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XY–CONVEX FREE POLYNOMIALS REVISITED

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Abstract. In this article, by using the matrix-valued analog of a factorization property of free polynomials, we offer an alternate approach to the structure of matrix-valued hermitian free polynomials that are xy-convex.

1. Introduction

The purpose of this article is twofold. Firstly, to observe that the factorization property proved in [7, Theorem 3.3] extends naturally to the matrix-valued setting. Secondly, as an application, to present an alternate and conceptually different proof of a characterization of matrix-valued xy-convex hermitian free polynomials given in [1, Theorem 1.2], by following a conceptually similar plan to the proof of the scalar-valued version in [7, Theorem 1.4]. Further, it is also pointed out heuristically why the structure of xy-convex free polynomials, i.e., [1, Theorem 1.2] might fail to imply the factorization property in [7, Theorem 3.3].

We begin by recalling some definitions. For simplicity and for the convenience of the reader, we adopt many of the same notations used in [7] and [1].

Let $\chi_1,\dots,\chi_k$ be freely noncommuting variables. Given a word

$$w = \chi_{i_1}\cdots\chi_{i_\ell} \tag{1.1}$$

in these variables and $T \in \mathbb{S}_n(\mathbb{C}^k)$, let

$$w(T) = T^w = T_{i_1}\cdots T_{i_\ell}.$$

Let $\mathscr{W}$ denote the collection of words in the variables χ. A $d\times d$ *matrix-valued free polynomial* is an expression of the form,

$$p(\chi) = \sum_{w\in\mathscr{W}} p_w w,$$

where the sum is finite and the $p_w \in M_d(\mathbb{C})$. The free polynomial p is naturally *evaluated* at $T \in \mathbb{S}_n(\mathbb{C}^k)$ as

$$p(T) = \sum p_w T^w.$$

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There is a natural *involution* * on free polynomials that reverses the order of products in words so that, for w in equation (1.1),

$$w^* = \chi_{i_\ell} \cdots \chi_{i_1};$$

and such that

$$p^* = \sum p_w^* w^*.$$

This involution is compatible with the adjoint operation on matrices,

$$p(T)^* = p^*(T).$$

A free polynomial p is *hermitian* if $p^* = p$; equivalently, if $p(T)^* = p(T)$ for all n and $T \in \mathbb{S}_n(\mathbb{C}^k)$.

From here on we often omit the adjectives matrix and free and simply refer to matrix-valued free polynomials as polynomials, particularly when there is no possibility of confusion.

Since the involution fixes the variables, $\chi_j^* = \chi_j$, we refer to $\chi_1, \ldots, \chi_k$ as *hermitian variables*.

Given $m \times n$ matrices $A_0, A_1, \ldots, A_{\mathtt{g}}, B_1, \ldots, B_{\mathtt{h}}, C_{pq}, 1 \leqslant p \leqslant \mathtt{g}, 1 \leqslant q \leqslant \mathtt{h}$, the expression

$$L(x,y) = A_0 - \sum_{j=1}^{\mathtt{g}} A_j x_j - \sum_{k=1}^{\mathtt{h}} B_k y_k - \sum_{p,q=1}^{\mathtt{g},\mathtt{h}} C_{pq} x_p y_q,$$

is called an $m \times n$ *matrix-valued* xy*-pencil*. When all the coefficient matrices are hermitian, then L is called a *hermitian* xy*-pencil*.

1.1. Factorization

Given a pair of block 2×2 matrices $A = (A_{i,j})$ and $B = (B_{i,j})$ define

$$A \circledast B = \left(A_{i,j} \otimes B_{i,j}\right).$$

Thus $A \circledast B$ is a mix of Schur product $(*)$ and tensor product $(\otimes)$. It is known as the *Khatri-Rao product*.

Let, for $j = 1, 2 \ldots, 2\mu$,

$$s_j = \begin{pmatrix} s_{j,0} & s_{j,1} \\ s_{j,1}^* & s_{j,2} \end{pmatrix},$$

where $\{s_{j,k} : 1 \leqslant j \leqslant 2\mu, 0 \leqslant k \leqslant 2\}$ are freely noncommuting variables with $s_{j,0}$ and $s_{j,2}$ being hermitian. For notational purposes, let

$$s_0 = \begin{pmatrix} \emptyset & 0 \\ 0 & \emptyset \end{pmatrix}.$$

where $\emptyset$ denotes the empty word.

Suppose $p = \sum_{j,k=0}^{2\mu} p_{j,k} x_j x_k$, is a $2\ell \times 2\ell$ hermitian matrix polynomial of degree at most two in 2μ hermitian freely noncommuting variables $x_1, \ldots, x_{2\mu}$, where, for notational purposes $x_0 = \emptyset$. In particular, each $p_{j,k} \in M_\ell(\mathbb{C}) \otimes M_2(\mathbb{C})$ and $p_{j,k}^* = p_{k,j}$. Let $\mathscr{E}p$ denote the matrix polynomial in the variables $\{s_{j,0}, s_{j,1}, s_{j,2} : 1 \leqslant j \leqslant 2\mu\}$ defined by

$$\mathscr{E}p(s) = \sum_{j,k=0}^{2\mu} p_{j,k} \circledast s_j s_k.$$

Such a polynomial is naturally evaluated at a 2μ-tuple $S = (S_1, \ldots, S_{2\mu})$ of block 2×2 hermitian matrices,

$$S_j = \begin{pmatrix} S_{j,0} & S_{j,1} \\ S_{j,1}^* & S_{j,2} \end{pmatrix} \in \mathbb{S}_{n+m}(\mathbb{C}^{2\mu}) \subseteq M_{n+m}(\mathbb{C}) = \begin{pmatrix} M_n(\mathbb{C}) & M_{n,m}(\mathbb{C}) \\ M_{m,n}(\mathbb{C}) & M_m(\mathbb{C}) \end{pmatrix}, \tag{1.2}$$

using $\circledast$ as

$$\mathscr{E}p(S) = \sum_{j,k=0}^{2\mu} p_{j,k} \circledast S_j S_k \in M_\ell(\mathbb{C}) \otimes \begin{pmatrix} M_n(\mathbb{C}) & M_{n,m}(\mathbb{C}) \\ M_{m,n}(\mathbb{C}) & M_m(\mathbb{C}) \end{pmatrix}.$$

The first observation in this article is that the following matrix-valued analog of the factorization property proved in [7, Theorem 3.3] holds.

THEOREM 1.1. *Suppose* $\rho = \sum_{j,k=0}^{2\mu} \rho_{j,k} x_j x_k$ *is a hermitian* $2d\mu \times 2d\mu$ *polynomial, where*

$$\rho_{j,k} = \begin{pmatrix} (\rho_{j,k})_{1,1} & (\rho_{j,k})_{1,2} \\ (\rho_{j,k})_{2,1} & (\rho_{j,k})_{2,2} \end{pmatrix},$$

with $(\rho_{j,k})_{a,b} \in M_d(\mathbb{C}) \otimes M_\mu(\mathbb{C})$ *for all* $a, b \in \{1,2\}$.

If $\mathscr{E}\rho(S) \succeq 0$ *for all positive integers* m, n *and* $S \in \mathbb{S}_{n+m}(\mathbb{C}^{2\mu})$, *then there exists an* $N \leqslant 2(2\mu+1)(2d\mu)$ *and* $q_0, q_1, \ldots, q_{2\mu} \in M_{N,d\mu}(\mathbb{C}) \otimes M_{1,2}(\mathbb{C})$ *such that*

$$\begin{aligned} q_j^* q_k &= \rho_{j,k}, \quad 1 \leqslant j,k \leqslant 2\mu, \\ q_0^* q_k + q_k^* q_0 &= \rho_{k,0} + \rho_{0,k}, \quad 1 \leqslant k \leqslant 2\mu \\ (q_0^* q_0)_{a,a} &= (\rho_{0,0})_{a,a} \in M_d(\mathbb{C}) \otimes M_\mu(\mathbb{C}), \quad a = 1,2. \end{aligned} \tag{1.3}$$

In particular, letting q *denote the* $N \times 2d\mu$ *matrix polynomial* $q = \sum_{j=0}^{2\mu} q_j x_j$, *there is an* $r_1 \in M_d(\mathbb{C}) \otimes M_\mu(\mathbb{C})$ *such that*

$$\rho = q^* q + r, \quad \text{where} \quad r = \begin{pmatrix} 0 & r_1 \\ r_1^* & 0 \end{pmatrix}.$$

1.2. Convexity

The two notions of convexity considered in this article are described for free polynomials. They involve partitioning the freely noncommuting variables into two classes $x_1, \ldots, x_\mu$ and $y_1, \ldots, y_\mu$.

1.2.1. Partial convexity

A $d \times d$ matrix-valued hermitian polynomial $p(x,y)$ is *convex in* y if for each positive integer n, each $X \in \mathbb{S}_n(\mathbb{C}^\mu)$, each $R,S \in \mathbb{S}_n(\mathbb{C}^\mu)$ and each $0 < t < 1$, one has

$$p(X,tR+(1-t)S) \preceq tp(X,R)+(1-t)p(X,S).$$

Partial convexity in the x-variables is defined analogously. A canonical example of a convex in y polynomial is a hermitian polynomial that is affine linear in y. For more details and results on (partial) convexity of free polynomials, please see [2], [3], [4], [5] and [6].

The following alternate characterization of convexity in y can be found in [9], [10] and [1]. A tuple $((X,Y),V)$, where $(X,Y) \in \mathbb{S}_n(\mathbb{C}^\mu) \times \mathbb{S}_n(\mathbb{C}^\mu)$ and $V : \mathbb{C}^m \to \mathbb{C}^n$ is an isometry, is an x^2*-pair* if $V^*X_i^2V = (V^*X_iV)^2$ for each $1 \leqslant i \leqslant \mu$. Equivalently $((X,Y),V)$ is an x^2-pair if $\operatorname{ran} V$ reduces X. A result from [9], [10] and [1] is that a hermitian polynomial p is convex in y, or x^2*-convex*, if and only if

$$p(V^*(X,Y)V) \preceq (I_d \otimes V^*)p(X,Y)(I_d \otimes V)$$

for all x^2-pairs $((X,Y),V)$.

1.2.2. xy-convexity

A tuple $((X,Y),V)$, where $(X,Y) \in \mathbb{S}_n(\mathbb{C}^\mu) \times \mathbb{S}_n(\mathbb{C}^\mu)$ and $V : \mathbb{C}^m \to \mathbb{C}^n$ is an isometry, such that $V^*(X_iY_j)V = V^*X_iVV^*Y_jV$, for all i,j, is an xy*-pair*. A hermitian matrix-valued free polynomial $p(x,y)$ is xy*-convex* if

$$p(V^*(X,Y)V) \preceq (I_d \otimes V)^*p(X,Y)(I_d \otimes V)$$

for all xy-pairs $((X,Y),V)$.

A main result in [1, Theorem 1.2] states that a hermitian $d \times d$ matrix-valued free polynomial $p(x,y)$ is xy-convex if and only if $p(x,y)$ is separately partially convex, i.e., partially convex in both x as well as y.

The main contribution in this article is an application of Theorem 1.1, which is an alternate and conceptually different proof of the following result.

THEOREM 1.2. [1, Theorem 1.2] *Suppose that $p(x,y)$ is a hermitian $d \times d$ matrix-valued free polynomial. The following statements are equivalent.*

(i) p is xy-convex.

(ii) There exists a hermitian $d \times d$ matrix-valued xy-pencil λ, a positive integer N and an $N \times d$ matrix-valued xy-pencil Λ such that

$$p(x,y) = \lambda(x,y) + \Lambda(x,y)^*\Lambda(x,y).$$

When $d = \mu = 1$, Theorem 1.2 reduces to [7, Theorem 1.4].

2. The proofs

This section contains the proofs of Theorem 1.1 and Theorem 1.2 in Subsections 2.2 and 2.4 respectively. To a point, the proofs parallel those of [7, Theorem 3.3] and [7, Theorem 1.4]. In subsections 2.1 & 2.2, and 2.3 the factorization result [7, Theorem 3.3] is extended to the matrix case in any number of variables; and xy-convexity of a polynomial is translated into positivity information on a type of Hessian. While more involved, these tasks in principle follow the same lines as in [7]. In Subsection 2.4 the positivity of the Hessian feeds into the factorization result and it is at this point – reading off the conclusion of Theorem 1.2 from the factorization – that the ad-hoc approach in [7] is replaced by a more conceptual argument. We begin with the following observations, which are matrix-valued analogs of their counterparts from [7].

LEMMA 2.1. *Let* $V^* = \begin{pmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \end{pmatrix} \in M_{2n,2(n+m)}(\mathbb{C})$, $\tau = \left(\tau_{i,j}\right)_{i,j=1}^2 \in M_{d\mu}(\mathbb{C}) \otimes M_2(\mathbb{C})$, $R = \left(R_{i,j}\right)_{i,j=1}^2 \in M_{n+m}(\mathbb{C}) \otimes M_2(\mathbb{C})$, *where* $R_{i,j} = \left(R_{i,j}^{a,b}\right)_{a,b=1}^2 \in M_{n+m}(\mathbb{C})$ *and* $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \in M_{n+m}(\mathbb{C}) \otimes M_2(\mathbb{C})$.

(i) $(I_{d\mu} \otimes V)^* \left[\tau \circledast R\right] (I_{d\mu} \otimes V) = \left(\tau_{i,j} \otimes R_{i,j}^{1,1}\right)_{i,j=1}^2 = \tau \circledast \widetilde{R}$, *where* $\widetilde{R} = \left(R_{i,j}^{1,1}\right)_{i,j=1}^2$.

(ii) $(I_{d\mu} \otimes D) \left[\tau \circledast R\right] (I_{d\mu} \otimes D) = \tau \circledast (DRD)$.

Proof. To prove (i), observe that

$$\tau \circledast R = \left(\tau_{i,j} \otimes R_{i,j}\right)_{i,j=1}^2$$
$$= \left(\begin{array}{cc|cc} \tau_{1,1} \otimes R_{1,1}^{1,1} & \tau_{1,1} \otimes R_{1,1}^{1,2} & \tau_{1,2} \otimes R_{1,2}^{1,1} & \tau_{1,2} \otimes R_{1,2}^{1,2} \\ \tau_{1,1} \otimes R_{1,1}^{2,1} & \tau_{1,1} \otimes R_{1,1}^{2,2} & \tau_{1,2} \otimes R_{1,2}^{2,1} & \tau_{1,2} \otimes R_{1,2}^{2,2} \\ \hline \tau_{2,1} \otimes R_{2,1}^{1,1} & \tau_{2,1} \otimes R_{2,1}^{1,2} & \tau_{2,2} \otimes R_{2,2}^{1,1} & \tau_{2,2} \otimes R_{2,2}^{1,2} \\ \tau_{2,1} \otimes R_{2,1}^{2,1} & \tau_{2,1} \otimes R_{2,1}^{2,2} & \tau_{2,2} \otimes R_{2,2}^{2,1} & \tau_{2,2} \otimes R_{2,2}^{2,2} \end{array}\right).$$

Thus, $(I_{d\mu} \otimes V)^* \left[\tau \circledast R\right] (I_{d\mu} \otimes V) = \left(\tau_{i,j} \otimes R_{i,j}^{1,1}\right)_{i,j=1}^2 = \tau \circledast \widetilde{R}$.

To prove (ii), observe that

$$\begin{aligned} (I_{d\mu} \otimes D) \left[\tau \circledast R\right] (I_{d\mu} \otimes D) &= \begin{pmatrix} (I_{d\mu} \otimes D_1) & 0 \\ 0 & (I_{d\mu} \otimes D_2) \end{pmatrix} \\ &\quad \begin{pmatrix} \tau_{1,1} \otimes R_{1,1} & \tau_{1,2} \otimes R_{1,2} \\ \tau_{2,1} \otimes R_{2,1} & \tau_{2,2} \otimes R_{2,2} \end{pmatrix} \begin{pmatrix} (I_{d\mu} \otimes D_1) & 0 \\ 0 & (I_{d\mu} \otimes D_2) \end{pmatrix} \\ &= \begin{pmatrix} \tau_{1,1} \otimes D_1 R_{1,1} D_1 & \tau_{1,2} \otimes D_1 R_{1,2} D_2 \\ \tau_{2,1} \otimes D_2 R_{2,1} D_1 & \tau_{2,2} \otimes D_2 R_{2,2} D_2 \end{pmatrix} \\ &= \tau \circledast (DRD). \quad \square \end{aligned} \tag{2.1}$$

2.1. A completely positive map

As a general principle, factorizations correspond to completely positive maps, a theme pursued in this subsection. For the remainder of this subsection the hypotheses of Theorem 1.1 are in force.

Let $\{e_1, e_2\}$ denote the standard orthonormal basis for $\mathbb{C}^2$ and $\langle x_1, \ldots, x_{2\mu} \rangle_1$ denote the words in those letters of length at most one. i.e.,

$$\langle x_1, \ldots, x_{2\mu} \rangle_1 := \{x_0, x_1, x_2, \ldots, x_{2\mu}\}.$$

We will view $\mathbb{C}^{2\mu+1}$ as the span of $\langle x_1, \ldots, x_{2\mu} \rangle_1$ with $\langle x_1, \ldots, x_{2\mu} \rangle_1$ as an orthonormal basis and elements of $M_{2\mu+1}(\mathbb{C})$ as matrices indexed by $\langle x_1, \ldots, x_{2\mu} \rangle_1 \times \langle x_1, \ldots, x_{2\mu} \rangle_1$. Thus $x_j x_k^*$ are the matrix units for $M_{2\mu+1}(\mathbb{C})$.

Let $\mathscr{S} \subset M_2(\mathbb{C}) \otimes M_{2\mu+1}(\mathbb{C})$ denote the subspace of matrices

$$T = \left(T_{\alpha,\beta}\right)_{\alpha,\beta \in \langle x_1, \ldots, x_{2\mu} \rangle_1},$$

where T_{x_0,x_0} is a diagonal matrix and $T_{\beta,x_0} = T_{x_0,\beta}$ for $\beta \in \langle x_1, \ldots, x_{2\mu} \rangle_1$. It is easily seen that $\mathscr{S}$ is an operator system in $M_2(\mathbb{C}) \otimes M_{2\mu+1}(\mathbb{C})$. Define $\psi : \mathscr{S} \to M_{d\mu}(\mathbb{C}) \otimes M_2(\mathbb{C})$ by

$$\psi((T_{\alpha,\beta})) = \sum_{\alpha,\beta \in \langle x_1, \ldots, x_{2\mu} \rangle_1} \rho_{\alpha,\beta} \circledast T_{\alpha,\beta}. \tag{2.2}$$

PROPOSITION 2.2. *The map ψ in equation (2.2) is completely positive (cp).*

A proof of the scalar-valued version of Proposition 2.2 is given in [7, Proposition 3.4]. With suitable modifications along with Lemma 2.1, the same proof can be made to work for our matrix-valued setting as well. Hence we skip the proof.

2.2. Proof of Theorem 1.1

A proof of the scalar-valued version of the factorization result Theorem 1.1 is given in [7, Theorem 3.3]. With suitable modifications, the same proof can be adapted to the present matrix-valued setting. Hence we skip the proof. We point out that the main tool for the proof is Proposition 2.2, which guarantees that the Choi matrix of the map ψ factors as F^*F for some matrix F.

2.3. The xy-Hessian and xy-convex polynomials

In this section xy-convexity of a polynomial is reinterpreted as positivity of a Hessian. While the construction is entirely algebraic, it is motivated by the usual geometry of the second derivative. Proposition 2.3 below is the xy-convex analog of Proposition 2.1 from [1].

PROPOSITION 2.3. [7, Proposition 4.1] *A triple* $((X,Y),V)$ *is an xy-pair if and only if, up to unitary equivalence, it has the block form*

$$X_j = \begin{pmatrix} X_{0j} & A_j & 0 \\ A_j^* & * & * \\ 0 & * & * \end{pmatrix}, \quad Y_k = \begin{pmatrix} Y_{0k} & 0 & C_k \\ 0 & * & * \\ C_k^* & * & * \end{pmatrix}, \quad V = \begin{pmatrix} I & 0 & 0 \end{pmatrix}^*, \tag{2.3}$$

$1 \leqslant j,k \leqslant \mu$. *Also, a* $d \times d$ *matrix-valued hermitian free polynomial* $p(x,y))$ *is xy-convex if and only if*

$$(I_d \otimes V)^* p(X,Y)(I_d \otimes V) - p(X_0,Y_0) \succeq 0$$

for each xy-pair $((X,Y),V)$ *of the form of equation* (2.3).

Let $\mathscr{L}$ denote the set of words in x,y of degree at most two in both x and y, but excluding those of the forms $x_j x_i y_k y_m$ and $y_m y_k x_i x_j$.

LEMMA 2.4. *Suppose* $p(x,y)$ *is a* $d \times d$ *matrix-valued hermitian polynomial. If* $p(x,y)$ *is xy-convex, then* p *is convex in both* x *and* y *separately. Hence* $p \in M_d(\mathbb{C}) \otimes span(\mathscr{L})$.

Proof. To show that p is convex in x and y separately, argue as in in the proof of [7, Lemma 4.3] by simply replacing the triple (X_1,X_2,Y) used in that proof by some triple $(X^1,X^2,Y) \in \mathbb{S}_n(\mathbb{C}^\mu)$. That $p \in M_d(\mathbb{C}) \otimes span(\mathscr{L})$ follows from [1, Corollary 2.8]. $\square$

Let $\{s_{0j}, t_{0j}, \alpha_j, \beta_{k,j}, \gamma_j, \delta_{k,j} : 0 \leqslant j \leqslant \mu, 0 \leqslant k \leqslant 2\}$ denote freely noncommuting variables with $s_{0j}, t_{0k}, \beta_{0j}, \beta_{2j}, \delta_{0k}, \delta_{2k}$ being hermitian. In view of Proposition 2.3, let

$$s_j = \begin{pmatrix} s_{0j} & \begin{pmatrix} \alpha_j & 0 \end{pmatrix} \\ \begin{pmatrix} \alpha_j^* \\ 0 \end{pmatrix} & \begin{pmatrix} \beta_{0j} & \beta_{1j} \\ \beta_{1j}^* & \beta_{2j} \end{pmatrix} \end{pmatrix}, \quad t_k = \begin{pmatrix} t_{0k} & \begin{pmatrix} 0 & \gamma_k \end{pmatrix} \\ \begin{pmatrix} 0 \\ \gamma_k^* \end{pmatrix} & \begin{pmatrix} \delta_{0k} & \delta_{1k} \\ \delta_{1k}^* & \delta_{2k} \end{pmatrix} \end{pmatrix},$$

and

$$V = \begin{pmatrix} \emptyset & 0 & 0 \end{pmatrix}^*.$$

The following notations will be adopted for the remainder of the article: $s_0 = (s_{01}, \dots, s_{0\mu})$, $t_0 = (t_{01}, \dots, t_{0\mu})$, $\alpha = (\alpha_1 \dots, \alpha_\mu)$, $\gamma = (\gamma_1, \dots, \gamma_\mu)$, $\beta_1 = (\beta_{11}, \dots, \beta_{1\mu})$, $\beta_2 = (\beta_{21}, \dots, \beta_{2\mu})$, $\delta_0 = (\delta_{01}, \dots, \delta_{0\mu})$ and $\delta_1 = (\delta_{11}, \dots, \delta_{1\mu})$.

The *xy-Hessian* of the $d \times d$ matrix-valued polynomial $p(x,y)$, denoted $H^{xy}p$, is the quadratic in α, γ part of $(I_d \otimes V^*)p(s,t)(I_d \otimes V) - p(V^*(s,t)V) = V^*p(s,t)V - p(s_0,t_0)$.

In particular, for $p \in M_d(\mathbb{C}) \otimes \operatorname{span}\mathscr{L}$,

$$\begin{aligned} H^{xy}p &:= (I_d \otimes V)^* p(s,t)(I_d \otimes V) - p(V^*(s,t)V) \\ &= (I_d \otimes V)^* p(s,t)(I_d \otimes V) - p(s_0,t_0). \end{aligned}$$

LEMMA 2.5. *If* $p=\sum_{u\in\mathscr{L}} p_u u$ *with* $p_u\in M_d(\mathbb{C})$, *then* $H^{xy}p$ *is a function of the variables* $\{\alpha,\gamma,s_0,t_0,\delta_0,\delta_1,\beta_1,\beta_2\}$ *and is given by*

$$\begin{aligned}
H^{xy}p = \sum_{j,k,\ell,m=1}^{\mu} & \{[p_{x_jx_\ell}\alpha_j\alpha_\ell^* + p_{y_ky_m}\gamma_k\gamma_m^*] \\
&+ [p_{x_jy_kx_\ell}\alpha_j\delta_{0k}\alpha_\ell^* + p_{y_kx_jy_m}\gamma_k\beta_{2j}\gamma_m^* + p_{x_jy_ky_m}(s_{0j}\gamma_k\gamma_m^* + \alpha_j\delta_{1k}\gamma_m^*) \\
&+ p_{y_my_kx_j}(\gamma_m\gamma_k^* s_{0j} + \gamma_m\delta_{1k}^*\alpha_j^*) \\
&+ p_{x_jx_\ell y_k}(\alpha_j\alpha_\ell^* t_{0k} + \alpha_j\beta_{1\ell}\gamma_k^*) + p_{y_kx_\ell x_j}(t_{0k}\alpha_\ell\alpha_j^* + \gamma_k\beta_{1\ell}^*\alpha_j^*)] \\
&+ [p_{x_jy_ky_mx_\ell}(s_{0j}\gamma_k\gamma_m^* s_{0\ell} + \alpha_j\delta_{1k}\gamma_m^* s_{0\ell} + s_{0j}\gamma_k\delta_{1m}^*\alpha_\ell^* + \alpha_j(\delta_{0k}\delta_{0m} + \delta_{1k}\delta_{1m}^*)\alpha_\ell^*) \\
&+ p_{x_jy_kx_\ell y_m}(\alpha_j\delta_{0k}\alpha_\ell^* t_{0m} + \alpha_j\delta_{0k}\beta_{1\ell}\gamma_m^* + s_{0j}\gamma_k\beta_{2\ell}\gamma_m^* + \alpha_j\delta_{1k}\beta_{2\ell}\gamma_m^*) \\
&+ p_{y_mx_\ell y_kx_j}(t_{0m}\alpha_\ell\delta_{0k}\alpha_j^* + \gamma_m\beta_{1\ell}^*\delta_{0k}\alpha_j^* + \gamma_m\beta_{2\ell}\gamma_k^* s_{0j} + \gamma_m\beta_{2\ell}\delta_{1k}^*\alpha_j^*) \\
&+ p_{y_kx_jx_\ell y_m}(t_{0k}\alpha_j\alpha_\ell^* t_{0m} + \gamma_k\beta_{1j}^*\alpha_\ell^* t_{0m} + t_{0k}\alpha_j\beta_{1\ell}\gamma_m^* + \gamma_k(\beta_{1j}^*\beta_{1\ell} + \beta_{2j}\beta_{2\ell})\gamma_m^*)]\}.
\end{aligned}$$

Alternatively, with $\emptyset$ *denoting the empty word,*

$$\begin{aligned}
H^{xy}p = \sum_{a,b,g,h=1}^{\mu} & (I_d\alpha_a)\left(p_{x_ax_b}\emptyset + \sum_{r,s=1}^{\mu}\{p_{x_ay_rx_b}\delta_{0r} + p_{x_jy_ry_sx_b}(\delta_{0r}\delta_{0s} + \delta_{1r}\delta_{1s}^*)\}\right)(I_d\alpha_b^*) \\
&+ (I_d\alpha_a)\left(p_{x_ax_by_h}\emptyset + \sum_{r=1}^{\mu} p_{x_ay_rx_by_h}\delta_{0r}\right)(I_d\alpha_b^* t_{0h}) \\
&+ (I_dt_{0g}\alpha_a)\left(p_{y_gx_ax_b}\emptyset + \sum_{r=1}^{\mu} p_{y_gx_ay_rx_b}\delta_{0r}\right)(I_d\alpha_b^*) + (I_dt_{0g}\alpha_a)\left(p_{y_gx_ax_by_h}\emptyset\right)(I_d\alpha_b^* t_{0h}) \\
&+ (I_d\alpha_a)\left(\sum_{r=1}^{\mu} p_{x_ay_ry_b}\delta_{1r} + p_{x_ax_ry_b}\beta_{1r} + p_{x_ay_rx_sy_b}(\delta_{0r}\beta_{1s} + \delta_{1r}\beta_{2s})\right)(I_d\gamma_b^*) \\
&+ (I_d\gamma_a)\left(\sum_{r=1}^{\mu} p_{y_ay_rx_b}\delta_{1r}^* + p_{y_ax_rx_b}\beta_{1r}^* + p_{y_ax_ry_sx_b}(\beta_{1r}^*\delta_{0s} + \beta_{2r}\delta_{1s})\right)(I_d\alpha_b^*) \\
&+ (I_d\alpha_a)\left(\sum_{r=1}^{\mu} p_{x_ay_ry_bx_h}\delta_{1r}\right)(I_d\gamma_b^* s_{0h}) + (I_ds_{0g}\gamma_a)\left(\sum_{r=1}^{\mu} p_{x_gy_ay_rx_b}\delta_{1r}^*\right)(I_d\alpha_b^*) \\
&+ (I_dt_{0g}\alpha_a)\left(\sum_{r=1}^{\mu} p_{y_gx_ax_ry_b}\beta_{1r}\right)(I_d\gamma_b^*) + (I_d\gamma_a)\left(\sum_{r=1}^{\mu} p_{y_ax_rx_by_h}\beta_{1r}^*\right)(I_d\alpha_b^* t_{0h}) \\
&+ (I_d\gamma_a)\left(p_{y_ay_b}\emptyset + \sum_{r=1}^{\mu} p_{y_ax_ry_b}\beta_{2r} + p_{y_ax_rx_sy_b}(\beta_{1r}^*\beta_{1s} + \beta_{2r}\beta_{2s})\right)(I_d\gamma_b^*)
\end{aligned}$$

$$+(I_d\gamma_a)\left(p_{y_ay_bx_h}\emptyset+\sum_{r=1}^{\mu}p_{y_ax_ry_bx_h}\beta_{2r}\right)I_d\gamma_b^*s_{0h}$$
$$+(I_ds_{0g}\gamma_a)\left(p_{x_gy_ay_b}\emptyset+\sum_{r=1}^{\mu}p_{x_gy_ax_ry_b}\beta_{2r}\right)(I_d\gamma_b^*)+(I_ds_{0g}\gamma_a)\left(p_{x_gy_ay_bx_h}\emptyset\right)(I_d\gamma_b^*s_{0h}).$$

Proof. Follows from direct computation. □

The *xy-border vector* $\mathscr{V}=\mathscr{V}(s_0,t_0,\alpha,\gamma)$ is the row vector-valued free polynomial $\mathscr{V}=(\mathscr{V}_1\quad \mathscr{V}_2\quad \cdots\quad \mathscr{V}_\mu)$, where

$$\mathscr{V}_a=\left(I_d\,\alpha_a\;I_d\,t_{01}\alpha_a\;\cdots\;I_d\,t_{0\mu}\alpha_a\;I_d\,\gamma_a\;I_d\,s_{01}\gamma_a\;\cdots\;I_d\,s_{0\mu}\gamma_a),\right)$$

$1\leqslant a\leqslant\mu$.

For $1\leqslant a,b\leqslant\mu$, let $\mathscr{M}_{a,b}(\beta_1,\beta_2,\delta_0,\delta_1)$ denote the matrix polynomial given by

$$(\mathscr{M}_{a,b})_{1,1}=\left(\begin{array}{c|c} \begin{array}{l} p_{x_ax_b}\emptyset+\displaystyle\sum_{r,s=1}^{\mu}\{p_{x_ay_rx_b}\delta_{0r}\\ +p_{x_jy_ry_sx_b}(\delta_{0r}\delta_{0s}+\delta_{1r}\delta_{1s}^*)\}\end{array} & \left(\left(\begin{array}{l}p_{x_ax_by_h}\emptyset\\ +\displaystyle\sum_{r=1}^{\mu}p_{x_ay_rx_by_h}\delta_{0r}\end{array}\right)_{1,h}\right)_{h=1}^{\mu} \\ \hline \left(\left(\begin{array}{l}p_{y_gx_ax_b}\emptyset\\ +\displaystyle\sum_{r=1}^{\mu}p_{y_gx_ay_rx_b}\delta_{0r}\end{array}\right)_{g,1}\right)_{g=1}^{\mu} & \left(p_{y_gx_ax_by_h}\emptyset\right)_{g,h=1}^{\mu}\end{array}\right)$$

$$(\mathscr{M}_{a,b})_{1,2}=\left(\begin{array}{c|c} \begin{array}{l} \displaystyle\sum_{r,s=1}^{\mu}[p_{x_ay_ry_b}\delta_{1r}\\ +p_{x_ax_ry_b}\beta_{1r}\\ +p_{x_ay_rx_sy_b}(\delta_{0r}\beta_{1s}+\delta_{1r}\beta_{2s})]\end{array} & \left(\left(\displaystyle\sum_{r=1}^{\mu}p_{x_ay_ry_bx_h}\delta_{1r}\right)_{1,h}\right)_{h=1}^{\mu} \\ \hline \left(\left(\displaystyle\sum_{r=1}^{\mu}p_{y_gx_ax_ry_b}\beta_{1r}\right)_{g,1}\right)_{g=1}^{\mu} & 0_{\mu\times\mu}\end{array}\right)$$

$$(\mathscr{M}_{a,b})_{2,1}=\left(\begin{array}{c|c} \begin{array}{l} \displaystyle\sum_{r,s=1}^{\mu}[p_{y_ay_rx_b}\delta_{1r}^*\\ +p_{y_ax_rx_b}\beta_{1r}^*\\ +p_{y_ax_ry_sx_b}(\beta_{1r}^*\delta_{0s}+\beta_{2r}\delta_{1s})]\end{array} & \left(\left(\displaystyle\sum_{r=1}^{\mu}p_{y_ax_rx_by_h}\beta_{1r}^*\right)_{1,h}\right)_{h=1}^{\mu} \\ \hline \left(\left(\displaystyle\sum_{r=1}^{\mu}p_{x_gy_ay_rx_b}\delta_{1r}^*\right)_{g,1}\right)_{g=1}^{\mu} & 0_{\mu\times\mu}\end{array}\right)$$

and

$$(\mathscr{M}_{a,b})_{2,2} = \left(\begin{array}{c|c} \begin{array}{c} p_{y_ay_b}\emptyset + \sum\limits_{r,s=1}^{\mu} p_{y_ax_ry_b}\beta_{2r} \\ + p_{y_ax_rx_sy_b}(\beta_{1r}^*\beta_{1s} + \beta_{2r}\beta_{2s}) \end{array} & \left(\left(\begin{array}{c} p_{y_ay_bx_h}\emptyset \\ + \sum\limits_{r=1}^{\mu} p_{y_ax_ry_bx_h}\beta_{2r} \end{array}\right)_{1,h}\right)_{h=1}^{\mu} \\ \hline \left(\left(\begin{array}{c} p_{x_gy_ay_b}\emptyset \\ + \sum\limits_{r=1}^{\mu} p_{x_gy_ax_ry_b}\beta_{2r} \end{array}\right)_{g,1}\right)_{g=1}^{\mu} & \left(p_{x_gy_ay_bx_h}\emptyset\right)_{g,h=1}^{\mu} \end{array}\right),$$

where $1 \leqslant g,h,a,b \leqslant \mu$. In particular, $\mathscr{M}_{a,b}$ is a block 2×2 matrix with each block entry being a $(\mu+1)\times(\mu+1)$ matrix of $d \times d$ matrix-valued free polynomials. The matrix

$$\mathscr{M} = (\mathscr{M}_{a,b})_{a,b=1}^{\mu},$$

is called the xy-*middle matrix* for p.

LEMMA 2.6. *If $p(x,y)$ is an xy-convex polynomial, then*

$$H^{xy}p = \mathscr{V}\mathscr{M}\mathscr{V}^*.$$

Proof. Since p is xy-convex, Lemma 2.4 implies $p \in M_d(\mathbb{C}) \otimes \mathscr{L}$. From here, the result follows from a direct computation by combining the definitions of $\mathscr{M}$ and $\mathscr{V}$ above with the definition of $H^{xy}p$. □

PROPOSITION 2.7. *If $p(x,y)$ is xy-convex, then*

$$\mathscr{M}(B_1,B_2,D_0,D_1) \succeq 0$$

for all μ-tuples B_1,B_2,D_0,D_1 of matrices of compatible sizes.

Proof. Since p is xy-convex, it follows from Proposition 2.3 that $H^{xy}p$, the xy-Hessian, takes positive semidefinite values. Given $M,N \in \mathbb{N}$, and

(a) matrices $D_{01},\dots,D_{0\mu} \in M_M(\mathbb{C})$, matrices $B_{21},\dots,B_{2\mu} \in M_N(\mathbb{C})$, and matrices $B_{11},\dots,B_{1\mu},D_{11}, \dots,D_{1\mu} \in M_{M,N}(\mathbb{C})$;

(b) vectors $w_{k,j}^{\ell} \in \mathbb{C}^M$ and $v_{k,j}^{\ell} \in \mathbb{C}^N$ for $0 \leqslant \ell \leqslant \mu$, $1 \leqslant j \leqslant d$ and $1 \leqslant k \leqslant \mu$,

let $g = \oplus_{k=1}^{\mu} g_k$, where $g_k \in \mathbb{C}^{\mu+1} \otimes ((\mathbb{C}^d \otimes \mathbb{C}^M) \oplus (\mathbb{C}^d \otimes \mathbb{C}^N))$ is given by

$$g_k = (\sum_{u=0}^{\mu}\sum_{j=1}^{d} \mathsf{e}_u \otimes e_j \otimes w_{k,j}^u) \oplus (\sum_{u=0}^{\mu}\sum_{j=1}^{d} \mathsf{e}_u \otimes e_j \otimes v_{k,j}^u),$$

and where $\{\mathtt{e}_0,\mathtt{e}_1,\dots,\mathtt{e}_\mu\}$ and $\{e_1,\dots,e_d\}$ denote the standard orthonormal basis for $\mathbb{C}^{\mu+1}$ and $\mathbb{C}^d$ respectively.

Using [1, proposition 2.5], choose a non-zero vector $h\in\mathbb{C}^{\mu+1}$ and matrices $U_{01},\dots,U_{0\mu}$ and $W_{01},\dots,W_{0\mu}$ from $\mathbb{S}_{\mu+1}(\mathbb{C})$ such that the sets $\{h,U_{01}h,\dots,U_{0\mu}h\}$ and $\{h,W_{01}h,\dots,W_{0\mu}h\}$ are bases for $\mathbb{C}^{\mu+1}$. Set $W_{00}=U_{00}=I_{\mu+1}$. Using linear independence, choose, for each $k\in\{1,2,\dots,\mu\}$, matrices $A_{k,j}\in M_{\mu+1,M}(\mathbb{C})$ and $C_{k,j}\in M_{\mu+1,N}(\mathbb{C})$ such that $A^*_{k,j}W_{0\ell}h=w^\ell_{k,j}$ and $C^*_{k,j}U_{0\ell}h=v^\ell_{k,j}$. Define $A^*_k=(A^*_{k,1}\ A^*_{k,2}\ \dots\ A^*_{k,d})$, and $C^*_k=\begin{pmatrix}C^*_{k,1} & C^*_{k,2} & \dots & C^*_{k,d}\end{pmatrix}$. Thus $A^*_k\in M_{M,d(\mu+1)}(\mathbb{C})$ and $C^*_k\in M_{N,d(\mu+1)}(\mathbb{C})$. Let $X_{0k}=U_{0k}\oplus\cdots\oplus U_{0k}\in M_{d(\mu+1)}(\mathbb{C})$ and $Y_{0k}=W_{0k}\oplus\cdots\oplus W_{0k}\in M_{d(\mu+1)}(\mathbb{C})$.

Let $X_0=(X_{01},\dots,X_{0\mu})$, let $Y_0=(Y_{01},\dots,Y_{0\mu})$ and observe that the tuples $X_0,Y_0\in\mathbb{S}_{d(\mu+1)}(\mathbb{C}^\mu)$. Let A and C denote the tuples $(A_1,\dots,A_\mu)$, and $(C_1,\dots,C_\mu)$ respectively. Recall the definition of the xy-border vector $\mathscr{V}$. By Lemma 2.6 it follows that,

$$\begin{aligned}\langle\mathscr{M}(B_1,B_2,D_0,D_1)g,g\rangle &= [\mathscr{V}(X_0,Y_0,A,C)h]^*\mathscr{M}(B_1,B_2,D_0,D_1)[\mathscr{V}(X_0,Y_0,A,C)h]\\ &= \langle H^{xy}p(X_0,Y_0,A,C,B_1,B_2,D_0,D_1)h,h\rangle\succeq 0,\end{aligned}$$

and the proof is complete. □

2.4. Proof of Theorem 1.2

Theorem 1.1 and Proposition 2.7 are combined in this section to complete the proof of Theorem 1.2. A simple and direct proof of the implication $(ii)\Rightarrow(i)$ of Theorem 1.2 is given in [1, Proposition 1.3].

Proof of $(i)\Rightarrow(ii)$. Let

$$\sigma=\left(\begin{pmatrix}\beta_{01} & \beta_{11}\\ \beta^*_{11} & \beta_{21}\end{pmatrix},\dots,\begin{pmatrix}\beta_{0\mu} & \beta_{1\mu}\\ \beta^*_{1\mu} & \beta_{2\mu}\end{pmatrix},\begin{pmatrix}\delta_{01} & \delta_{11}\\ \delta^*_{11} & \delta_{21}\end{pmatrix},\dots,\begin{pmatrix}\delta_{0\mu} & \delta_{1\mu}\\ \delta^*_{1\mu} & \delta_{2\mu}\end{pmatrix}\right).$$

Let $1\leqslant a,b\leqslant\mu$ and

$$Q^{1}=Q(\sigma)=\left(Q_{i,j}\right)^2_{i,j=1}$$

denote the $2d\mu\times 2d\mu$ matrix polynomial whose block entries are given by $Q_{i,j}=((Q_{i,j})_{a,b})^\mu_{a,b=1}$ where $(Q_{i,j})_{a,b}$ is the $(1,1)$ entry of the matrix $(\mathscr{M}_{a,b})_{i,j}$ and $\mathscr{M}$ is the xy-middle matrix for p. Thus Q equals

$$\left(\begin{array}{c|c}\begin{pmatrix}p_{x_ax_b}\emptyset+\displaystyle\sum_{r,s=1}^{\mu}\{p_{x_ay_rx_b}\delta_{0r}\\ +p_{x_jy_ry_sx_b}(\delta_{0r}\delta_{0s}+\delta_{1r}\delta^*_{1s})\}\end{pmatrix}^{\mu}_{a,b=1} & \begin{pmatrix}\displaystyle\sum_{r,s=1}^{\mu}p_{x_ay_ry_b}\delta_{1r}+p_{x_ax_ry_b}\beta_{1r}\\ +p_{x_ay_rx_sy_b}(\delta_{0r}\beta_{1s}+\delta_{1r}\beta_{2s})\end{pmatrix}^{\mu}_{a,b=1}\\ \hline \begin{pmatrix}\displaystyle\sum_{r,s=1}^{\mu}p_{y_ay_rx_b}\delta^*_{1r}+p_{y_ax_rx_b}\beta^*_{1r}\\ +p_{y_ax_ry_sx_b}(\beta^*_{1r}\delta_{0s}+\beta_{2r}\delta_{1s})\end{pmatrix}^{\mu}_{a,b=1} & \begin{pmatrix}p_{y_ay_b}\emptyset+\displaystyle\sum_{r,s=1}^{\mu}p_{y_ax_ry_b}\beta_{2r}\\ +p_{y_ax_rx_sy_b}(\beta^*_{1r}\beta_{1s}+\beta_{2r}\beta_{2s})\end{pmatrix}^{\mu}_{a,b=1}\end{array}\right).$$

[1] This is the reduced xy-hessian of the polynomial p. See section A.4 in [8] for more details.

Given $S=(S_1,S_2,\dots,S_{2\mu})\in\mathbb{S}_{n+m}(\mathbb{C}^{2\mu})$ of the block form of equation (1.2), let $B_1:=(S_{1,1},\dots,S_{\mu,1})$, $B_2:=(S_{1,2},\dots,S_{\mu,2})$, $D_0:=(S_{\mu+1,0},\dots,S_{2\mu,0})$ and $D_1:=(S_{\mu+1,1}\dots,S_{2\mu,1})$. Observe, by Proposition 2.7, that $\mathscr{M}(B_1,B_2,D_0,D_1)\succeq 0$. Further,

$$Q(S)=J^*\left[\mathscr{M}(B_1,B_2,D_0,D_1)\right]J,$$

for an appropriately chosen isometry J. Thus

$$Q(S)\succeq 0. \tag{2.4}$$

Recall the notation $(x,y)=(x_1,\dots,x_\mu,y_1,\dots,y_\mu)=(x_1,\dots,x_\mu,x_{\mu+1},\dots,x_{2\mu})$. Let $1\leqslant a,b,r,s\leqslant\mu$ and let x_0 denote the empty word. Define a $2d\mu\times 2d\mu$ matrix-valued polynomial P by

$$P(x_1,\dots,x_{2\mu})=\sum_{j,k=0}^{2\mu}P_{j,k}x_jx_k,$$

where the $P_{j,k}\in(M_d\otimes M_\mu)\otimes M_2$ are given by

$$P_{j,k}=\begin{cases}
\left(\begin{array}{c|c}\left(p_{x_ax_b}\right)_{a,b=1}^{\mu} & 0\\ \hline 0 & \left(p_{y_ay_b}\right)_{a,b=1}^{\mu}\end{array}\right); & (j,k)=(0,0)\\[2ex]
\frac{1}{2}\left(\begin{array}{c|c}0 & \left(p_{x_ax_ry_b}\right)_{a,b=1}^{\mu}\\ \hline \left(p_{y_ax_rx_b}\right)_{a,b=1}^{\mu} & \left(p_{y_ax_ry_b}\right)_{a,b=1}^{\mu}\end{array}\right); & (j,k)\in\{(r,0),(0,r)\}\\[2ex]
\frac{1}{2}\left(\begin{array}{c|c}\left(p_{x_ay_rx_b}\right)_{a,b=1}^{\mu} & \left(p_{x_ay_ry_b}\right)_{a,b=1}^{\mu}\\ \hline \left(p_{y_ay_rx_b}\right)_{a,b=1}^{\mu} & 0\end{array}\right); & (j,k)\in\{(\mu+r,0),(0,\mu+r)\}\\[2ex]
\left(\begin{array}{c|c}\left(p_{x_ay_ry_sx_b}\right)_{a,b=1}^{\mu} & 0\\ \hline 0 & 0\end{array}\right); & (j,k)=(\mu+r,\mu+s)\\[2ex]
\left(\begin{array}{c|c}0 & 0\\ \hline \left(p_{y_ax_ry_sx_b}\right)_{a,b=1}^{\mu} & 0\end{array}\right); & (j,k)=(r,\mu+s)\\[2ex]
\left(\begin{array}{c|c}0 & \left(p_{x_ay_rx_sy_b}\right)_{a,b=1}^{\mu}\\ \hline 0 & 0\end{array}\right); & (j,k)=(\mu+r,s)\\[2ex]
\left(\begin{array}{c|c}0 & 0\\ \hline 0 & \left(p_{y_ax_rx_sy_b}\right)_{a,b=1}^{\mu}\end{array}\right); & (j,k)=(r,s).
\end{cases}$$

Observe that P is a hermitian polynomial and $\mathscr{E}P(\sigma)=Q(\sigma)$. Thus, by equation (2.4), it follows that $\mathscr{E}P(S)=Q(S)\succeq 0$ for all tuples $S=(S_1,\dots,S_{2\mu})\in\mathbb{S}_{n+m}(\mathbb{C}^{2\mu})$ of 2×2 block hermitian matrices.

By Theorem 1.1, there exists an $N \leqslant 2(2\mu+1)(2d\mu)$ and $q_j \in M_{N\times 2d\mu}(\mathbb{C})$ for $0 \leqslant j \leqslant 2\mu$ such that

$$\begin{aligned} q_j^* q_k &= P_{j,k}, \quad 1 \leqslant j,k \leqslant 2\mu, \\ q_0^* q_k + q_k^* q_0 &= P_{k,0} + P_{0,k}, \quad 1 \leqslant k \leqslant 2\mu \\ (q_0^* q_0)_{\alpha,\alpha} &= (P_{0,0})_{\alpha,\alpha} \in M_d \otimes M_\mu, \quad 1 \leqslant \alpha \leqslant 2. \end{aligned}$$

Let $q(x_1,\dots,x_{2\mu}) = \sum_{r=0}^{2\mu} q_r x_r$. and note, in terms of x,y,

$$q(x,y) = q_0 x_0 + \sum_{r=1}^{\mu} (q_r x_r + q_{\mu+r} y_r).$$

A simple computation shows that

$$q(x,y)^* q(x,y) = P(x,y) + R,$$

where $R = \left(\begin{array}{c|c} 0 & (q_0^* q_0)_{1,2} \\ \hline (q_0^* q_0)_{2,1} & 0 \end{array}\right) \in (M_d(\mathbb{C}) \otimes M_\mu(\mathbb{C})) \otimes M_2(\mathbb{C})$. Note that $(q_0^* q_0)_{1,2}^* = (q_0^* q_0)_{2,1}$. Let $\{\mathsf{e}_1,\dots,\mathsf{e}_{2\mu}\}$ denote the standard orthonormal basis for $\mathbb{C}^{2\mu}$. Define the $2d\mu \times d$ matrix-valued polynomial in $\eta(x,y)$ by

$$\eta(x,y) := \sum_{j=1}^{\mu} (\mathsf{e}_j \otimes I_d) x_j + (\mathsf{e}_{\mu+j} \otimes I_d) y_j$$

and

$$\Lambda(x,y) := q(x,y)\,\eta(x,y).$$

Since $q_j^* q_j = P_{j,j}$, it follows that $q_j(\mathsf{e}_j \otimes I_d) = 0$. Hence $\Lambda(x,y)$ is an xy-pencil.

Recall the set of words $\mathscr{L}$ from Subsection 2.3. Let $\mathscr{L}_*$ denote the words in $\mathscr{L}$ of degree two in either x or y and verify

$$\begin{aligned} \Lambda(x,y)^* \Lambda(x,y) &= \eta(x,y)^* q(x,y)^* q(x,y)\,\eta(x,y) \\ &= \eta(x,y)^* [P(x,y) + R]\,\eta(x,y) \\ &= \sum_{w \in \mathscr{L}_*} p_w w + \sum_{r,s=1}^{\mu} \Big((q_0^* q_0)_{1,2}\Big)_{r,s} x_r y_s + \Big((q_0^* q_0)_{2,1}\Big)_{r,s} y_r x_s. \end{aligned}$$

Thus

$$p(x,y) = \lambda(x,y) + \Lambda(x,y)^* \Lambda(x,y),$$

where

$$\lambda(x,y) = \sum_{w \in \mathscr{L} \setminus \mathscr{L}_*} p_w w - \left(\sum_{r,s=1}^{\mu} \Big((q_0^* q_0)_{1,2}\Big)_{r,s} x_r y_s + \Big((q_0^* q_0)_{2,1}\Big)_{r,s} y_r x_s \right).$$

Since $\lambda(x,y)$ is a hermitian xy-pencil, the proof is complete. $\square$

We conclude this article with the following observation. Recall the notations used in Sections 1 and 2.

REMARK 2.8. As shown above, Theorem 1.1 implies Theorem 1.2. We conjecture that Theorem 1.2 does not imply Theorem 1.1, based on the following observation. Fix $d = \mu = 1$ and suppose that $\mathscr{E}\rho(S) = Q(S) \succeq 0$. Recall that $Q(\sigma)$ is totally general (up to the choice made in the unitary equivalence stated in [7, Proposition 4.1]) for the 2×2 matrix polynomial $P(x,y)$ constructed out of the xy-Hessian of a polynomial in $span(\mathscr{L})$. Let $q(x,y)$ be a free polynomial whose *reduced* xy-hessian is $Q(\sigma)$. It follows from Section A.4 in [8] that $q(x,y)$ is xy-convex. An application of Theorem 1.1 yields $q(x,y) = \lambda(x,y) + \Lambda(x,y)^*\Lambda(x,y)$ for some (scalar-valued) xy-pencil $\lambda(x,y)$ and a $N\times 1$ matrix-valued xy-pencil $\Lambda(x,y)$. In particular $q(x,y) \in span(\mathscr{L})$. If we use this structure of $q(x,y)$ to obtain the desired factorization of the polynomial $\rho(x,y)$, we see that it places restrictions on the coefficients of $\rho(x,y)$. This happens because $Q(\sigma)$ only depends on four variables (and not on β_0 and δ_2), whereas $\mathscr{E}\rho(\sigma)$ depends on six variables, strongly supporting the conjecture that $Q(\sigma)$ is far from the most general 2×2 matrix polynomial $\rho(x,y) \in span(\mathscr{L})$ for which $\mathscr{E}\rho(\sigma) \succeq 0$ for all σ.

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REFERENCES

[1] S. BALASUBRAMANIAN, N. HOTWANI AND S. MCCULLOUGH, *Bilinear matrix inequalities and polynomials in several freely noncommuting variables*, J. Math. Anal. Appl. **526** (1), p. 127196, 2023.
[2] H. DYM, J. W. HELTON AND S. MCCULLOUGH, *Irreducible non-commutative defining polynomials for convex sets have degree four or less*, Indiana Univ. Math. J. **56** (2007), 1189–1232.
[3] H. DYM, J. W. HELTON AND S. MCCULLOUGH, *The Hessian of a non-commutative polynomial has numerous negative eigenvalues*, Journal D'Analyse Mathematique **102** (2007), 29–76.
[4] H. DYM, J. W. HELTON AND S. MCCULLOUGH, *NC polys with convex level slices*, Indiana University Mathematics Journal, vol. 66, no. 6 (2017), pp. 2071–2135.
[5] D. M. HAY, J. W. HELTON, A. LIM AND S. MCCULLOUGH, *Non commutative Partial Matrix Convexity*, Indiana University Mathematics Journal, vol. 57, no. 6 (2008): 2815–42.
[6] J. W. HELTON AND S. MCCULLOUGH, *Convex non-commutative polynomials have degree two or less*, SIAM J. Matrix Anal. Appl. **25** (2004), 1124–1139.
[7] MICHAEL JURY, IGOR KLEP, MARK E. MANCUSO, SCOTT MCCULLOUGH, JAMES ELDRED PASCOE, *Noncommutative partially convex rational functions*, Rev. Mat. Iberoam. **38** (2022), no. 3, pp. 731–759.
[8] MICHAEL JURY, IGOR KLEP, MARK E. MANCUSO, SCOTT MCCULLOUGH, JAMES ELDRED PASCOE, *Noncommutative partially convex rational functions*, https://arxiv.org/pdf/2005.01086.
[9] M. JURY, I. KLEP, M. E. MANCUSO, S. MCCULLOUGH, J. PASCOE, *Noncommutative Partial Convexity Via Γ-Convexity*, J. Geom. Anal. **31**, 3137–3160 (2021).
[10] J. E. PASCOE, RYAN TULLY-DOYLE, *The royal road to automatic noncommutative real analyticity, monotonicity, and convexity*, arXiv:1907.05875.
[11] V. PAULSEN, *Completely bounded maps and operator algebras*, Cambridge University Press, 2003.

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