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THE BEREZIN RADIUS AND THE BEREZIN NORM ASSOCIATED WITH THE TENSOR PRODUCT

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Abstract. The Berezin range of a bounded operator A acting on a reproducing kernel Hilbert space $\mathscr{H}$ is the set $\mathbf{Ber}(A) := \{\langle A\hat{k}_\tau, \hat{k}_\tau\rangle : \tau \in \Theta\}$, where $\hat{k}_\tau$ is the normalized reproducing kernel for $\mathscr{H}$ at $\tau \in \Theta$. The Berezin radius (number) and the Berezin norms of an operator A are defined by $\mathbf{ber}(A) := \sup_{\tau\in\Theta}\left|\langle A\hat{k}_\tau, \hat{k}_\tau\rangle\right|$, $\|A\|_{\mathbf{ber},1} := \sup_{\tau,\mu\in\Theta}\left|\langle A\hat{k}_\tau, \hat{k}_\mu\rangle\right|$, and $\|A\|_{\mathbf{ber},2} := \sup_{\tau\in\Theta}\left\|A\hat{k}_\tau\right\|$ respectively. In this paper, we obtain some Berezin radius upper bounds for Hilbert space operators involving the tensor product. Moreover, the obtained upper bounds have been compared with the previously known bounds to demonstrate their reliability.

1. Introduction

Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ be the C^*-algebra of all bounded linear operators defined on $\mathscr{H}$. In the case, when $\dim\mathscr{H} = n$, we identify $\mathscr{L}(\mathscr{H})$ with the matrix algebra $\mathbb{M}_n(\mathbb{C})$ of all $n\times n$ matrices with entries in the complex field. An operator $A \in \mathscr{L}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geqslant 0$ for all $x \in \mathscr{H}$ and in this case we write $A \geqslant 0$. For self-adjoint operators $A, B \in \mathscr{L}(\mathscr{H})$, we say $B \geqslant A$ if $B - A \geqslant 0$. For an operator $A \in \mathscr{L}(\mathscr{H})$, the operator $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of A. For a bounded linear operator A on a Hilbert space $\mathscr{H}$, the numerical range $W(A)$ is defined by $W(A) = \{\langle Ax, x\rangle : x \in \mathscr{H}\ \&\ \|x\| = 1\}$. Moreover, the numerical radius is defined by $w(A) = \sup_{\|x\|=1}|\langle Ax, x\rangle|$.

The tensor product $\mathscr{H}\otimes\mathscr{H}$ of a Hilbert space $\mathscr{H}$ is the completion of the inner product space consisting of elements of the form $\sum_{i=1}^n x_i \otimes y_i$ with $x_i, y_i \in \mathscr{H}$ for any $n \geqslant 1$ under the inner product $\langle x\otimes u, y\otimes v\rangle = \langle x, y\rangle\langle u, v\rangle$ for all $x, y, u, v \in \mathscr{H}$. Let A and B be bounded linear operators on a Hilbert space $\mathscr{H}$. The tensor product of A and B is denoted by $A\otimes B$ on $\mathscr{H}\otimes\mathscr{H}$ and is defined by $(A\otimes B)(x\otimes y) = Ax\otimes By$ for every $x, y \in \mathscr{H}$. Furthermore, for operators $A, B, C, D \in \mathscr{L}(\mathscr{H})$, the tensor product has the following properties:

(1) $\langle (A\otimes B)(x\otimes u), (y\otimes v)\rangle = \langle Ax, y\rangle\langle Bu, v\rangle$ for all $x, y, u, v \in \mathscr{H}$;

(2) $(A\otimes B)(C\otimes D) = AC\otimes BD$;

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(3) $(A\otimes B)^* = A^*\otimes B^*$;

(4) $|A\otimes B| = |A|\otimes|B|$;

(5) $\|A\otimes B\| = \|A\|\|B\|$;

(6) If A and B are positive, then $(A\otimes B)^r = A^r\otimes B^r$ for all $r\geqslant 0$;

(7) If A and B are invertible, then $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$.

From the property (1), it is clear that the tensor product of positive operators is positive. Utilizing the propersties (6) and (7) of the tensor product, we have $(A\otimes B)^r = A^r\otimes B^r$ for all positive invertible operators A and B and $r\in\mathbb{R}$. Especially important are the operators $A_1\otimes A_2\otimes\cdots\otimes A_n$, which are k-fold tensor product of operators $A_i\in\mathscr{L}(\mathscr{H})$ $(1\leqslant i\leqslant n)$. Such a product will be written more briefly as $\otimes_{i=1}^n A_i$. For more information about the tensor product, see [4, 26] and references therein.

A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta\subseteq\mathbb{C}$ such that the evaluation functionals $\varphi_\tau(f) = f(\tau)$, $\tau\in\Theta$, are continuous on $\mathscr{H}$. Then, by the Riesz representation theorem, there is a unique element $k_\tau\in\mathscr{H}$ such that $f(\tau) = \langle f, k_\tau\rangle$ for all $f\in\mathscr{H}$ and every $\tau\in\Theta$. The function k on $\Theta\times\Theta$ defined by $k(z,\tau) = k_\tau(z)$ is called the reproducing kernel of $\mathscr{H}$, see [1]. It was shown that $k_\tau(z)$ can be represented by $k_\tau(z) = \sum_{n=1}^{\infty}\overline{e_n(\tau)}e_n(z)$ for any orthonormal basis $\{e_n\}_{n\geqslant 1}$ of $\mathscr{H}$. For example, for the Hardy-Hilbert space $\mathscr{H}^2 = \mathscr{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z\in\mathbb{C} : |z| < 1\}$, $\{z^n\}_{n\geqslant 1}$ is an orthonormal basis. Therefore, the reproducing kernel of $\mathscr{H}^2$ is the function $k_\tau(z) = \sum_{n=1}^{\infty}\overline{\tau}_n z^n = (1-\overline{\tau}z)^{-1}$, $\tau\in\mathbb{D}$. Let $\hat{k}_\tau = \frac{k_\tau}{\|k_\tau\|}$ be the normalized reproducing kernel of the space $\mathscr{H}$ (RKHS). For a given bounded linear operator A on $\mathscr{H}$, the Berezin symbol (or Berezin transform) of A is the bounded function $\widetilde{A}$ on Θ defined by $\widetilde{A}(\tau) = \langle A\hat{k}_\tau(z), \hat{k}_\tau(z)\rangle$, $\tau\in\Theta$. An important property of the Berezin symbol is that for all $A, B\in\mathscr{L}(\mathscr{H})$, if $\widetilde{A}(\tau) = \widetilde{B}(\tau)$ for all $\tau\in\Theta$, then $A = B$ (at least when $\mathscr{H}$ consists of analytic functions, see Zhu [27]). For more details, see [14, 19, 20]. So, the map $A\to\widetilde{A}$ is injective [12]. The Berezin set (range), the Berezin radius (number), and the Berezin norms of an operator A are defined, respectively, by

$$\mathbf{Ber}(A) := \{\langle A\hat{k}_\tau, \hat{k}_\tau\rangle : \tau\in\Theta\}, \qquad \mathbf{ber}(A) := \sup_{\tau\in\Theta}\left|\langle A\hat{k}_\tau, \hat{k}_\tau\rangle\right|,$$
$$\|A\|_{\mathbf{ber},1} := \sup_{\tau,\mu\in\Theta}\left|\langle A\hat{k}_\tau, \hat{k}_\mu\rangle\right|, \quad \text{and} \quad \|A\|_{\mathbf{ber},2} := \sup_{\tau\in\Theta}\left\|A\hat{k}_\tau\right\|.$$

Clearly, by the above definition and the Cauchy-Schwartz inequality, we have

$$\mathbf{ber}(A)\leqslant\|A\|_{\mathbf{ber},1}\leqslant\|A\|_{\mathbf{ber},2}\leqslant\|A\| \quad \text{for} \quad A\in\mathscr{L}(\mathscr{H}). \tag{1}$$

Moreover, for $A, B\in\mathscr{L}(\mathscr{H})$, it is clear from the above definitions of the Berezin radius and the Berezin norms that the following properties hold:

(1) $\mathbf{ber}(\lambda A) = |\lambda|\mathbf{ber}(A)$ for all $\lambda \in \mathbb{C}$;

(2) $\mathbf{ber}(A+B) \leqslant \mathbf{ber}(A) + \mathbf{ber}(B)$;

(3) $\|\lambda T\|_{\mathbf{ber,i}} = |\lambda| \, \|T\|_{\mathbf{ber,i}}$ $(i = 1,2)$ for all $\lambda \in \mathbb{C}$;

(4) $\|A+B\|_{\mathbf{ber,i}} \leqslant \|A\|_{\mathbf{ber,i}} + \|B\|_{\mathbf{ber,i}}$ $(i = 1,2)$;

(5) $\mathbf{ber}(A) = \mathbf{ber}(A^*)$ and $\|A\|_{\mathbf{ber,1}} = \|A^*\|_{\mathbf{ber,1}}$.

For all positive operators $A \in \mathscr{L}(\mathscr{H})$, we have $\mathbf{ber}(A) = \|A\|_{\mathbf{ber,1}}$, see [5].

It is clear that the Berezin transform $\widetilde{A}$ is the bounded function on Θ whose values lie in the numerical range of the operator A, and hence $\mathbf{Ber}(A) \subseteq W(A)$ and $\mathbf{ber}(A) \leqslant w(A)$ for all $A \in \mathscr{L}(\mathscr{H})$. Karaev [21] showed that for $A = S \otimes S \in \mathscr{L}(\mathscr{H}^2)$, where S is the shift operator defined by $Sf(z) = zf(z)$ on the Hardy-Hilbert space $\mathscr{H}^2 = \mathscr{H}^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, we have $\widetilde{A}(\lambda) = |\lambda|^2(1-|\lambda|^2)$, and thus $\mathbf{Ber}(A) = \left[0, \frac{1}{4}\right] \subsetneq [0,1] = W(A)$ and $\mathbf{ber}(A) = \frac{1}{4} < 1 = w(A)$. For more information about the Berezin radius and the Berezin norm, see [11, 16, 20, 23] and references therein.

Recently, the authors [16, Theorem 2.5] presented the following Young type inequality

$$\mathbf{ber}^r(A^*B) \leqslant \frac{1}{2} \left\| |A|^{2r} + |B|^{2r} \right\|_{\mathbf{ber},1} \tag{2}$$

for all $A, B \in \mathscr{L}(\mathscr{H})$ and all $r \geqslant 1$, which is associated with the Berezin radius and the Berezin norm and involves the product of operators. Moreover, we have

$$\mathbf{ber}^r(A) \leqslant \frac{1}{2} \left\| |A|^{2rs} + |A^*|^{2r(1-s)} \right\|_{\mathbf{ber},1} \tag{3}$$

for $0 \leqslant s \leqslant 1$ and $r \geqslant 1$, see [2]. There are several generalizations and refinements of the inequalities (2) and (3), that have been proven recently, see [2, 15, 18, 25].

For operators $A, B \in \mathscr{L}(\mathscr{H})$, the authors [2] showed that some refinements of the inequality (2) as follows

$$\begin{aligned} \mathbf{ber}^r(A^*B) &\leqslant \frac{1}{2^{r\mu}p}\mathbf{ber}^{r(1-\mu)}(A^*B) \left\| |A|^2 + |B|^2 \right\|_{\mathbf{ber},1}^{r\mu} \\ &\quad + \frac{1}{2^{r(1-\nu)}q}\mathbf{ber}^{r\nu}(A^*B) \left\| |A|^2 + |B|^2 \right\|_{\mathbf{ber},1}^{r(1-\nu)} \\ &\leqslant \frac{1}{2} \left\| |A|^{2r} + |B|^{2r} \right\|_{\mathbf{ber},1} \end{aligned} \tag{4}$$

and

$$\mathbf{ber}^r(A^*B) \leqslant \int_0^1 \frac{1}{2^{r\mu}}\mathbf{ber}^{r(1-\mu)}(A^*B) \left\| |A|^2 + |B|^2 \right\|_{\mathbf{ber},1}^{r\mu} d\mu \leqslant \frac{1}{2} \left\| |A|^{2r} + |B|^{2r} \right\|_{\mathbf{ber},1} \tag{5}$$

for all $1 \leqslant r \leqslant 2$, $\mu, \nu \in [0,1]$, and $p,q > 0$ with $\frac{1}{p}+\frac{1}{q}=1$. Also, in the same work, they showed that

$$\begin{aligned}
&\mathbf{ber}^2(A)\\
&\leqslant \frac{1}{2^{\mu+1}p}\mathbf{ber}^{(1-\mu)}(A^2)\left\||A|^2+|A^*|^2\right\|_{\mathbf{ber},1}^{\mu}+\frac{1}{2^{2-\nu}p}\mathbf{ber}^{\nu}(A^2)\left\||A|^2+|A^*|^2\right\|_{\mathbf{ber},1}^{1-\nu}\\
&\quad+\frac{1}{4}\left\||A|^2+|A^*|^2\right\|_{\mathbf{ber},1}\\
&\leqslant \frac{1}{2}\left\||A|^2+|A^*|^2\right\|_{\mathbf{ber},1}
\end{aligned} \tag{6}$$

for $\mu,\nu \in [0,1]$ and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$.

In the present paper, we establish some Berezin radius upper bounds for Hilbert space operators involving the tensor product. Moreover, the obtained upper bounds have been compared with the previously known bounds to demonstrate their reliability.

2. Main results

In this section, some inequalities involving the tensor product of operators. To prove our Berezin radius inequalities, we need several known lemmas. The first lemma is McCarthy's inequality for positive operators.

LEMMA 1. [9] (McCarthy's inequality) *Let $A \in \mathscr{L}(\mathscr{H})$ be positive. Then for all unit vectors $x \in \mathscr{H}$, we have*

$$\langle Ax,x\rangle^r \leqslant \langle A^r x,x\rangle,$$

where $r \geqslant 1$. This inequality is reversed for $0<r\leqslant 1$.

In the following lemma, we give the mixed Schwarz inequality, which can be found in [22].

LEMMA 2. *Let $A \in \mathscr{L}(\mathscr{H})$ and let $x,y \in \mathscr{H}$. Then*

$$|\langle Ax,y\rangle|^2 \leqslant \left\langle |A|^{2s}x,x\right\rangle\left\langle |A^*|^{2(1-s)}y,y\right\rangle \quad \textit{for all } 0 \leqslant s \leqslant 1.$$

LEMMA 3. [2] *Let $A \in \mathscr{L}(\mathscr{H})$. Then*

(1) If A is positive and $r \geqslant 1$, then $\|A\|_{\mathbf{ber},1}^r \leqslant \|A^r\|_{\mathbf{ber},1}$;

(2) If A is positive and $0 \leqslant r \leqslant 1$, then $\|A^r\|_{\mathbf{ber},1} \leqslant \|A\|_{\mathbf{ber},1}^r$.

REMARK 1. For the operator norm, we have $\|A^r\| = \|A\|^r$ for all positive operators A and $r \geqslant 0$. In contrast to the the operator norm, the 1-Berezin norm dose not have this property. For instance, let $\{e_1, e_1\}$ be the standard orthonormal basis for $\mathbb{C}^2$

as a RKHS on the set $\Theta = \{1,2\}$. If we put the positive-definite matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$. Then we have

$$\|A\|^2_{\mathbf{ber},1} = 9 \quad \lneqq \quad \|A^2\|_{\mathbf{ber},1} = 10$$

and

$$\sqrt{\|A\|_{\mathbf{ber},1}} \approx 1.73 \quad \gneqq \quad \|\sqrt{A}\|_{\mathbf{ber},1} = 1.7013.$$

REMARK 2. For the operator norm, we have $\| \, |A| \, \| = \|A\|$ for all operators A. In contrast to the the operator norm, the 1-Berezin norm dose not have this property. For instance, let $\{e_1, e_1\}$ be the standard orthonormal basis for $\mathbb{C}^2$ as a RKHS on the set $\Theta = \{1,2\}$. Then for the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$, we have

$$\|A\|_{\mathbf{ber},1} = 3 \quad \lneqq \quad \| \, |A| \, \|_{\mathbf{ber},1} \approx 3.15,$$

and for the matrix $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{2} \end{bmatrix}$, we have

$$\|A\|_{\mathbf{ber},1} = 0.5 \quad \gneqq \quad \| \, |A| \, \|_{\mathbf{ber},1} \approx 0.49.$$

Therefore, $\|A\|_{\mathbf{ber},1}$ and $\| \, |A| \, \|_{\mathbf{ber},1}$ are not comparable in general.

REMARK 3. The Berezin radius and the Berezin norm have several properties that are similar to the numerical radius and the operator norm, respectively. In contrast to the numerical radius, the Berezin radius is not weakly unitarily invariant, i.e., there exist an operator $A \in \mathscr{L}(\mathscr{H})$ and a unitary operator $U \in \mathscr{L}(\mathscr{H})$ such that $\mathbf{ber}(U^*AU) \neq \mathbf{ber}(A)$. For instance, let $\{e_1, e_1\}$ be the standard orthonormal basis for $\mathbb{C}^2$ as a RKHS on the set $\Theta = \{1,2\}$. If we put $A = \begin{bmatrix} 2 & 1 \\ 5 & 0 \end{bmatrix}$ and $U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, then

$$\mathbf{ber}(U^*AU) = 4 \neq \mathbf{ber}(A) = 2.$$

Moreover, the operator norm is unitarily invariant, i.e., $\|U^*AV\| = \|A\|$ for all $A \in \mathscr{L}(\mathscr{H})$ and all unitary operators $U, V \in \mathscr{L}(\mathscr{H})$. In contrast to the operator norm, the Berezin norm dose not have this property. To see this, consider $A = \begin{bmatrix} 2 & 5 \\ 7 & 0 \end{bmatrix}$, $U = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$, and $V = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then we have

$$9.89 \approx \|U^*AV\|_{\mathbf{ber},1} \neq \|A\|_{\mathbf{ber},1} = 7.$$

For more information, see [2] and [3] and references therein.

In the following proposition, we have a result for the 1-Berezin norm and the 2-Berezin norm.

PROPOSITION 1. Let $A \in \mathscr{L}(\mathscr{H})$. Then

(1) $\| |A| \|_{\mathbf{ber},1} \leqslant \|A\|_{\mathbf{ber},2}$;

(2) $\|A\|_{\mathbf{ber},2} = \| |A|^2 \|_{\mathbf{ber},1}^{\frac{1}{2}}$.

Proof. Applying Proposition 3(1), we have

$$\| |A| \|_{\mathbf{ber},1}^2 \leqslant \| |A|^2 \|_{\mathbf{ber},1} = \mathbf{ber}(|A|^2) = \sup_{\tau\in\Theta} \left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle = \sup_{\tau\in\Theta} \|A\hat{k}_\tau\|^2 = \|A\|_{\mathbf{ber},2}^2.$$

Hence, we have the first result. For the second result, we have

$$\begin{aligned} \|A\|_{\mathbf{ber},2}^2 &= \sup_{\tau\in\Theta} \|A\hat{k}_\tau\|^2 = \sup_{\tau\in\Theta} \left\langle A\hat{k}_\tau, A\hat{k}_\tau \right\rangle = \sup_{\tau\in\Theta} \left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau \right\rangle \\ &= \mathbf{ber}(|A|^2) = \| |A|^2 \|_{\mathbf{ber},1}. \quad \square \end{aligned}$$

Let us consider the finite dimensional setting. Let $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $\Theta = \{1, \ldots, n\}$. We can consider $\mathbb{C}^n$ as the set of all functions mapping $\Theta \to \mathbb{C}$ by $z(i) = z_i$. Letting e_i be the ith standard basis vector for $\mathbb{C}^n$ under the standard inner product, we can view $\mathbb{C}^n$ as an RKHS with kernel $k(i,j) = \langle e_j, e_i \rangle$. Note that $k_i = \hat{k}_i$ for every $i = 1, \ldots, n$. For any $n \times n$ complex matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$, we have $\langle Ae_i, e_i \rangle = a_{ii}$. Thus, the Berezin range, the Berezin radius, and the Berezin norms of A are simply by

$$\mathbf{Ber}(A) = \{a_{ii} : i = 1, \ldots, n\}, \qquad \mathbf{ber}(A) = \max_{1 \leqslant i \leqslant n} |a_{ii}|,$$

$$\|A\|_{\mathbf{ber},1} = \max_{1 \leqslant i,j \leqslant n} |a_{ij}| \quad \text{and} \quad \|A\|_{\mathbf{ber},2} = \max_{1 \leqslant j \leqslant n} \left(\sum_{i=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}},$$

respectively, see [7] and references therein. Let us consider $\{e_i\}_{i=1}^n$ be the standard basis for $\mathbb{C}^n$ under the standard inner product as an RKHS with kernel $k(i,j) = \langle e_j, e_i \rangle$. Then, $\{e_i \otimes e_j\}_{i,j=1}^n$ is basis for $\mathbb{C}^n \otimes \mathbb{C}^n$ as the RKHS. Hence, for two matrices $A = (a_{ij})_{1 \leqslant i,j \leqslant n}$ and $B = (b_{ij})_{1 \leqslant i,j \leqslant n}$, we have

$$\langle (A \otimes B)(e_i \otimes e_j), (e_i \otimes e_j) \rangle = \langle Ae_i, e_i \rangle \langle Be_j, e_j \rangle = a_{ii} b_{jj} \qquad \text{for all } 1 \leqslant i,j \leqslant n.$$

Therefore, we have

$$\begin{aligned} \mathbf{Ber}(A \otimes B) &= \left\{ \langle (A \otimes B)(e_i \otimes e_j), (e_i \otimes e_j) \rangle : 1 \leqslant i,j \leqslant n \right\} \\ &= \left\{ \langle Ae_i, e_i \rangle \langle Be_j, e_j \rangle : 1 \leqslant i,j \leqslant n \right\} \\ &= \{ a_{ii} b_{jj} : 1 \leqslant i,j \leqslant n \} \\ &\subseteq \mathbf{Ber}(A)\mathbf{Ber}(B) \end{aligned}$$

for two arbitrary matrices $A, B \in \mathbb{M}_n(\mathbb{C})$. As a consequence of the above equation, we have the Berezin radius and the Berezin norms for k-fold tensor product of $A_i \in \mathbb{M}_n(\mathbb{C})$ $(1 \leqslant i \leqslant n)$ as follows.

PROPOSITION 2. Let $A_i \in \mathbb{M}_n(\mathbb{C})$ $(1 \leqslant i \leqslant n)$. Then

(1) $\mathbf{ber}\left(\otimes_{i=1}^n A_i\right) \leqslant \prod_{i=1}^n \mathbf{ber}(A_i)$;

(2) $\left\|\otimes_{i=1}^n A_i\right\|_{\mathbf{ber},1} \leqslant \prod_{i=1}^n \|A_i\|_{\mathbf{ber},1}$;

(3) $\left\|\otimes_{i=1}^n A_i\right\|_{\mathbf{ber},2} \leqslant \prod_{i=1}^n \|A_i\|_{\mathbf{ber},2}$.

Proof. We have

$$\begin{aligned}\mathbf{ber}(\otimes_{i=1}^n A_i) &= \sup_{\otimes_{j=1}^n e_j} \left|\left\langle \left(\otimes_{i=1}^n A_i\right)\left(\otimes_{j=1}^n e_i\right), \left(\otimes_{i=1}^n e_i\right)\right\rangle\right| \\ &= \sup_{\otimes_{i=1}^n e_i} \left|\left\langle \otimes_{i=1}^n A_i e_i, \otimes_{i=1}^n e_i\right\rangle\right| \\ &= \sup_{\otimes_{i=1}^n e_i} \prod_{i=1}^n |\langle A_i e_i, e_i\rangle| \qquad \text{(by the property (1) of the tensor product)} \\ &\leqslant \prod_{i=1}^n \sup_{e_i} |\langle A_i e_i, e_i\rangle| = \prod_{i=1}^n \mathbf{ber}(A_i).\end{aligned}$$

Hence, we have the first result. The proof of the second result is similar to part (1). For part (3), we have

$$\begin{aligned}\|\otimes_{i=1}^n A_i\|_{\mathbf{ber},2} &= \sup_{\otimes_{i=1}^n e_i} \left\|\left(\otimes_{i=1}^n A_i\right)\left(\otimes_{i=1}^n e_i\right)\right\| \\ &= \sup_{\otimes_{i=1}^n e_i} \|\otimes_{i=1}^n A_i e_i\| \\ &= \sup_{\otimes_{i=1}^n e_i} \prod_{i=1}^n \|A_i e_i\| \\ &= \prod_{i=1}^n \sup_{e_i} \|A_i e_i\| \leqslant \prod_{i=1}^n \|A_i\|_{\mathbf{ber},2}. \quad \square\end{aligned}$$

In the next result, we have an upper bound for the Berezin radius of the tensor product.

PROPOSITION 3. Let $A, B \in \mathscr{L}(\mathscr{H})$. Then

$$\begin{aligned}\mathbf{ber}^r(AB \otimes BA) &\leqslant \frac{1}{2^{r\mu} p}\mathbf{ber}^{r(1-\mu)}(AB \otimes BA)\left\||A|^2 \otimes |B|^2 + |B^*|^2 \otimes |A^*|^2\right\|_{\mathbf{ber},1}^{r\mu} \\ &\quad + \frac{1}{2^{r(1-\nu)} q}\mathbf{ber}^{r\nu}(AB \otimes BA)\left\||A|^2 \otimes |B|^2 + |B^*|^2 \otimes |A^*|^2\right\|_{\mathbf{ber},1}^{r(1-\nu)} \\ &\leqslant \frac{1}{2}\left\||A|^{2r} \otimes |B|^{2r} + |B^*|^{2r} \otimes |A^*|^{2r}\right\|_{\mathbf{ber},1}\end{aligned}$$

for all $1 \leqslant r \leqslant 2$, $\mu, \nu \in [0,1]$, and $p, q > 0$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Assume that $1 \leqslant r \leqslant 2$, $\mu, \nu \in [0,1]$, and $p,q > 0$ with $\frac{1}{p}+\frac{1}{q}=1$. Then we have

$$\begin{aligned}
\mathbf{ber}^r(AB\otimes BA) &= \mathbf{ber}^r((A\otimes B)(B\otimes A))\\
&\leqslant \frac{1}{2^{r\mu}p}\mathbf{ber}^{r(1-\mu)}((A\otimes B)(B\otimes A))\left\||A\otimes B|^2+|(B\otimes A)^*|^2\right\|_{\mathbf{ber},1}^{r\mu}\\
&\quad+\frac{1}{2^{r(1-\nu)}q}\mathbf{ber}^{r\nu}((A\otimes B)(B\otimes A))\left\||A\otimes B|^2+|(B\otimes A)^*|^2\right\|_{\mathbf{ber},1}^{r(1-\nu)}\\
&\qquad\qquad \text{(by the first inequality in (4))}\\
&= \frac{1}{2^{r\mu}p}\mathbf{ber}^{r(1-\mu)}(AB\otimes BA)\left\||A|^2\otimes|B|^2+|B^*|^2\otimes|A^*|^2\right\|_{\mathbf{ber},1}^{r\mu}\\
&\quad+\frac{1}{2^{r(1-\nu)}q}\mathbf{ber}^{r\nu}(AB\otimes BA)\left\||A|^2\otimes|B|^2+|B^*|^2\otimes|A^*|^2\right\|_{\mathbf{ber},1}^{r(1-\nu)}\\
&\qquad\qquad \text{(by the properties of the tensor product)}\\
&\leqslant \frac{1}{2}\left\||A\otimes B|^{2r}+|(B\otimes A)^*|^{2r}\right\|_{\mathbf{ber},1}\\
&\qquad\qquad \text{(by the second inequality in (4))}\\
&= \frac{1}{2}\left\||A|^{2r}\otimes|B|^{2r}+|B^*|^{2r}\otimes|A^*|^{2r}\right\|_{\mathbf{ber},1}. \quad \square
\end{aligned}$$

Taking $r=2$ and $\mu=\nu=\frac{1}{2}$ in Proposition 3, we have the next result.

COROLLARY 1. *Let* $A,B\in\mathscr{L}(\mathscr{H})$. *Then*

$$\begin{aligned}
\mathbf{ber}^2(AB\otimes BA) &\leqslant \mathbf{ber}(AB\otimes BA)\left\||A|^2\otimes|B|^2+|B^*|^2\otimes|A^*|^2\right\|_{\mathbf{ber},1}\\
&\leqslant \frac{1}{2}\left\||A|^4\otimes|B|^4+|B^*|^4\otimes|A^*|^4\right\|_{\mathbf{ber},1}.
\end{aligned}$$

PROPOSITION 4. Let $A,B\in\mathscr{L}(\mathscr{H})$. Then

$$\begin{aligned}
\mathbf{ber}^r(AB\otimes BA) &\leqslant \int_0^1 \frac{1}{2^{r\mu}}\mathbf{ber}^{r(1-\mu)}(AB\otimes BA)\left\||A|^2\otimes|B|^2+|B^*|^2\otimes|A^*|^2\right\|_{\mathbf{ber},1}^{r\mu}\,d\mu\\
&\leqslant \frac{1}{2}\left\||A|^{2r}\otimes|B|^{2r}+|B^*|^{2r}\otimes|A^*|^{2r}\right\|_{\mathbf{ber},1}
\end{aligned}$$

for all $1\leqslant r\leqslant 2$.

Proof. Assume that $1\leqslant r\leqslant 2$. Using the inequality (5), we have

$$\begin{aligned}
\mathbf{ber}^r(AB\otimes BA) &= \mathbf{ber}^r((A\otimes B)(B\otimes A))\\
&\leqslant \int_0^1 \frac{1}{2^{r\mu}}\mathbf{ber}^{r(1-\mu)}((A\otimes B)(B\otimes A))\left\||A\otimes B|^2+|(B\otimes A)^*|^2\right\|_{\mathbf{ber},1}^{r\mu}\,d\mu\\
&= \int_0^1 \frac{1}{2^{r\mu}}\mathbf{ber}^{r(1-\mu)}(AB\otimes BA)\left\||A|^2\otimes|B|^2+|B^*|^2\otimes|A^*|^2\right\|_{\mathbf{ber},1}^{r\mu}\,d\mu\\
&\leqslant \frac{1}{2}\left\||A|^{2r}\otimes|B|^{2r}+|B^*|^{2r}\otimes|A^*|^{2r}\right\|_{\mathbf{ber},1},
\end{aligned}$$

as required. □

Recall that if $X=U|X|$ be the polar decomposition of $X\in\mathscr{L}(\mathscr{H})$, then for $0\leqslant \nu\leqslant 1$, the ν-Aluthge transform is defined by $\widetilde{X_\nu}=|X|^\nu U|X|^{1-\nu}$. As a consequence of Proposition 4, we have the following result for the Aluthge transform.

COROLLARY 2. *Let $X\in\mathscr{L}(\mathscr{H})$. Then*

$$\begin{aligned}&\mathbf{ber}^r(X\otimes\widetilde{X_\nu})\\&\leqslant\int_0^1\frac{1}{2^{r\mu}}\mathbf{ber}^{r(1-\mu)}(X\otimes\widetilde{X_\nu})\left\||X|^{2(1-\nu)}\otimes|X|^{2\nu}+|X|^{2\nu}\otimes|X^*|^{2(1-\nu)}\right\|_{\mathbf{ber},1}^{r\mu}d\mu\\&\leqslant\frac{1}{2}\left\||X|^{2r(1-\nu)}\otimes|X|^{2r\nu}+|X|^{2r\nu}\otimes|X^*|^{2r(1-\nu)}\right\|_{\mathbf{ber},1}\end{aligned}$$

for all $0\leqslant\nu\leqslant 1$ *and* $1\leqslant r\leqslant 2$.

Proof. Assume that $X=U|X|$ be the polar decomposition of X. Putting $A=U|X|^{1-\nu}$ and $B=|X|^\nu$, in Proposition 4. Then $AB=X$, $BA=\widetilde{X_\nu}$, $|B|^2=|B^*|^2=|X|^{2\nu}$. Using the properties of the polar decomposition [8, p.58], we have

$$|A^*|^2=U|X|^{2(1-\nu)}U^*=|X^*|^{2(1-\nu)}\quad\text{and}\quad |A|^2=|X|^{1-\nu}U^*U|X|^{1-\nu}=|X|^{2(1-\nu)}.$$

Hence, we get

$$\begin{aligned}&\mathbf{ber}^r(X\otimes\widetilde{X_\nu})\\&\leqslant\int_0^1\frac{1}{2^{r\mu}}\mathbf{ber}^{r(1-\mu)}(X\otimes\widetilde{X_\nu})\left\||X|^{2(1-\nu)}\otimes|X|^{2\nu}+|X|^{2\nu}\otimes|X^*|^{2(1-\nu)}\right\|_{\mathbf{ber},1}^{r\mu}d\mu\\&\leqslant\frac{1}{2}\left\||X|^{2r(1-\nu)}\otimes|X|^{2r\nu}+|X|^{2r\nu}\otimes|X^*|^{2r(1-\nu)}\right\|_{\mathbf{ber},1}\end{aligned}$$

for all $1\leqslant r\leqslant 2$ as required. □

THEOREM 1. *Let $A,B\in\mathscr{L}(\mathscr{H})$. Then*

$$\begin{aligned}\mathbf{ber}^2(A\otimes B)\leqslant\frac{1}{4}&\left\||A|^{4s}\otimes|B|^{4s}+|A^*|^{4(1-s)}\otimes|B^*|^{4(1-s)}\right\|_{\mathbf{ber},1}\\&+\frac{1}{2}\mathbf{ber}\left(\Re\left(|A|^{2s}|A^*|^{2(1-s)}\otimes|B|^{2s}|B^*|^{2(1-s)}\right)\right)\end{aligned}$$

for $0\leqslant s\leqslant 1$. In particular,

$$\mathbf{ber}^2(A\otimes B)\leqslant\frac{1}{4}\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\|_{\mathbf{ber},1}+\frac{1}{2}\|\Re\left(|A||A^*|\otimes|B||B^*|\right)\|_{\mathbf{ber},1}.$$

Proof. Assume that $\hat{\mathbf{k}}_\tau \in \mathscr{H} \otimes \mathscr{H}$ is the normalized reproducing kernel. Then, utilizing Lemma 2, we have

$$\begin{aligned}\left|\langle (A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\rangle\right| &\leqslant \left\langle |A\otimes B|^{2s}\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle^{\frac{1}{2}}\left\langle |(A\otimes B)^*|^{2(1-s)}\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle^{\frac{1}{2}}\\ &\leqslant \frac{1}{2}\left\langle\left(|A\otimes B|^{2s}+|(A\otimes B)^*|^{2(1-s)}\right)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\\ &\qquad \text{(by the arithmetic geometric inequality).}\end{aligned}$$

Hence,

$$\begin{aligned}\left|\langle (A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\rangle\right|^2 &\leqslant \frac{1}{4}\left\langle\left(|A\otimes B|^{2s}+|(A\otimes B)^*|^{2(1-s)}\right)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle^2\\ &\leqslant \frac{1}{4}\left\langle\left(|A\otimes B|^{2s}+|(A\otimes B)^*|^{2(1-s)}\right)^2\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\\ &\qquad \text{(by Lemma 1)}\\ &\leqslant \frac{1}{4}\left\langle\left(|A|^{4s}\otimes|B|^{4s}+|A^*|^{4(1-s)}\otimes|B^*|^{4(1-s)}\right)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\\ &\qquad \text{(by the property (6) of the tensor product)}\\ &\quad+\frac{1}{2}\left\langle\Re\left(|A|^{2s}|A^*|^{2(1-s)}\otimes|B|^{2s}|B^*|^{2(1-s)}\right)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\\ &\leqslant \frac{1}{4}\mathbf{ber}\left(|A|^{4s}\otimes|B|^{4s}+|A^*|^{4(1-s)}\otimes|B^*|^{4(1-s)}\right)\\ &\quad+\frac{1}{2}\mathbf{ber}\left(\Re\left(|A|^{2s}|A^*|^{2(1-s)}\otimes|B|^{2s}|B^*|^{2(1-s)}\right)\right)\\ &= \frac{1}{4}\left\||A|^{4s}\otimes|B|^{4s}+|A^*|^{4(1-s)}\otimes|B^*|^{4(1-s)}\right\|_{\mathbf{ber},1}\\ &\quad+\frac{1}{2}\mathbf{ber}\left(\Re\left(|A|^{2s}|A^*|^{2(1-s)}\otimes|B|^{2s}|B^*|^{2(1-s)}\right)\right)\\ &\qquad \text{(since } \mathbf{ber}(X)=\|X\|_{\mathbf{ber},1} \text{ for all positive operators } X).\end{aligned}$$

Then, by taking the supremum over all $\hat{\mathbf{k}}_\tau \in \mathscr{H}\otimes\mathscr{H}$, we get the first inequality. For the second inequality, it is enough to put $s=\frac{1}{2}$ in the first result. □

THEOREM 2. *Let $A,B\in\mathscr{L}(\mathscr{H})$. Then*

$$\begin{aligned}&\mathbf{ber}^2(A\otimes B)\\ &\leqslant \frac{1}{2^{\mu+1}p}\mathbf{ber}^{(1-\mu)}(A^2\otimes B^2)\left\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right\|_{\mathbf{ber},1}^{\mu}\\ &\quad+\frac{1}{2^{2-\nu}p}\mathbf{ber}^{\nu}(A^2\otimes B^2)\left\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right\|_{\mathbf{ber},1}^{1-\nu}\\ &\quad+\frac{1}{4}\left\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right\|_{\mathbf{ber},1}\\ &\leqslant \frac{1}{2}\left\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right\|_{\mathbf{ber},1}\end{aligned}$$

for $\mu,\nu\in[0,1]$ and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Assume that $\mu,\nu\in[0,1]$ and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$. Then, using the inequality (6), we have

$$\begin{aligned}
&\mathbf{ber}^2(A\otimes B)\\
&\leqslant \frac{1}{2^{\mu+1}p}\mathbf{ber}^{(1-\mu)}((A\otimes B)^2)\left\||A\otimes B|^2+|(A\otimes B)^*|^2\right\|_{\mathbf{ber},1}^{\mu}\\
&\quad+\frac{1}{2^{2-\nu}p}\mathbf{ber}^{\nu}((A\otimes B)^2)\left\||(A\otimes B|^2+|(A\otimes B)^*|^2\right\|_{\mathbf{ber},1}^{1-\nu}\\
&\quad+\frac{1}{4}\left\||A\otimes B|^2+|(A\otimes B)^*|^2\right\|_{\mathbf{ber},1}\\
&=\frac{1}{2^{\mu+1}p}\mathbf{ber}^{(1-\mu)}(A^2\otimes B^2)\left\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right\|_{\mathbf{ber},1}^{\mu}\\
&\quad+\frac{1}{2^{2-\nu}p}\mathbf{ber}^{\nu}(A^2\otimes B^2)\left\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right\|_{\mathbf{ber},1}^{1-\nu}\\
&\quad+\frac{1}{4}\left\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right\|_{\mathbf{ber},1}\\
&\leqslant\frac{1}{2}\left\||A\otimes B|^2+|(A\otimes B)^*|^2\right\|_{\mathbf{ber},1}\\
&=\frac{1}{2}\left\||A|^2\otimes|B|^2+|A^*|^2\otimes|B^*|^2\right\|_{\mathbf{ber},1},
\end{aligned}$$

as required. □

In the next result, we obtain a lower bound for the Berezin radius involving the tensor product.

PROPOSITION 5. Let $A,B\in\mathscr{L}(\mathscr{H})$. Then

$$\mathbf{ber}(A\otimes B)\geqslant\frac{1}{\sqrt{2}}\max\left\{\mathbf{ber}\left(\Re(A\otimes B)+\Im(A\otimes B)\right),\mathbf{ber}\left(\Re(A\otimes B)-\Im(A\otimes B)\right)\right\}.$$

Proof. Assume that $\hat{\mathbf{k}}_\tau\in\mathscr{H}\otimes\mathscr{H}$ is the normalized reproducing kernel. Then, utilizing Lemma 2, we have

$$\begin{aligned}
\left|\left\langle(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|^2&\geqslant\left\langle\Re(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle^2+\left\langle\Im(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle^2\\
&\geqslant\frac{1}{2}\left(\left|\left\langle\Re(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|+\left|\left\langle\Im(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|\right)^2\\
&\geqslant\frac{1}{2}\left|\left\langle\Re(A\otimes B)\pm\Im(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|^2.
\end{aligned}$$

Hence,

$$\mathbf{ber}(A\otimes B)\geqslant\left|\left\langle(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|\geqslant\frac{1}{\sqrt{2}}\left|\left\langle\Re(A\otimes B)\pm\Im(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|$$

Therefore, we get

$$\mathbf{ber}(A\otimes B)\geqslant\frac{1}{\sqrt{2}}\max\left\{\left|\left\langle\Re(A\otimes B)+\Im(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|,\left|\left\langle\Re(A\otimes B)-\Im(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|\right\}$$

Then, by taking the supremum over all $\hat{\mathbf{k}}_\tau\in\mathscr{H}\otimes\mathscr{H}$, we get the desired inequality. □

THEOREM 3. *Let $A,B\in\mathscr{L}(\mathscr{H})$. Then*

$$0\leqslant\|A\otimes B\|^2_{\mathbf{ber},2}-\mathbf{ber}^2(A\otimes B)\leqslant\inf_{\eta\in\mathbb{C}}\left\{\|A\otimes B-\eta I\otimes I\|^2_{\mathbf{ber},2}-c^2_{\mathbf{ber}}(A\otimes B-\eta I\otimes I)\right\},$$

where $c_{\mathbf{ber}}(X)=\inf_{\tau\in\Theta}\left|\left\langle X\hat{k}_\tau,\hat{k}_\tau\right\rangle\right|$.

Proof. Assume that $\hat{\mathbf{k}}_\tau\in\mathscr{H}\otimes\mathscr{H}$ is the normalized reproducing kernel. First note that for all $\eta\in\mathbb{C}$, we have

$$\left\|(A\otimes B)\hat{\mathbf{k}}_\tau\right\|^2-\left|\left\langle(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|^2=\left\|(A\otimes B-\eta I\otimes I)\hat{\mathbf{k}}_\tau\right\|^2-\left|\left\langle(A\otimes B-\eta I\otimes I)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|^2.$$

Hence,

$$\begin{aligned}\left\|(A\otimes B)\hat{\mathbf{k}}_\tau\right\|^2-\left|\left\langle(A\otimes B)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|^2&=\left\|(A\otimes B-\eta I\otimes I)\hat{\mathbf{k}}_\tau\right\|^2-\left|\left\langle(A\otimes B-\eta I\otimes I)\hat{\mathbf{k}}_\tau,\hat{\mathbf{k}}_\tau\right\rangle\right|^2\\&\leqslant\|A\otimes B-\eta I\otimes I\|^2_{\mathbf{ber},2}-c^2_{\mathbf{ber}}(A\otimes B-\eta I\otimes I).\end{aligned}$$

Taking the supremum over all $\hat{\mathbf{k}}_\tau\in\mathscr{H}\otimes\mathscr{H}$, we get

$$\|A\otimes B\|^2_{\mathbf{ber},2}-\mathbf{ber}^2(A\otimes B)\leqslant\|A\otimes B-\eta I\otimes I\|^2_{\mathbf{ber},2}-c^2_{\mathbf{ber}}(A\otimes B-\eta I\otimes I).$$

It follows from $\eta\in\mathbb{C}$ is arbitrary that

$$\|A\otimes B\|^2_{\mathbf{ber},2}-\mathbf{ber}^2(A\otimes B)\leqslant\inf_{\eta\in\mathbb{C}}\left\{\|A\otimes B-\eta I\otimes I\|^2_{\mathbf{ber},2}-c^2_{\mathbf{ber}}(A\otimes B-\eta I\otimes I)\right\},$$

as required. □

3. Upper bounds for the Berezin radius

In the present section, we obtain the upper bounds for the Berezin radius of bounded linear operators including tensor product of operators. For this goal, we need the following two lemmas. The first one is known as Busano's inequality and the second one is known the weighted arithmetic-geometric mean inequality.

LEMMA 4. [6] *Let $x,y,e\in\mathscr{H}$, and let $\|e\|=1$. Then*

$$\left|\langle x,e\rangle\langle e,y\rangle\right|\leqslant\frac{1}{2}\left[|\langle x,y\rangle|+\|x\|\|y\|\right].$$

LEMMA 5. [17] *If* $a,b \geqslant 0$ *and* $0 \leqslant \alpha \leqslant 1$, *then*

$$a^{\alpha}b^{1-\alpha} \leqslant \alpha a + (1-\alpha)b.$$

THEOREM 4. *Let* $A \in \mathscr{L}(\mathscr{H})$. *Then*

$$\mathbf{ber}^2(A) \leqslant \frac{1}{2}\Big\{\mathbf{ber}^2(A^2) + \min_{t\in[0,1]} \big[(t\|A\|^2_{\mathbf{ber},2} + (1-t)\|A^*\|^2_{\mathbf{ber},2}) \\ ((1-t)\|A\|^2_{\mathbf{ber},2} + t\|A^*\|^2_{\mathbf{ber},2})\big] + \mathbf{ber}(A^2)(\|A\|^2_{\mathbf{ber},2} + \|A^*\|^2_{\mathbf{ber},2})\Big\}^{\frac{1}{2}}.$$

Proof. Let $\tau \in \Theta$ be arbitrary. Then we have

$$\begin{aligned}
\left|\widetilde{A^2}(\tau)\right| &= \left|\langle A\hat{k}_\tau, \hat{k}_\tau\rangle^2\right| \\
&= \left|\left\langle A\hat{k}_\tau, \hat{k}_\tau\right\rangle \left\langle \hat{k}_\tau, A^*\hat{k}_\tau\right\rangle\right| \\
&\leqslant \frac{1}{2}\left[\left|\left\langle A\hat{k}_\tau, A^*\hat{k}_\tau\right\rangle\right| + \|A\hat{k}_\tau\|\,\|A^*\hat{k}_\tau\|\right] \\
&\leqslant \frac{1}{2}\Big\{\left|\left\langle A\hat{k}_\tau, A^*\hat{k}_\tau\right\rangle\right|^2 + \left\|A\hat{k}_\tau\right\|^2\left\|A^*\hat{k}_\tau\right\|^2 + 2\left|\left\langle A\hat{k}_\tau, A^*\hat{k}_\tau\right\rangle\right|\|A\hat{k}_\tau\|\,\|A^*\hat{k}_\tau\|\Big\}^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\Big\{\left|\left\langle A^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|^2 + \left|\left\langle A^*A\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|\left|\left\langle AA^*\hat{k}_\tau, \hat{k}_\tau\right\rangle\right| \\
&\quad + 2\left|\left\langle A^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|\left|\left\langle (AA^*+A^*A)\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|\Big\}^{\frac{1}{2}} \\
&= \frac{1}{2}\Big\{\left|\left\langle A^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|^2 + \left|\left\langle A^*A\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|^t\left|\left\langle AA^*\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|^{1-t}\left|\left\langle A^*A\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|^{1-t} \\
&\quad \times\left|\left\langle AA^*\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|^t + 2\left|\left\langle A^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|\left|\left\langle (AA^*+A^*A)\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|\Big\}^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\Big\{\left|\left\langle A^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|^2 + \left\langle (tAA^*+(1-t)A^*A)\hat{k}_\tau, \hat{k}_\tau\right\rangle\left\langle ((1-t)AA^*+tA^*A)\hat{k}_\tau, \hat{k}_\tau\right\rangle \\
&\quad + 2\left|\left\langle A^2\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|\left|\left\langle (AA^*+A^*A)\hat{k}_\tau, \hat{k}_\tau\right\rangle\right|\Big\}^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\Big\{\mathbf{ber}^2(A^2) + \big[(t\|A\|^2_{\mathbf{ber},2} + (1-t)\|A^*\|^2_{\mathbf{ber},2}) \\
&\quad ((1-t)\|A\|^2_{\mathbf{ber},2} + t\|A^*\|^2_{\mathbf{ber},2})\big] + \mathbf{ber}(A^2)(\|A\|^2_{\mathbf{ber},2} + \|A^*\|^2_{\mathbf{ber},2})\Big\}^{\frac{1}{2}}.
\end{aligned}$$

Taking the supremum over all $\tau \in \Theta$, we get for all $t \in [0,1]$ that

$$\mathbf{ber}^2(A) \leqslant \frac{1}{2}\Big\{\mathbf{ber}^2(A^2) + \big[(t\|A\|^2_{\mathbf{ber},2} + (1-t)\|A^*\|^2_{\mathbf{ber},2}) \\ ((1-t)\|A\|^2_{\mathbf{ber},2} + t\|A^*\|^2_{\mathbf{ber},2})\big] + \mathbf{ber}(A^2)(\|A\|^2_{\mathbf{ber},2} + \|A^*\|^2_{\mathbf{ber},2})\Big\}^{\frac{1}{2}}$$

and considering the minimum over $t \in [0,1]$, we arrive at

$$\mathbf{ber}^2(A) \leqslant \frac{1}{2}\Big\{\mathbf{ber}^2(A^2) + \min_{t\in[0,1]}\big[(t\|A\|^2_{\mathbf{ber},2} + (1-t)\|A^*\|^2_{\mathbf{ber},2}) \\ ((1-t)\|A\|^2_{\mathbf{ber},2} + t\|A^*\|^2_{\mathbf{ber},2})\big] + \mathbf{ber}(A^2)(\|A\|^2_{\mathbf{ber},2} + \|A^*\|^2_{\mathbf{ber},2})\Big\}^{\frac{1}{2}}$$

as required. This completes the proof. □

THEOREM 5. *Let $A \in \mathscr{L}(\mathscr{H})$. Then*

$$\mathbf{ber}(A^*A \otimes AA^*) \leqslant \frac{1}{4}\mathbf{ber}\left((A^*A)^2 + (AA^*)^2\right) + \frac{1}{2}\mathbf{ber}\left(A^*A^2A^*\right). \tag{7}$$

Proof. Let $\tau \in \Theta$ be arbitrary. It follows from Lemma 4 that

$$\begin{aligned}
\widetilde{A^*A \otimes AA^*}(\tau) &= \left\langle (A^*A \otimes AA^*)\hat{k}_\tau \otimes \hat{k}_\tau, \hat{k}_\tau \otimes \hat{k}_\tau \right\rangle \\
&= \left\langle A^*A\hat{k}_\tau \otimes AA^*\hat{k}_\tau, \hat{k}_\tau \otimes \hat{k}_\tau \right\rangle \\
&= \left\langle A^*A\hat{k}_\tau, \hat{k}_\tau \right\rangle \left\langle AA^*\hat{k}_\tau, \hat{k}_\tau \right\rangle \\
&\leqslant \frac{1}{2}\left[\left\|A^*A\hat{k}_\tau\right\| \left\|AA^*\hat{k}_\tau\right\| + \left|\left\langle AA^*\hat{k}_\tau, AA^*\hat{k}_\tau \right\rangle\right|\right] \\
&\leqslant \frac{1}{4}\left(\left\|A^*A\hat{k}_\tau\right\|^2 + \left\|AA^*\hat{k}_\tau\right\|^2\right) + \frac{1}{2}\left|\left\langle A^*A^2A^*\hat{k}_\tau, \hat{k}_\tau \right\rangle\right| \\
&= \frac{1}{4}\left\langle \left((A^*A)^2 + (AA^*)^2\right)\hat{k}_\tau, \hat{k}_\tau \right\rangle + \frac{1}{2}\left|\left\langle A^*A^2A^*\hat{k}_\tau, \hat{k}_\tau \right\rangle\right| \\
&\leqslant \frac{1}{4}\mathbf{ber}\left((A^*A)^2 + (AA^*)^2\right) + \frac{1}{2}\mathbf{ber}\left(A^*A^2A^*\right).
\end{aligned}$$

Taking the supremum over $\tau \in \Omega$, we arrive at the required inequality (7). The proof is completed. □

In order to state our next results, we need the following lemma, see [24].

LEMMA 6. *Let $x, y, e \in \mathbb{H}$ with $\|e\| = 1$. Then*

$$|\langle x, e\rangle\langle e, y\rangle| \leqslant \frac{1}{2}\sqrt{3\|x\|^2\|y\|^2 + \|x\|\|y\||\langle x, y\rangle|}.$$

THEOREM 6. *Let $A, B \in \mathscr{L}(\mathscr{H})$. Then*

$$\mathbf{ber}^2(A \otimes B) \leqslant \frac{3}{8}\mathbf{ber}\left(|A|^4 + |B|^4\right) + \frac{1}{8}\mathbf{ber}\left(|A|^2 + |B|^2\right)\mathbf{ber}(BA). \tag{8}$$

Proof. Let $\tau \in \Theta$ be arbitrary. By replacing $e = \hat{k}_\tau$, $x = A\hat{k}_\tau$, and $y = A^*\hat{k}_\tau$ in Lemma 6, we obtain

$$\begin{aligned}
&\left|\widetilde{A \otimes B}(\tau)\right|^2 \\
&= \left|\langle A\hat{k}_\tau, \hat{k}_\tau\rangle \langle B\hat{k}_\tau, \hat{k}_\tau\rangle\right|^2 \\
&\leqslant \frac{3}{4}\left\|A\hat{k}_\tau\right\|^2 \left\|B^*\hat{k}_\tau\right\|^2 + \frac{1}{4}\left\|A\hat{k}_\tau\right\| \left\|B^*\hat{k}_\tau\right\| \left|\langle BA\hat{k}_\tau, \hat{k}_\tau\rangle\right| \\
&= \frac{3}{4}\left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau\right\rangle \left\langle |B^*|^2 \hat{k}_\tau, \hat{k}_\tau\right\rangle + \frac{1}{4}\sqrt{\left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau\right\rangle \left\langle |B^*|^2 \hat{k}_\tau, \hat{k}_\tau\right\rangle} \left|\langle BA\hat{k}_\tau, \hat{k}_\tau\rangle\right| \\
&\leqslant \frac{3}{8}\left(\left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau\right\rangle^2 + \left\langle |B^*|^2 \hat{k}_\tau, \hat{k}_\tau\right\rangle^2\right) \\
&\quad + \frac{1}{8}\left(\left\langle |A|^2 \hat{k}_\tau, \hat{k}_\tau\right\rangle + \left\langle |B^*|^2 \hat{k}_\tau, \hat{k}_\tau\right\rangle\right) \left|\langle BA\hat{k}_\tau, \hat{k}_\tau\rangle\right| \\
&\leqslant \frac{3}{8}\left\langle \left(|A|^4 + |B^*|^4\right) \hat{k}_\tau, \hat{k}_\tau\right\rangle + \frac{1}{8}\left\langle \left(|A|^4 + |B^*|^4\right) \hat{k}_\tau, \hat{k}_\tau\right\rangle \left|\langle BA\hat{k}_\tau, \hat{k}_\tau\rangle\right|.
\end{aligned}$$

This clearly implies inequality (8). □

COROLLARY 3. *Let $A, B \in \mathscr{L}(\mathscr{H})$. Then*

$$\mathbf{ber}^4(B^*A) \leqslant \frac{3}{8}\mathbf{ber}\left(|A|^8 + |B|^8\right) + \frac{1}{8}\mathbf{ber}\left(|A|^4 + |B|^4\right)\mathbf{ber}(|B|^2|A|^2).$$

Proof. Replacing A by $|A|^2$ and B by $|B|^2$ in the proof of Theorem 8, respectively, we obtain

$$\begin{aligned}
\left(\langle |A|^2\hat{k}_\tau, \hat{k}_\tau\rangle \langle |B|^2\hat{k}_\tau, \hat{k}_\tau\rangle\right)^2 \leqslant\ & \frac{3}{8}\left\langle \left(|A|^8 + |B|^8\right) \hat{k}_\tau, \hat{k}_\tau\right\rangle \\
& + \frac{1}{8}\left\langle \left(|A|^4 + |B|^4\right) \hat{k}_\tau, \hat{k}_\tau\right\rangle \left|\langle |B|^2|A|^2\hat{k}_\tau, \hat{k}_\tau\rangle\right|. \qquad (9)
\end{aligned}$$

On the other hand, it follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}
\left|\langle B^*A\hat{k}_\tau, \hat{k}_\tau\rangle\right|^4 &= \left|\langle A\hat{k}_\tau, B\hat{k}_\tau\rangle\right|^4 \\
&\leqslant \|A\hat{k}_\tau\|^4 \|B\hat{k}_\tau\|^4 \\
&= \left(\left|\langle A\hat{k}_\tau, A\hat{k}_\tau\rangle\right| \left|\langle B\hat{k}_\tau, B\hat{k}_\tau\rangle\right|\right)^2 \\
&= \left(\left|\langle |A|^2\hat{k}_\tau, \hat{k}_\tau\rangle\right| \left|\langle |B|^2\hat{k}_\tau, \hat{k}_\tau\rangle\right|\right)^2. \qquad (10)
\end{aligned}$$

Combining (9) and (10), we have

$$\begin{aligned}
\left|\langle B^*A\hat{k}_\tau, \hat{k}_\tau\rangle\right|^4 \leqslant\ & \frac{3}{8}\left\langle \left(|A|^8 + |B|^8\right) \hat{k}_\tau, \hat{k}_\tau\right\rangle \\
& + \frac{1}{8}\left\langle \left(|A|^4 + |B|^4\right) \hat{k}_\tau, \hat{k}_\tau\right\rangle \left|\langle |B|^2|A|^2\hat{k}_\tau, \hat{k}_\tau\rangle\right|.
\end{aligned}$$

Taking the supremum over $\tau \in \Omega$ in the latter inequality, we have that

$$\mathbf{ber}^4(B^*A) \leqslant \frac{3}{8}\mathbf{ber}\left(|A|^8+|B|^8\right)+\frac{1}{8}\mathbf{ber}\left(|A|^4+|B|^4\right)\mathbf{ber}(|B|^2|A|^2).$$

This proves the corollary. □

The following results is proven in [10, Theorem 3.1].

THEOREM 7. *Let $A \in \mathscr{L}(\mathscr{H})$. Then*

$$\mathbf{ber}(A) \leqslant \left(\|A\|_{\mathbf{ber},2} - \inf_{\tau\in\Omega}\left\|(A-\widetilde{A}(\tau))\hat{k}_\tau\right\|^2\right)^{\frac{1}{2}}.$$

In the following proposition, we refine the general inequality

$$\mathbf{ber}(A\otimes B) \leqslant \|A\|_{\mathbf{ber},2}\|B\|_{\mathbf{ber},2}.$$

PROPOSITION 6. Let $A,B \in \mathscr{L}(\mathscr{H})$. Then

$$\mathbf{ber}(A\otimes B) \leqslant \left(\|A\|_{\mathbf{ber},2} - \inf_{\tau\in\Omega}\left\|(A-\widetilde{A}(\tau))\hat{k}_\tau\right\|^2\right)^{\frac{1}{2}}\left(\|B\|_{\mathbf{ber},2} - \inf_{\tau\in\Omega}\left\|(B-\widetilde{B}(\tau))\hat{k}_\tau\right\|^2\right)^{\frac{1}{2}}.$$

Proof. Since $\mathbf{ber}(A\otimes B) \leqslant \mathbf{ber}(A)\mathbf{ber}(B)$, the proof is immediate from Theorem 7. □

For the related results see [10], [13] and [24].

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