,\varphi,n}$.

An analytic self-map φ of $\mathbb{D}$ has an *angular derivative* at ζ on the unit circle if there is η on the unit circle so that $(\varphi(z)-\eta)/(z-\zeta)$ has a finite nontangential limit as $z \to \zeta$. This limit is denoted by $\varphi'(\zeta)$ in case it exists as a finite complex number. By the Julia-Carathéodory Theorem (see, e.g., [3, Theorem 2.44] or [13, Chapter 4]),

$$|\varphi'(\zeta)| = \liminf_{z \to \zeta} \frac{1-|\varphi(z)|}{1-|z|},$$

where the liminf is taken as z approaches ζ unrestrictedly in $\mathbb{D}$. Moreover, the value $|\varphi'(\zeta)|$ is strictly greater than 0 (see [3, p. 50]).

An elliptic automorphism of $\mathbb{D}$ is the one-to-one analytic map of the unit disk onto itself with one fixed point in the unit disk and a second fixed point in the complement of the closed disk. For an analytic self-map φ of $\mathbb{D}$, a point $\zeta \in \overline{\mathbb{D}}$ is called a *fixed point* of φ if $\lim_{r \to 1} \varphi(r\zeta) = \zeta$. For every analytic self-map φ of $\mathbb{D}$ that is neither the identity nor an elliptic automorphism of $\mathbb{D}$, there exists a unique point $w \in \overline{\mathbb{D}}$ so that the iterates φ_n of φ converges to w uniformly on compact subsets of the unit disk. The point w called the *Denjoy-Wolff point* of φ is a fixed point of φ.

2. Compactness of $D_{\psi,\varphi,n}$

In this section, we will study the basic results on the compactness of the weighted composition–differentiation operator of order n on the Hardy space. Although recent literature has been published on weighted composition–differentiation operators, none of them has studied the compactness of these operators on the Hardy space, as far as we know. First, we state the following lemma which will be used in the proof of Proposition 2:

LEMMA 1. [10, Lemma 2.3] *If an operator $D_{\psi,\varphi,n}$ is bounded on H^2, then*

$$D^*_{\psi,\varphi,n}(K_w) = \overline{\psi(w)}K^{[n]}_{\varphi(w)}.$$

We say that a function ψ is bounded away from zero near the unit circle if there are $\delta > 0$ and $\varepsilon > 0$ such that $|\psi(z)| > \varepsilon$ for $\delta < |z| < 1$.

The following result shows that the boundedness of $D_{\psi,\varphi,n}$ leads to φ having no angular derivatives a.e on $\partial\mathbb{D}$ in case ψ is not identically zero:

PROPOSITION 2. *Suppose $D_{\psi,\varphi,n}$ is bounded on H^2. If $\liminf_{r\to 1}|\psi(r\zeta)| > 0$ for some ζ on the unit circle, then the angular derivative of φ at ζ does not exist. In addition, if ψ is not identically zero, then φ does not have the angular derivative a.e. on $\partial\mathbb{D}$. Moreover, if ψ is bounded away from zero near the unit circle, then φ has the angular derivative nowhere and the Denjoy-Wolff point of φ is inside the open unit disk.*

Proof. Suppose that $D_{\psi,\varphi,n}$ is bounded on H^2. For $\zeta \in \partial\mathbb{D}$, we can see that

$$\left\|D_{\psi,\varphi,n}\right\|^2 \geqslant \left\|D^*_{\psi,\varphi,n}\frac{K_{r\zeta}}{\|K_{r\zeta}\|}\right\|^2 = |\psi(r\zeta)|^2\frac{\left\|K^{[n]}_{\varphi(r\zeta)}\right\|^2}{\|K_{r\zeta}\|^2} \tag{2.1}$$

by Lemma 1. Observe that

$$\begin{aligned}
\|K^{[n]}_{\varphi(r\zeta)}\|^2 &= \sum_{j=n}^{\infty}\left(|\varphi(r\zeta)|^2\right)^{j-n}\left(\frac{j!}{(j-n)!}\right)^2 \\
&= \sum_{j=0}^{\infty}\left((j+n)(j+n-1)\dots(j+1)\right)^2\left(|\varphi(r\zeta)|^2\right)^j \\
&\geqslant \frac{1}{(2n)!}\sum_{j=0}^{\infty}(j+2n)\dots(j+1)\left(|\varphi(r\zeta)|^2\right)^j \\
&= \frac{1}{\left(1-|\varphi(r\zeta)|^2\right)^{2n+1}}
\end{aligned} \tag{2.2}$$

(note that it is easy to see that $\sum_{j=0}^{\infty}\frac{(j+2n)!}{j!}x^j=\frac{(2n)!}{(1-x)^{2n+1}}$ for $|x|<1$). We infer from (2.1) and (2.2) that

$$\|D_{\psi,\varphi,n}\|^2\geqslant|\psi(r\zeta)|^2\frac{1-|r\zeta|^2}{\left(1-|\varphi(r\zeta)|^2\right)^{2n+1}}. \tag{2.3}$$

If $\liminf_{r\to 1}|\psi(r\zeta)|>0$ for some ζ on the unit circle, then φ does not have the angular derivative at ζ by (2.3) and [3, Theorem 2.44]. Since $D_{\psi,\varphi,n}$ is bounded and ψ is not identically zero, the map ψ belongs to H^2 and also $\lim_{r\to 1}\psi(r\zeta)$ exists and is nonzero for almost all $\zeta\in\partial\mathbb{D}$ (see [12, Theorem 17.18]). Hence φ does not have the angular derivative a.e. on $\partial\mathbb{D}$. If ψ is bounded away from zero near the unit circle, then the angular derivative of φ cannot exist anywhere and so φ cannot have its Denjoy-Wolff point on the unit circle. Therefore, the Denjoy-Wolff point of φ must be inside the open unit disk. □

REMARK 3. Note that for a bounded operator $D_{\psi,\varphi,n}$, by substituting w for $r\zeta$ in (2.1) and (2.2), we obtain

$$\sup_{w\in\mathbb{D}}|\psi(w)|^2\frac{1-|w|^2}{\left(1-|\varphi(w)|^2\right)^{2n+1}}<\infty. \tag{2.4}$$

Now suppose that $D_{\psi,\varphi,n}$ is compact on H^2. As the fact that $\left\{\frac{K_w}{\|K_w\|}\right\}$ tends to zero weakly as $|w|\to 1^-$, we obtain

$$\lim_{|w|\to 1^-}\left\|D^*_{\psi,\varphi,n}\frac{K_w}{\|K_w\|}\right\|^2=0.$$

By the similar ideas used in the proofs of (2.1) and (2.2), we have

$$\lim_{|w|\to 1^{-1}}|\psi(w)|^2\frac{1-|w|^2}{\left(1-|\varphi(w)|^2\right)^{2n+1}}=0. \tag{2.5}$$

As a consequence of Proposition 2, for a compact operator $D_{\psi,\varphi,n}$ with a nonzero function ψ, the function φ does not have the angular derivative a.e. on the unit circle. This result will be improved in the next proposition, which makes φ send the unit circle to the unit disk almost everywhere:

PROPOSITION 4. *Suppose that ψ is not identically zero. If $D_{\psi,\varphi,n}$ is compact on H^2, then $|\varphi|<1$ a.e. on $\partial\mathbb{D}$.*

Proof. Suppose that there is a measurable set $E\subseteq\partial\mathbb{D}$ with $m(E)>0$ so that $|\varphi(\zeta)|=1$ for each $\zeta\in E$. Since $D_{\psi,\varphi,n}$ is compact and the sequence $\{z^m\}$ converges weakly to 0 as $m\to\infty$, we have $\|D_{\psi,\varphi,n}(z^m)\|\to 0$ as $m\to\infty$. Observe that

$$\begin{aligned}\|D_{\psi,\varphi,n}(z^m)\|^2&=\int_0^{2\pi}m^2\dots(m-n+1)^2|\psi(e^{i\theta})|^2\left(|\varphi(e^{i\theta})|^2\right)^{m-n}\frac{d\theta}{2\pi}\\&\geqslant\int_E|\psi(\zeta)|^2dm(\zeta)\end{aligned}$$

for each $m > n$. Since $\|D_{\psi,\varphi,n}(z^m)\| \to 0$ as $m \to \infty$, we can see that $\int_E |\psi(\zeta)|^2 dm(\zeta)$ must be zero. It follows that ψ is identically zero (note that $\psi \in H^2$ and see [12, Theorem 17.18]), which is a contradiction. □

The next result gives a condition under which the operator $D_{\psi,\varphi,n}$ is compact:

THEOREM 5. *Suppose that $D_{\varphi,n}$ is bounded on H^2 and $\psi \in H^2$. If*

$$\lim_{\delta \to 0^+} \left(\operatorname{ess\,sup}\{|\psi(\eta)|^2 : \eta \in \partial\mathbb{D}, |\varphi(\eta)| \geqslant 1-\delta\} \right) = 0, \tag{2.6}$$

then $D_{\psi,\varphi,n}$ is compact.

Proof. Let $\{h_m\}$ be a sequence in H^2 converging weakly to zero. The sequence $\{h_m\}$ is bounded in norm and converges to zero uniformly on compact subsets of $\mathbb{D}$ by [2, Proposition 8.15, p. 130]. Without loss of generality, we assume that $\|h_m\| \leqslant 1$ for each m. Since (2.6) holds, for arbitrary $\varepsilon > 0$, we can choose $\delta > 0$ so that

$$\operatorname{ess\,sup}\left\{|\psi(\eta)|^2 : \eta \in \partial\mathbb{D}, |\varphi(\eta)| \geqslant 1-\delta\right\} < \varepsilon. \tag{2.7}$$

Let $A_\delta = \{\eta : \eta \in \partial\mathbb{D}, |\varphi(\eta)| \geqslant 1-\delta\}$. We can see that

$$\begin{aligned}\|D_{\psi,\varphi,n}h_m\| = &\int_{A_\delta} \left|\psi(\eta)h_m^{(n)}(\varphi(\eta))\right|^2 dm(\eta)\\ &+ \int_{A_\delta^c} \left|\psi(\eta)h_m^{(n)}(\varphi(\eta))\right|^2 dm(\eta).\end{aligned}$$

We can see that

$$\begin{aligned}\int_{A_\delta} \left|\psi(\eta)h_m^{(n)}(\varphi(\eta))\right|^2 dm(\eta) &\leqslant \varepsilon \|D_{\varphi,n}(h_m)\|\\ &\leqslant \varepsilon \|D_{\varphi,n}\|.\end{aligned} \tag{2.8}$$

Since $\{h_m\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$, we observe that $\{h_m^{(n)}\}$ converges to zero uniformly on all compact subsets of $\mathbb{D}$ by [1, Theorem 2.1, p. 151]. Then there is a sufficiently large integer N, such that for $n \geqslant N$, we obtain

$$\int_{A_\delta^c} \left|\psi(\eta)h_m^{(n)}(\varphi(\eta))\right|^2 dm(\eta) \leqslant \varepsilon \|\psi\|^2. \tag{2.9}$$

From (2.8) and (2.9), we conclude that $\|D_{\psi,\varphi,n}h_m\| \to 0$ as $m \to \infty$, so $D_{\psi,\varphi,n}$ is compact. □

If φ is a map in H^2, then for almost all θ

$$\varphi(e^{i\theta}) = \lim_{r \to 1} \varphi(re^{i\theta}) \tag{2.10}$$

exists (see [3, Theorem 2.2]). In the following corollary, we assume that for all θ, $\varphi(e^{i\theta})$ exists:

COROLLARY 6. *Suppose that φ is an analytic self-map of $\mathbb{D}$ which is defined on the unit circle as (2.10). Assume that φ is continuous on $\partial\mathbb{D}$, $|\varphi|<1$ a.e. on $\partial\mathbb{D}$, and $\psi\in A(\mathbb{D})$. Let $A=\{\zeta\in\partial\mathbb{D}:|\varphi(\zeta)|=1\}$. Suppose that $D_{\varphi,n}$ is bounded on H^2. If $\psi(\zeta)=0$ for each $\zeta\in A$, then $D_{\psi,\varphi,n}$ is compact on H^2.*

Proof. Suppose that $\psi(\zeta)=0$ for each $\zeta\in A$. Note that $N_\delta=\operatorname{ess\,sup}\{|\psi(\zeta)|^2: \zeta\in\partial\mathbb{D},|\varphi(\zeta)|\geqslant 1-\delta\}$ decreases as $\delta\to 0^+$. If $\lim_{\delta\to 0^+}N_\delta=0$, then Theorem 5 implies that $D_{\psi,\varphi,n}$ is compact. Suppose that

$$\lim_{\delta\to 0^+}N_\delta=a \tag{2.11}$$

for some $a>0$. Since ψ is continuous on $\partial\mathbb{D}$, there is $0<\delta<1$ so that $|\psi(x)-\psi(y)|<a/4$ if $x,y\in\partial\mathbb{D}$ and $|x-y|<\delta$. There is an open set $V\subseteq\partial\mathbb{D}$ such that $A\subseteq V$ and $m(V)<\delta$ because $m(A)=0$. Hence $V=\bigcup_{j=1}^\infty V_j$, where for each j, V_j is an open arc and $V_j\bigcap A\neq\emptyset$. Therefore, $|\psi(x)|<a/4$ for any $x\in V$. By (2.11), there is $\tilde{\delta}>0$ such that $N_\delta>a/2$ for each $\delta<\tilde{\delta}$. Let $\delta_n=\tilde{\delta}/n$. Then there exists $\zeta_n\in\partial\mathbb{D}\setminus V$ such that $|\varphi(\zeta_n)|\geqslant 1-\delta_n$ and $|\psi(\zeta_n)|^2>a/2$. We can see that $|\varphi(\zeta_n)|\to 1$. There is a subsequence $\{\zeta_{n_k}\}$ so that $\zeta_{n_k}\to\zeta_0$ for some $\zeta_0\in\partial\mathbb{D}\setminus V$. The continuity of φ on $\partial\mathbb{D}$ shows that $|\varphi(\zeta_0)|=1$. Then $\zeta_0\in V$, which is a contradiction. □

Now we give an example for the previous corollary:

EXAMPLE 7. Let

$$\varphi(z)=\frac{\sigma(z)^{\frac{1}{2n+1}}-1}{\sigma(z)^{\frac{1}{2n+1}}+1}$$

where $\sigma(z)=\frac{1+z}{1-z}$. We know that φ is an analytic self-map of $\mathbb{D}$ which fixes the points 1 and -1 and sends all points of $\partial\mathbb{D}$ into $\mathbb{D}$ (see [13, p. 27]). By [11, Examples], we have $1-|\varphi(z)|^2\approx|1-z|^{\frac{1}{2n+1}}$ for z near 1. Therefore, $\sup_{w\in\mathbb{D}}\frac{1-|w|}{\left(1-|\varphi(w)|\right)^{2n+1}}<\infty$. We conclude that $D_{\varphi,n}$ is bounded and noncompact from [9, Corollary 2.3]. Let $\psi(z)=1-z^2$. We infer from Corollary 6 that $D_{\psi,\varphi,n}$ is compact on H^2.

3. The upper estimate on the norm of $D_{\psi,\varphi,n}$

In this section, as an application of Section 2, we estimate the upper bound on the norm of a class of compact operators $D_{\psi,\varphi,n}$. We begin with a few easy observations that help us in the proof of Theorem 10. Throughout this section, we assume that $0^0=1$.

LEMMA 8. *Let n be a positive integer, and $\alpha_k>0$ for each $0\leqslant k\leqslant n$. Then for $0\leqslant x<1$, the following statements hold:*

(a) $\sum_{k=0}^n\frac{\alpha_k x^k}{(1-x)^{n+k+1}}\leqslant\frac{\sum_{k=0}^n\alpha_k}{(1-x)^{2n+1}}$.

(b) There exists a positive number β such that $\sum_{k=0}^n\frac{\alpha_k x^k}{(1-x)^{n+k+1}}\geqslant\frac{\beta}{(1-x)^{2n+1}}$.

Proof. (a) We can see that

$$\sum_{k=0}^{n} \frac{\alpha_k x^k}{(1-x)^{n+k+1}} = \frac{\sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k}}{(1-x)^{2n+1}}.$$

Since $0 \leqslant x < 1$ and $\alpha_k > 0$, we conclude that $\sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k} \leqslant \sum_{k=0}^{n} \alpha_k$. Hence, the conclusion follows.

(b) We have

$$(1-x)^{2n+1} \sum_{k=0}^{n} \frac{\alpha_k x^k}{(1-x)^{n+k+1}} = \sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k} > 0.$$

Because $\sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k}$ is a continuous function on $[0,1]$, there exists a positive number β such that $\sum_{k=0}^{n} \alpha_k x^k (1-x)^{n-k} \geqslant \beta$. Consequently, the result follows. □

LEMMA 9. *Let n be a positive integer. Then*

$$\sum_{m=n}^{\infty} \left[m(m-1)\dots(m-n+1)\right]^2 x^{m-n} = \left(n!\right)^2 \sum_{k=0}^{n} \frac{(n+k)!}{\left(k!\right)^2 (n-k)!} \frac{x^k}{(1-x)^{n+k+1}}$$

for $0 \leqslant x < 1$.

Proof. See [14, Lemma 1] and the general Leibniz rule. □

Let A be a linear operator on a separable Hilbert space H. The *Hilbert-Schmidt norm* of A is given by

$$\|A\|_{HS} = \left(\sum_{n=1}^{\infty} \|Ae_n\|^2\right)^{1/2}, \tag{3.1}$$

where $\{e_n\}$ is an orthonormal basis of H. The Hilbert-Schmidt norm is independent of the choice of the basis and $\|A\|_{HS} \geqslant \|A\|$.

In the next theorem, we find the upper bound for the norm of $D_{\psi,\varphi,n}$ by using the previous lemmas and the fact that the Hilbert-Schmidt norm of an operator is greater than or equal to the norm of that operator:

THEOREM 10. *Suppose that $D_{\psi,\varphi,n}$ is a bounded operator with $\|\varphi\|_\infty < 1$ and ψ is not identically zero. Then*

$$\|D_{\psi,\varphi,n}\| < \alpha^{1/2} \frac{\|\psi\|}{\left(1 - \|\varphi\|_\infty^2\right)^{\frac{2n+1}{2}}},$$

where $\alpha = \sum_{k=0}^{n} \frac{(n!)^2 (n+k)!}{(k!)^2 (n-k)!}$. *Moreover, if $\varphi \equiv p$ for some $p \in \mathbb{D}$, then* $\|D_{\psi,\varphi,n}\| = \|\psi\| \|K_p^{[n]}\|$.

Proof. Lemmas 8 and 9, and [12, Theorem 1.27] imply that

$$\begin{aligned}
\|D_{\psi,\varphi,n}\|_{HS}^2 &= \sum_{m=0}^{\infty} \left\|D_{\psi,\varphi,n} z^m\right\|^2 \\
&= \sum_{m=n}^{\infty} \left\|m(m-1)\dots(m-n+1)\psi\varphi^{m-n}\right\|^2 \\
&= \sum_{m=n}^{\infty} \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \left|m(m-1)\dots(m-n+1)\psi(re^{i\theta})\varphi^{m-n}(re^{i\theta})\right|^2 d\theta \\
&= \lim_{r\to 1} \sum_{m=n}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} \left|m(m-1)\dots(m-n+1)\psi(re^{i\theta})\varphi^{m-n}(re^{i\theta})\right|^2 d\theta \\
&= \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \sum_{m=n}^{\infty} \left|m(m-1)\dots(m-n+1)\psi(re^{i\theta})\varphi^{m-n}(re^{i\theta})\right|^2 d\theta \\
&= \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \sum_{k=0}^{n} \frac{(n!)^2(n+k)!}{(k!)^2(n-k)!} \frac{\left|\psi(re^{i\theta})\right|^2\left|\varphi(re^{i\theta})\right|^{2k} d\theta}{\left(1-\left|\varphi(re^{i\theta})\right|^2\right)^{n+k+1}} \\
&\leqslant \lim_{r\to 1} \frac{1}{2\pi}\int_0^{2\pi} \frac{\alpha|\psi(re^{i\theta})|^2 d\theta}{\left(1-\left|\varphi(re^{i\theta})\right|^2\right)^{2n+1}},
\end{aligned} \tag{3.2}$$

where $\alpha = \sum_{k=0}^{n} \frac{(n!)^2(n+k)!}{(k!)^2(n-k)!}$ (note that the interchange of limit and summation is justified by [3, Corollary 2.23] and using Lebesgue's Monotone Convergence Theorem with counting measure). Since $\|D_{\psi,\varphi,n}\| \leqslant \|D_{\psi,\varphi,n}\|_{HS}$, we have $\|D_{\psi,\varphi,n}\| \leqslant \alpha^{1/2}\|\psi\|/\left(1-\|\varphi\|_\infty^2\right)^{\frac{2n+1}{2}}$ by (3.2). Now suppose that $\|D_{\psi,\varphi,n}\| = \alpha^{1/2}\|\psi\|/\left(1-\|\varphi\|_\infty^2\right)^{\frac{2n+1}{2}}$ for some functions φ and ψ. Due to (3.2), we obtain $\|D_{\psi,\varphi,n}\| = \|D_{\psi,\varphi,n}\|_{HS}$. Because $\|\varphi\|_\infty < 1$ and $\psi \in H^2$, Theorem 5 dictates that the operator $D_{\psi,\varphi,n}$ is compact. We have

$$0 = \|D_{\psi,\varphi,n}\|_e < \|D_{\psi,\varphi,n}\| = \|D_{\psi,\varphi,n}\|_{HS}.$$

It follows immediately from [7, Proposition 2.2] that there exists a function f with $\|f\| = 1$ so that $\|D_{\psi,\varphi,n} f\| = \|D_{\psi,\varphi,n}\|$. There is a basis $\{e_m : m \in \mathbb{N}\}$ for H^2 that $e_1 = f$. Since $\|D_{\psi,\varphi,n}\| = \|D_{\psi,\varphi,n}\|_{HS}$, we can see that $D_{\psi,\varphi,n} e_m = 0$ for each $m > 1$ by (3.1). Then $D_{\psi,\varphi,n}$ is of rank 1, which can occur if and only if φ is a constant function. Suppose that $\varphi \equiv p$ for some $p \in \mathbb{D}$. We obtain

$$\begin{aligned}
\|D_{\psi,\varphi,n}\| &= \sup\left\{\|D_{\psi,\varphi,n} g\| : \|g\| = 1\right\} \\
&= \sup\left\{\left|g^{(n)}(p)\right|\|\psi\| : \|g\| = 1\right\} \\
&= \|\psi\|\|K_p^{[n]}\| \\
&= \|\psi\|\left[\sum_{j=n}^{\infty} \left(|p|^2\right)^{j-n}\left(\frac{j!}{(j-n)!}\right)^2\right]^{1/2} \\
&= \|\psi\| n!\left[\sum_{k=0}^{n} \frac{(n+k)!}{(k!)^2(n-k)!} \frac{\left(|p|^2\right)^k}{\left(1-|p|^2\right)^{n+k+1}}\right]^{1/2}
\end{aligned} \tag{3.3}$$

by Lemma 9. Assume that $p \neq 0$. We obtain

$$\frac{|p|^{2k}}{\left(1-|p|^2\right)^{n+k+1}} < \frac{1}{\left(1-|p|^2\right)^{2n+1}} \tag{3.4}$$

for each k that $0 \leqslant k \leqslant n$. Hence (3.3) and (3.4) imply that $\|D_{\psi,\varphi,n}\| < \alpha^{1/2}\|\psi\|/\left(1-|p|^2\right)^{(2n+1)/2}$. Therefore, φ must be the zero function, and we infer from (3.3) that $\|D_{\psi,\varphi,n}\| = n!\|\psi\| < \alpha^{1/2}\|\psi\|$, which is a contradiction. □

In the following example, we show that for some operators $D_{\psi,\varphi,n}$, the upper estimate on the norm of $D_{\psi,\varphi,n}$ found in Theorem 10 is less than the other estimates given in the previous literature (see [4, Proposition 3.6] and [5, Proposition 4]):

EXAMPLE 11. Suppose that $\varphi(z) = \frac{1}{m}z + a - \frac{1}{m}$, where m is a positive integer and that $\frac{1}{m} < a < 1$. We can see that $\|\varphi\|_\infty = a$. On the one hand, by [5, Proposition 4], we obtain $\|D_\varphi\| \leqslant \sqrt{2am-1}\lfloor\frac{1}{1-a}\rfloor a^{\lfloor\frac{1}{1-a}\rfloor-1}$, where $\lfloor\cdot\rfloor$ denotes the greatest integer function, but on the other hand, Theorem 10 dictates that $\|D_\varphi\| < \frac{\sqrt{3}}{(1-a^2)^{3/2}}$. For sufficiently large m, we can see that Theorem 10 gives the better upper estimate on the norm of D_φ. Similarly, since $\|\psi\| \leqslant \|\psi\|_\infty$ for each $\psi \in H^\infty$, Theorem 10 makes a more precise upper bound for $\|D_{\psi,\varphi}\|$ than [4, Proposition 3.6], when φ was defined as above.

4. Further question

In the case that $\psi \equiv 1$, it is worth noting that for a univalent self-map φ, (2.4) and (2.5) are the necessary and sufficient conditions for $D_{\varphi,n}$ to be bounded and compact, respectively (see [9, Corollary 2.3] and [11, Corollary 3.2]). In view of Remark 3, we conclude with a question for further consideration:

QUESTION. Which maps φ and ψ make (2.4) or (2.5) be the necessary and sufficient conditions for the boundedness and compactness of $D_{\psi,\varphi,n}$, respectively?

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