Operators and Matrices
Volume 19, Number 3 (2025), 353–372
doi:10.7153/oam-2025-19-23

PROPERTIES OF INTEGRAL OPERATORS ON BERGMAN–MORREY SPACES

RUISHEN QIAN* AND ZHONGHUA HE

(*Communicated by V. Bolotnikov*)

Abstract. For $0<q<\infty$ and $0<\eta<\infty$, the tent space $T_{q,\eta}(\mu)$ consists of all μ-measurable functions f such that

$$\|f\|_{T_{q,\eta}(\mu)}^{q}:=\sup_{I\subseteq\partial\mathbb{D}}\frac{1}{|I|^{\eta}}\int_{S(I)}|f(z)|^{q}d\mu(z)<\infty.$$

In this note, we study the boundedness and compactness of the inclusion mapping i from Bergman-Morrey Spaces $\mathscr{A}^{p,\lambda}$ to Tent Spaces $T_{q,\eta}(\mu)$. The boundedness and essential norm of Volterra integral operators from Bergman-Morrey Spaces $\mathscr{A}^{p,\lambda}$ to Bergman-Morrey Spaces $\mathscr{A}^{q,\eta}$ are also investigated in this paper, which generalized the main results in [31]. In the end, we investigated the closed range Volterra integral operators on Bergman-Morrey Spaces $\mathscr{A}^{p,\lambda}$.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Let $0<p<\infty$, $-2<q<\infty$, $0\leqslant s<\infty$. The space $F(p,q,s)$ (see [32]) is the space consisting of all $f\in H(\mathbb{D})$ such that

$$\|f\|_{F(p,q,s)}^{p}=|f(0)|^{p}+\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f'(z)|^{p}(1-|z|^{2})^{q}(1-|\sigma_a(z)|^{2})^{s}dA(z)<\infty,$$

where dA is the normalized Lebesgue area measure in $\mathbb{D}$ such that $A(\mathbb{D})=1$ and $\sigma_a(z)=\frac{a-z}{1-\bar{a}z}$. Especially, when $p=2$ and $s=1$, it gets the *BMOA* space (see [8]), the space of analytic functions in the Hardy space whose boundary functions have bounded mean oscillation. It is well known that $F(p,p\alpha-2,s)$ is equivalent to weighted Bloch space $\mathscr{B}^{\alpha}$ $(0<\alpha<\infty)$ for all $s>1$, where the weighted Bloch space $\mathscr{B}^{\alpha}$ (see [34]) is the class of all $f\in H(\mathbb{D})$ for which

$$\|f\|_{\mathscr{B}^{\alpha}}:=|f(0)|+\sup_{z\in\mathbb{D}}(1-|z|^{2})^{\alpha}|f'(z)|<\infty.$$

Mathematics subject classification (2020): 30H20, 47B38.
Keywords and phrases: Volterra integral operator, Bergman-Morrey spaces, essential norm, closed range.
Qian was supported by NNSF of China (no. 12226312, 12226306), Lingnan Normal University (no. LT2410), GuangDong Basic and Applied Basic Research Foundation (2024A1515012551) and Department of Education of Guangdong Province (no. 2023KTSCX072). He was supported by NNSF of China (no. 11971123).
* Corresponding author.

If $\alpha = 1$, we denote $\mathscr{B}^\alpha$ simply by $\mathscr{B}$, which is the well-known classical Bloch space.

Let $S(I)$ be the Carleson box based on I with

$$S(I) = \left\{ z \in \mathbb{D} : 1 - |I| < |z| < 1 \text{ and } \frac{z}{|z|} \in I \right\}.$$

If $I = \partial\mathbb{D}$, then $S(I) = \mathbb{D}$. For $0 < s < \infty$, we say that a non-negative measure μ on $\mathbb{D}$ is a s-Carleson measure if

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty.$$

When $s = 1$, it is the classical Carleson measure (see [35]). From [32], we see that $f \in F(p,q,s)$ if and only if $d\mu(z) = |f'(z)|^p (1-|z|^2)^{q+s} dA(z)$ is a bounded s-Carleson measure, where $0 < p < \infty$, $-2 < q < \infty$, $0 < s < \infty$ and $q + s > -1$.

Hardy space H^p $(0 < p < \infty$, see [5]) is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

Weighted Bergman space A^p_α $(0 < p < \infty,\ -1 < \alpha < \infty$, see [35]) consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha}^p = \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

As usual, let $A^p_0 = A^p$.

Let $0 < p < \infty$ and $0 < \lambda < 2$, the Bergman-Morrey space $\mathscr{A}^{p,\lambda}$ denoted the spaces of function $f \in H(\mathbb{D})$ satisfies

$$\|f\|_{\mathscr{A}^{p,\lambda}} = |f(0)| + \sup_{a \in \mathbb{D}} (1-|a|^2)^{\frac{2-\lambda}{p}} \|f \circ \sigma_a - f(a)\|_{A^p} < \infty.$$

The Bergman-Morrey space $\mathscr{A}^{p,\lambda}$ was introduced by Yang and Liu in [31]. And from [31], we can see that the Bergman-Morrey space $\mathscr{A}^{p,\lambda}$ is a special space of $F(p,q,s)$, that is, $\mathscr{A}^{p,\lambda} = F(p, p-\lambda, \lambda)$. Furthermore, we see that $F(p, p-\lambda, \lambda) = \mathscr{N}(p, -\lambda, \lambda)$ by [12]. Thus, when $p > 0$ and $0 < \lambda < 2$, we have

$$\mathscr{A}^{p,\lambda} = F(p, p-\lambda, \lambda) = \mathscr{N}(p, -\lambda, \lambda),$$

where $\mathscr{N}(p, -\lambda, \lambda)$ is the class of all $f \in H(\mathbb{D})$ for which

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f(z)|^p (1-|z|^2)^{-\lambda} (1-|\sigma_a(z)|^2)^\lambda dA(z) < \infty.$$

Associated with $f, g \in H(\mathbb{D})$, the Volterra integral operators are defined as follows

$$V_g f(z) = \int_0^z f(w) g'(w) dw, \quad S_g f(z) = \int_0^z f'(w) g(w) dw, \quad z \in \mathbb{D}.$$

For $0<q<\infty$ and $0<s<\infty$, the tent space $T_{q,s}(\mu)$ consists of all μ-measurable functions f such that

$$\|f\|^p_{T_{q,s}(\mu)} := \sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^s}\int_{S(I)} |f(z)|^q d\mu(z) < \infty.$$

The tent space $T_{2,s}(\mu) = T_2^\infty$ was introduced by by Xiao in [29] to studied Volterra integral operators acting on $\mathscr{Q}_s$ space, generlized by Pau and Zhao in [18] later. For more results on tent space and Volterra integral operators, we refer to [3, 4, 7, 19, 36].

Recently, Yang and Liu studied tent space and Volterra type operators acting on Bergman-Morrey space $\mathscr{A}^{p,\lambda}$ in [31]. They proved that V_g is bounded from $\mathscr{A}^{p,\lambda}$ to $\mathscr{A}^{p,\lambda}$ if and only if $g\in\mathscr{B}$. Motivated by their works, we prove that inclusion mapping $i:\mathscr{A}^{p,\lambda}\to T_{q,2-\frac{q(2-\lambda)}{p}}(\mu)$ is bounded if and only if μ is a 2-Carleson measure which generalized the main results in [31]. Furthermore, V_g is bounded from $\mathscr{A}^{p,\lambda}$ to $\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}$ if and only if $g\in\mathscr{B}$, where $0<q\leqslant p<\infty$ and $0<\lambda,\eta<2$. And essential norm related them are also investigated. In the end, we also investigated the closed range Volterra integral operators on Bergman-Morrey Spaces $\mathscr{A}^{p,\lambda}$.

Throughout this article, positive constants are denoted by C, they may differ from one occurrence to the other. We say that $f\lesssim g$ if there exists a constant C such that $f\leqslant Cg$. The symbol $f\approx g$ means that $f\lesssim g\lesssim f$.

2. Embedding from $\mathscr{A}^{p,\lambda}$ to $T_{q,\eta}(\mu)$

In this section, we will characterise the boundedness and compactness of the inclusion mapping $i:\mathscr{A}^{p,\lambda}\to T_{q,\eta}(\mu)$. For this purpose, we start this section by quoting some auxiliary results which will be used in the proofs of the main results of this paper.

LEMMA 1. ([31]) *Let $0<p<\infty$ and $0<\lambda<2$. Then $f\in\mathscr{A}^{p,\lambda}$ if and only if*

$$\sup_{I\subseteq\partial\mathbb{D}} \frac{1}{|I|^\lambda}\int_{S(I)} |f'(z)|^p(1-|z|^2)^p dA(z) < \infty.$$

LEMMA 2. ([31]) *Let $0<p<\infty$ and $0<\lambda<2$. Suppose that $f\in\mathscr{A}^{p,\lambda}$, then*

$$|f(z)| \lesssim \frac{\|f\|_{\mathscr{A}^{p,\lambda}}}{(1-|z|^2)^{\frac{2-\lambda}{p}}},\quad z\in\mathbb{D}.$$

LEMMA 3. ([31]) *Let $0<p<\infty$ and $0<\lambda<2$. Then*

$$f_a(z) = \frac{1-|a|^2}{(1-\overline{a}z)^{1+\frac{2-\lambda}{p}}} \in \mathscr{A}^{p,\lambda}$$

and

$$g_a(z) = \frac{1-|a|^2}{\overline{a}(1-\overline{a}z)^{1+\frac{2-\lambda}{p}}} \in \mathscr{A}^{p,\lambda},\quad a,z\in\mathbb{D}.$$

LEMMA 4. ([14]) *For* $0<r<1$, *let* $\chi_{\{z:|z|<r\}}(z)$ *be the characteristic function of the set* $\{z:|z|<r\}$. *If* μ *is a* p*-Carleson measure on* $\mathbb{D}$, *then* μ *is a vanishing* p*-Carleson measure if and only if* $\|\mu-\mu_r\|_p\to 0$ *as* $r\to 1^-$, *where* $d\mu_r=\chi_{\{z:|z|<r\}}d\mu$.

Now, we are going to prove our first result.

THEOREM 1. *Let* μ *be a positive Borel measure on* $\mathbb{D}$. *Suppose that* $0<p\leqslant q<\infty$ *and* $0<\frac{q}{p}\lambda\leqslant\eta<2$. *Then the inclusion mapping* $i:\mathscr{A}^{p,\lambda}\to T_{q,\eta}(\mu)$ *is bounded if and only if*

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^{\eta+\frac{q(2-\lambda)}{p}}}<\infty.$$

Proof. For any $I\in\partial\mathbb{D}$, let $a=(1-|I|)\zeta\in\mathbb{D}$, where ζ is the center of I. Then

$$1-|a|\approx|1-\overline{a}z|\approx|I|,\ \ z\in S(I).$$

Necessity. Suppose that the inclusion mapping $i:\mathscr{A}^{p,\lambda}\to T_{q,\eta}(\mu)$ is bounded. From Lemma 3, we see that

$$f_a(z)=\frac{1-|a|^2}{(1-\overline{a}z)^{1+\frac{2-\lambda}{p}}}\in\mathscr{A}^{p,\lambda}.$$

Then

$$|f_a(z)|\approx\frac{1}{|I|^{\frac{2-\lambda}{p}}}.$$

Thus,

$$\infty>\frac{1}{|I|^\eta}\int_{S(I)}|f_a(z)|^q d\mu(z)\approx\frac{\mu(S(I))}{|I|^{\eta+\frac{q(2-\lambda)}{p}}}.$$

Sufficiency. Suppose that $f\in\mathscr{A}^{p,\lambda}$ and

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^{\eta+\frac{q(2-\lambda)}{p}}}<\infty.$$

Then

$$\begin{aligned}&\frac{1}{|I|^\eta}\int_{S(I)}|f(z)|^q d\mu(z)\\ \leqslant&\frac{1}{|I|^\eta}\left(\int_{S(I)}|f(z)-f(a)|^q d\mu(z)+\int_{S(I)}|f(a)|^q d\mu(z)\right)\\ =:&M+N.\end{aligned}$$

Combine with Lemma 2, we obtain

$$|f(a)|\lesssim\frac{\|f\|_{\mathscr{A}^{p,\lambda}}}{|I|^{\frac{2-\lambda}{p}}}.$$

Thus,

$$N \lesssim \sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^{\eta+\frac{q(2-\lambda)}{p}}} < \infty.$$

Now, we estimat M. Since $0 < p \leqslant q < \infty$ and $0 < \frac{q}{p}\lambda \leqslant \eta < 2$. We have

$$\eta + \frac{q(2-\lambda)}{p} \geqslant \lambda + \frac{p(2-\lambda)}{p} = 2.$$

It is well known that μ is $\eta + \frac{q(2-\lambda)}{p}$-Carleson measure if and only if $A^p_{\frac{p\eta-q\lambda}{q}} \subseteq L^q(\mu)$ (see [35]). Note that $0 < \frac{q}{p}\lambda \leqslant \eta < 2$, we have $\mathscr{A}^{p,\lambda} \subseteq \mathscr{A}^p \subseteq \mathscr{A}^p_{\frac{p\eta-q\lambda}{q}}$. Then

$$\begin{aligned}
M &\lesssim (1-|a|^2)^{\frac{4q}{p}} \int_{S(I)} \left| \frac{f(z)-f(a)}{(1-\overline{a}z)^{4/p+\frac{\eta}{q}}} \right|^q d\mu(z) \\
&\leqslant (1-|a|^2)^{\frac{4q}{p}} \int_{\mathbb{D}} \left| \frac{f(z)-f(a)}{(1-\overline{a}z)^{4/p+\frac{\eta}{q}}} \right|^q d\mu(z) \\
&\lesssim (1-|a|^2)^{\frac{4q}{p}} \left(\int_{\mathbb{D}} \left| \frac{f(z)-f(a)}{(1-\overline{a}z)^{4/p+\frac{\eta}{q}}} \right|^p (1-|z|^2)^{\frac{p\eta-q\lambda}{q}} dA(z) \right)^{\frac{q}{p}} \\
&= (1-|a|^2)^{\frac{2q}{p}} \left((1-|a|^2)^2 \int_{\mathbb{D}} |f(z)-f(a)|^p \frac{(1-|a|^2)^2}{|1-\overline{a}z|^{4+\frac{p\eta}{q}}} (1-|z|^2)^{\frac{p\eta-q\lambda}{q}} dA(z) \right)^{\frac{q}{p}} \\
&\lesssim \left((1-|a|^2)^{(2-\lambda)} \int_{\mathbb{D}} |f(z)-f(a)|^p \frac{(1-|a|^2)^2}{|1-\overline{a}z|^4} dA(z) \right)^{\frac{q}{p}} \\
&= \left((1-|a|^2)^{(2-\lambda)} \|f\circ\varphi_a(z) - f(a)\|^p_{\mathscr{A}^p} \right)^{\frac{q}{p}} < \infty. \quad \square
\end{aligned}$$

THEOREM 2. *Let μ be a positive Borel measure on $\mathbb{D}$. Suppose that $\eta + \frac{q(2-\lambda)}{p} \geqslant 2$, $\eta + \frac{q(2-\lambda)}{p} - 2 < q \leqslant p < \infty$ and $0 < \lambda, \eta < 2$. Then the inclusion mapping i : $\mathscr{A}^{p,\lambda} \to T_{q,\eta}(\mu)$ is bounded if and only if*

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^{\eta+\frac{q(2-\lambda)}{p}}} < \infty.$$

Proof. Necessity. Similar to the proof of Theorem 1, thus we omitted the proof.
Sufficiency. Suppose that $f \in \mathscr{A}^{p,\lambda}$ and

$$\sup_{I \subseteq \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^{\eta+\frac{q(2-\lambda)}{p}}} < \infty.$$

Similar to the proof of Theorem 1, we only need to estimat M. Since μ is $\eta+\frac{q(2-\lambda)}{p}$-Carleson measure if and only if $A^q_{\eta+\frac{q(2-\lambda)}{p}-2} \subseteq L^q(\mu)$. Note that $p \geqslant q$, we have $\mathscr{A}^{p,\lambda} \subseteq \mathscr{A}^p \subseteq \mathscr{A}^q \subseteq \mathscr{A}^q_{\eta+\frac{q(2-\lambda)}{p}-2}$. Then

$$\begin{aligned}
M &\lesssim (1-|a|^2)^{2+\frac{q(2-\lambda)}{p}} \int_{S(I)} \frac{|f(z)-f(a)|^q}{|1-\overline{a}z|^{2+\frac{q(2-\lambda)}{p}+\eta}} d\mu(z) \\
&\leqslant (1-|a|^2)^{2+\frac{q(2-\lambda)}{p}} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^q}{|1-\overline{a}z|^{2+\frac{q(2-\lambda)}{p}+\eta}} d\mu(z) \\
&\lesssim (1-|a|^2)^{2+\frac{q(2-\lambda)}{p}} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^q}{|1-\overline{a}z|^{2+\frac{q(2-\lambda)}{p}+\eta}} (1-|z|^2)^{\eta+\frac{q(2-\lambda)}{p}-2} dA(z) \\
&\leqslant (1-|a|^2)^{2+\frac{q(2-\lambda)}{p}} \int_{\mathbb{D}} \frac{|f(z)-f(a)|^q}{|1-\overline{a}z|^4} dA(z) \\
&= (1-|a|^2)^{\frac{q(2-\lambda)}{p}} \|f\circ\varphi_a(z)-f(a)\|^q_{\mathscr{A}^q} \\
&\lesssim (1-|a|^2)^{\frac{q(2-\lambda)}{p}} \|f\circ\varphi_a(z)-f(a)\|^q_{\mathscr{A}^p} < \infty. \quad \square
\end{aligned}$$

THEOREM 3. *Let μ be a positive Borel measure on $\mathbb{D}$. Suppose that $0 < p \leqslant q < \infty$ and $0 < \frac{q}{p}\lambda \leqslant \eta < 2$. Then the inclusion mapping $i : \mathscr{A}^{p,\lambda} \to T_{q,\eta}(\mu)$ is compact if and only if*

$$\lim_{|I|\to 0} \frac{\mu(S(I))}{|I|^{\eta+\frac{q(2-\lambda)}{p}}} = 0.$$

Proof. Suppose that the inclusion mapping $i : \mathscr{A}^{p,\lambda} \to T_{q,\eta}(\mu)$ is compact. Given a sequence of arcs $\{I_n\}$ with $\lim_{n\to\infty}|I_n| = 0$. Denote the center of I_n by $e^{i\theta_n}$ and $a_n = (1-|I_n|)e^{i\theta_n}$. Let

$$f_n(z) := \frac{1-|a_n|^2}{(1-\overline{a_n}z)^{1+\frac{2-\lambda}{p}}}, \quad z \in \mathbb{D}.$$

It is clear that $\{f_n\}$ is bounded in $\mathscr{A}^{p,\lambda}$ and $\{f_n\}$ converges to zero uniformly on any compact subset of $\mathbb{D}$. Then $\lim_{n\to\infty}\|f_n\|_{T_{q,\eta}(\mu)} = 0$. Since

$$|f_n(z)| \approx (1-|a_n|)^{\frac{\lambda-2}{p}} \approx |I_n|^{\frac{\lambda-2}{p}}, \quad z \in S(I_n),$$

we obtain

$$\frac{\mu(S(I_n))}{|I_n|^{\eta+\frac{q(2-\lambda)}{p}}} \approx \frac{1}{|I_n|^\eta} \int_{S(I_n)} |f_n(z)|^q d\mu(z) \leqslant \|f_n\|^q_{T_{q,\eta}(\mu)} \to 0, \quad n \to \infty.$$

By the arbitrariness of $\{I_n\}$, we deduce that μ is a vanishing $(\eta+\frac{q(2-\lambda)}{p})$-Carleson measure.

Conversely, suppose that μ is a vanishing $(\eta+\frac{q(2-\lambda)}{p})$-Carleson measure, then μ is also a $(\eta+\frac{q(2-\lambda)}{p})$-Carleson measure and $\lim_{r\to 1^-}\|\mu-\mu_r\|_{\eta+\frac{q(2-\lambda)}{p}}=0$ by Lemma 4. It follows from the boundedness above, the inclusion mapping $i:\mathscr{A}^{p,\lambda}\to T_{q,\eta}(\mu)$ is bounded. Let $\{f_n\}$ be a bounded sequence in $\mathscr{A}^{p,\lambda}$ such that $\{f_n\}$ converges to zero uniformly on each compact subset of $\mathbb{D}$. We have

$$\begin{aligned}\frac{1}{|I|^\eta}\int_{S(I)}|f_n(z)|^q d\mu(z) &\lesssim \frac{1}{|I|^\eta}\int_{S(I)}|f_n(z)|^q d\mu_r(z)+\frac{1}{|I|^\eta}\int_{S(I)}|f_n(z)|^q d(\mu-\mu_r)(z)\\ &\lesssim \frac{1}{|I|^\eta}\int_{S(I)}|f_n(z)|^q d\mu_r(z)+\|\mu-\mu_r\|_{\eta+\frac{q(2-\lambda)}{p}}\|f_n\|^q_{\mathscr{A}^{p,\lambda}}\\ &\lesssim \frac{1}{|I|^\eta}\int_{S(I)}|f_n(z)|^q d\mu_r(z)+\|\mu-\mu_r\|_{\eta+\frac{q(2-\lambda)}{p}}\\ &\to 0,\end{aligned}$$

as $r\to 1^-$ and $n\to\infty$. Therefore, $\lim_{n\to\infty}\|f_n\|_{T_{q,\eta}(\mu)}=0$. This shows that the inclusion mapping $i:\mathscr{A}^{p,\lambda}\to T_{q,\eta}(\mu)$ is compact. □

THEOREM 4. *Let μ be a positive Borel measure on $\mathbb{D}$. Suppose that $\eta+\frac{q(2-\lambda)}{p}\geqslant 2$, $\eta+\frac{q(2-\lambda)}{p}-2<q\leqslant p<\infty$ and $0<\lambda,\eta<2$. Then the inclusion mapping $i:\mathscr{A}^{p,\lambda}\to T_{q,\eta}(\mu)$ is compact if and only if*

$$\lim_{|I|\to 0}\frac{\mu(S(I))}{|I|^{\eta+\frac{q(2-\lambda)}{p}}}=0.$$

Proof. The proof is similar to Theorem 3, thus we omitted the proof. □

From Theorem 1 or Theorem 2, we obtain [31, Theorem 3.3], that is

COROLLARY 1. *Let $0<p<\infty$ and $0<\lambda<2$. Suppose that μ is a positive Borel measure on $\mathbb{D}$. Then the inclusion mapping $i:\mathscr{A}^{p,\lambda}\to T_{p,\lambda}(\mu)$ is bounded if and only if*

$$\sup_{I\subseteq\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^2}<\infty.$$

3. Boundedness of Volterra integral operators $V_g(S_g):\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,\eta}$

In this section, we study the boundedness of Volterra integral operators $V_g(S_g):\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,\eta}$.

THEOREM 5. *Let $g\in H(\mathbb{D})$.*
(1). *Suppose that $0<p\leqslant q<\infty$ and $0<\frac{q}{p}\lambda\leqslant\eta<2$. Then V_g is bounded from $\mathscr{A}^{p,\lambda}$ to $\mathscr{A}^{q,\eta}$ if and only if $g\in\mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}$.*

(2). *Suppose that* $\eta+\frac{q(2-\lambda)}{p}\geqslant 2,\ \eta+\frac{q(2-\lambda)}{p}-2<q\leqslant p<\infty$ *and* $0<\lambda,\eta<2$. *Then* V_g *is bounded from* $\mathscr{A}^{p,\lambda}$ *to* $\mathscr{A}^{q,\eta}$ *if and only if* $g\in\mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}$.

Proof. (1). For any $I\in\partial\mathbb{D}$, let $a=(1-|I|)\zeta\in\mathbb{D}$, where ζ is the center of I. Then

$$1-|a|\approx|1-\overline{a}z|\approx|I|,\ \ z\in S(I).$$

Suppose that V_g is bounded from $\mathscr{A}^{p,\lambda}$ to $\mathscr{A}^{q,\eta}$. From Lemma 3, we see that

$$f_a(z)=\frac{1-|a|^2}{(1-\overline{a}z)^{1+\frac{2-\lambda}{p}}}\in\mathscr{A}^{p,\lambda}.$$

Combined with Lemma 1, we obtain

$$\begin{aligned}&\frac{1}{|I|^{\eta+\frac{q(2-\lambda)}{p}}}\int_{S(I)}|g'(z)|^q(1-|z|^2)^q dA(z)\\ \approx&\frac{1}{|I|^{\eta}}\int_{S(I)}|f_a(z)|^q|g'(z)|^q(1-|z|^2)^q dA(z)\\ =&\frac{1}{|I|^{\eta}}\int_{S(I)}|(V_gf_a)'(z)|^q(1-|z|^2)^q dA(z)\lesssim\|V_gf_a\|^p_{\mathscr{A}^{q,\eta}}<\infty.\end{aligned}$$

Thus, $g\in F(q,q-(\eta+\frac{q(2-\lambda)}{p}),\eta+\frac{q(2-\lambda)}{p})$. Note that $0<p\leqslant q<\infty$ and $0<\frac{q}{p}\lambda\leqslant\eta<2$, thus,

$$\eta+\frac{q(2-\lambda)}{p}\geqslant\lambda+\frac{p(2-\lambda)}{p}=2>1.$$

That is $F(q,q-(\eta+\frac{q(2-\lambda)}{p}),\eta+\frac{q(2-\lambda)}{p})=\mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}$.

On the other hand, suppose that $f\in\mathscr{A}^{p,\lambda}$ and $g\in\mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}$. Since

$$g\in\mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}=F\Big(q,q-(\eta+\frac{q(2-\lambda)}{p}),\eta+\frac{q(2-\lambda)}{p}\Big),$$

Then $d\mu_g(z)=:|g'(z)|^q(1-|z|^2)dA(z)^q$ is a $(\eta+\frac{q(2-\lambda)}{p})$-Carleson measure. Combined with Theorem 1, we have

$$\begin{aligned}&\frac{1}{|I|^{\eta}}\int_{S(I)}|(V_gf)'(z)|^q(1-|z|^2)^q dA(z)\\ =&\frac{1}{|I|^{\eta}}\int_{S(I)}|f(z)|^q|g'(z)|^q(1-|z|^2)^q dA(z)\\ =:&\frac{1}{|I|^{\eta}}\int_{S(I)}|f(z)|^q d\mu_g(z)\\ \lesssim&\Big((1-|a|^2)^{(2-\lambda)}\|f\circ\varphi_a(z)-f(a)\|^p_{\mathscr{A}^p}\Big)^{\frac{q}{p}}<\infty.\end{aligned}$$

(2). The proof is similar to (1), thus we omitted the proof. □

From Theorem 5, when $\eta = 2 - \frac{q(2-\lambda)}{p}$, we have

COROLLARY 2. *Let $0 < q \leqslant p < \infty$ and $0 < \lambda < 2$. Suppose that $g \in H(\mathbb{D})$. Then V_g is bounded from $\mathscr{A}^{p,\lambda}$ to $\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}$ if and only if $g \in \mathscr{B}$.*

We also easy to have one of main results in [31].

COROLLARY 3. *Let $0 < p < \infty$ and $0 < \lambda < 2$. Suppose that $g \in H(\mathbb{D})$. Then V_g is bounded from $\mathscr{A}^{p,\lambda}$ to $\mathscr{A}^{p,\lambda}$ if and only if $g \in \mathscr{B}$.*

The next result generalized [31, Theorem 4.3].

THEOREM 6. *Let $0 < q \leqslant p < \infty$ and $0 < \lambda < 2$. Suppose that $g \in H(\mathbb{D})$. Then S_g is bounded from $\mathscr{A}^{p,\lambda}$ to $\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}$ if and only if $g \in H^\infty$.*

Proof. Suppose that S_g is bounded from $\mathscr{A}^{p,\lambda}$ to $A^{q,2-\frac{q(2-\lambda)}{p}}$. By Lemma 3, we have

$$g_a(z) = \frac{1-|a|^2}{\overline{a}(1-\overline{a}z)^{1+\frac{2-\lambda}{p}}} \in \mathscr{A}^{p,\lambda}.$$

Noticed the fact that

$$1-|\sigma_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\overline{a}z|^2}$$

and

$$|1-\overline{a}z| \approx 1-|z|^2 \approx 1-|a|^2, \ \ z \in \mathbb{D}(a,r),$$

where $\mathbb{D}(a,r) = \{z : |\sigma_a(z)| < r\}$. Then

$$\begin{aligned}
\infty &> \|S_g g_a\|^q_{A^{q,2-\frac{q(2-\lambda)}{p}}} \\
&\gtrsim \int_{\mathbb{D}} |g_a'(z)|^q |g(z)|^q (1-|z|^2)^{q-(2-\frac{q(2-\lambda)}{p})} (1-|\sigma_a(z)|^2)^{2-\frac{q(2-\lambda)}{p}} dA(z) \\
&\gtrsim \int_{\mathbb{D}(a,r)} |g_a'(z)|^q |g(z)|^q (1-|z|^2)^{q-(2-\frac{q(2-\lambda)}{p})} (1-|\sigma_a(z)|^2)^{2-\frac{q(2-\lambda)}{p}} dA(z) \\
&\gtrsim \frac{1}{(1-|a|^2)^2} \int_{\mathbb{D}(a,r)} |g(z)|^q dA(z) \gtrsim |g(a)|^q.
\end{aligned}$$

Thus, $g \in H^\infty$.

On the other hand, let $g \in H^\infty$ and $f \in \mathscr{A}^{p,\lambda}$. Using the fact that $A^p \subseteq A^q$, when

$p \geqslant q$. We have $\|\cdot\|_{A^q} \lesssim \|\cdot\|_{A^p}$. We obtain

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(z)|^q |g(z)|^q (1-|z|^2)^{q-(2-\frac{q(2-\lambda)}{p})} (1-|\sigma_a(z)|^2)^{2-\frac{q(2-\lambda)}{p}} dA(z) \\
\leqslant &\sup_{z\in\mathbb{D}} |g(z)|^q \int_{\mathbb{D}} |f'(z)|^q (1-|z|^2)^{q-(2-\frac{q(2-\lambda)}{p})} (1-|\sigma_a(z)|^2)^{2-\frac{q(2-\lambda)}{p}} dA(z) \\
= &\sup_{z\in\mathbb{D}} |g(z)|^q \left((1-|a|^2)^{\frac{(2-\lambda)}{p}} \|f\circ\sigma_a - f(a)\|_{A^q} \right)^q \\
\lesssim &\sup_{z\in\mathbb{D}} |g(z)|^q \left((1-|a|^2)^{\frac{(2-\lambda)}{p}} \|f\circ\sigma_a - f(a)\|_{A^p} \right)^q. \quad \square
\end{aligned}$$

When $p=q$, from Theorem 6, we easy to have [31, Theorem 4.3].

COROLLARY 4. *Let* $0<p<\infty$ *and* $0<\lambda<2$. *Suppose that* $g\in H(\mathbb{D})$. *Then* S_g *is bounded from* $\mathscr{A}^{p,\lambda}$ *to* $\mathscr{A}^{p,\lambda}$ *if and only if* $g\in H^\infty$.

REMARK. Since multiplication operator

$$M_g f(z) := f(z)g(z) = f(0)g(0) + V_g f + S_g f.$$

Combined with Theorems 5 and 6, similar to the proof of [31, Corollary 4.6], we have the multiplication operator M_g is bounded from $\mathscr{A}^{p,\lambda}$ to $\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}$ if and only if $g\in H^\infty$.

4. Essential norm of V_g from $\mathscr{A}^{p,\lambda}$ to $\mathscr{A}^{q,\eta}$

Let us recall some definitions. Let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be Banach spaces and $T: X\to Y$ be a bounded linear operator. The essential norm of $T: X\to Y$, denoted by $\|T\|_{e,X\to Y}$, is defined by

$$\|T\|_{e,X\to Y} = \inf_K\{\|T-K\|_{X\to Y} : K \text{ is compact from } X \text{ to } Y\}.$$

It is easy to see that $T: X\to Y$ is compact if and only if $\|T\|_{e,X\to Y}=0$. Let A be a closed subspace of X. Given $f\in X$, the distance from f to A, denoted by $\mathrm{dist}_X(f,A)$, is defined by

$$\mathrm{dist}_X(f,A) = \inf_{g\in A}\|f-g\|_X.$$

LEMMA 5. [33] *If* $\alpha>0$ *and* $f\in\mathscr{B}^\alpha$, *then*

$$\limsup_{|z|\to 1^-}(1-|z|^2)^\alpha|f'(z)| \approx \mathrm{dist}_{\mathscr{B}^\alpha}(f,\mathscr{B}^\alpha_0) \approx \limsup_{r\to 1^-}\|f-f_r\|_{\mathscr{B}^\alpha}.$$

Here $f_r(z)=f(rz)$, $0<r<1$, $z\in\mathbb{D}$.

LEMMA 6. *Let* $g \in \mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}$.

(1). *Suppose that* $0 < p \leqslant q < \infty$ *and* $0 < \frac{q}{p}\lambda \leqslant \eta < 2$. *Then* $V_{g_r} : \mathscr{A}^{p,\lambda} \to \mathscr{A}^{q,\eta}$ *is compact;*

(2). *Suppose that* $\eta + \frac{q(2-\lambda)}{p} \geqslant 2,\ \eta + \frac{q(2-\lambda)}{p} - 2 < q \leqslant p < \infty$ *and* $0 < \lambda, \eta < 2$. *Then* $V_{g_r} : \mathscr{A}^{p,\lambda} \to \mathscr{A}^{q,\eta}$ *is compact.*

Proof. The proof is similar to Lemma 5.2 of [31], thus we omitted it. □

The following lemma is well known.

LEMMA 7. ([25]) *Let* X, Y *be two Banach spaces of analytic functions on* $\mathbb{D}$. *Suppose that*

(1) *The point evaluation functionals on* Y *are continuous.*

(2) *The closed unit ball of* X *is a compact subset of* X *in the topology of uniform convergence on compact sets.*

(3) $T : X \to Y$ *is continuous when* X *and* Y *are given the topology of uniform convergence on compact sets.*

Then, T *is a compact operator if and only if for any bounded sequence* $\{f_n\}$ *in* X *such that* $\{f_n\}$ *converges to zero uniformly on every compact set of* $\mathbb{D}$, *then the sequence* $\{Tf_n\}$ *converges to zero in the norm of* Y.

THEOREM 7. *Suppose that* $g \in H(\mathbb{D})$ *and* $V_g : \mathscr{A}^{p,\lambda} \to \mathscr{A}^{q,\eta}$ *is bounded. Suppose that* $0 < p \leqslant q < \infty,\ 0 < \frac{q}{p}\lambda \leqslant \eta < 2$ *or* $\eta + \frac{q(2-\lambda)}{p} \geqslant 2,\ \eta + \frac{q(2-\lambda)}{p} - 2 < q \leqslant p < \infty$ *and* $0 < \lambda, \eta < 2$. *Then*

$$\|V_g\|_{e,\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,\eta}} \approx \limsup_{|z|\to 1^-} (1-|z|^2)^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}} |f'(z)|$$

$$\approx \operatorname{dist}_{\mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}} \left(f, \mathscr{B}_0^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}\right).$$

Proof. By Lemma 6, $V_{g_r} : \mathscr{A}^{p,\lambda} \to \mathscr{A}^{q,\eta}$ is compact. Then

$$\|V_g\|_{e,\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,\eta}} \leqslant \|V_g - V_{g_r}\| = \|V_{g-g_r}\| \approx \|g - g_r\|_{\mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}}.$$

Using Lemma 5, we have

$$\|V_g\|_{e,\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,\eta}} \lesssim \limsup_{r\to 1^-} \|g - g_r\|_{\mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}}$$

$$\approx \operatorname{dist}_{\mathscr{B}^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}} \left(g, \mathscr{B}_0^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}\right).$$

On the other hand, let $\{a_n\}$ be a sequence in $\mathbb{D}$ such that $\lim_{n\to\infty}|a_n|=1$. For each n, let f_{a_n} be defined as in Lemma 3. Then $\{f_{a_n}\}$ is bounded in $\mathscr{A}^{p,\lambda}$. Furthermore, $\{f_{a_n}\}$ converges to zero uniformly on every compact subset of $\mathbb{D}$. Let $a_n=(1-|I_n|)\zeta\in\mathbb{D}$, where ζ is the center of I_n. Then

$$1-|a_n|\approx|1-\overline{a_n}z|\approx|I_n|,\ \ z\in S(I_n).$$

Given a compact operator $T:\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,\eta}$, by Lemma 7 we have $\lim_{n\to\infty}\|Tf_{a_n}\|_{\mathscr{A}^{q,\eta}}=0$. Therefore,

$$\begin{aligned}\|V_g-T\|^q&\gtrsim\limsup_{n\to\infty}\|(V_g-T)f_{a_n}\|^q_{\mathscr{A}^{q,\eta}}\\&\gtrsim\limsup_{n\to\infty}\left(\|V_gf_{a_n}\|^q_{\mathscr{A}^{q,\eta}}-\|Tf_{a_n}\|^q_{\mathscr{A}^{q,\eta}}\right)\\&=\limsup_{n\to\infty}\|V_gf_{a_n}\|^q_{\mathscr{A}^{q,\eta}}\\&\geqslant\limsup_{n\to\infty}\frac{1}{|I_n|^\eta}\int_{S(I_n)}|f_{a_n}(z)|^q|g'(z)|^q(1-|z|^2)^q dA(z)\\&\gtrsim\limsup_{n\to\infty}\frac{1}{|I_n|^{\eta+\frac{q(2-\lambda)}{p}}}\int_{S(I_n)}|g'(z)|^q(1-|z|^2)^q dA(z).\end{aligned}$$

Hence

$$\|V_g\|^q_{e,\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,\eta}}\gtrsim\limsup_{|a_n|\to1^-}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-\eta-\frac{q(2-\lambda)}{p}}(1-|\sigma_{a_n}(z)|^2)^{\eta+\frac{q(2-\lambda)}{p}}dA(z).$$

It follows from Lemma 5 and the arbitrariness of $\{a_n\}$ that

$$\begin{aligned}\|V_g\|^q_{e,\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,\eta}}&\gtrsim\limsup_{|a|\to1^-}\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-\eta-\frac{q(2-\lambda)}{p}}(1-|\sigma_a(z)|^2)^{\eta+\frac{q(2-\lambda)}{p}}dA(z)\\&\approx\limsup_{|z|\to1^-}\left((1-|z|^2)^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}|f'(z)|\right)^q.\quad\square\end{aligned}$$

The following result can be deduced by Theorem 3 directly.

COROLLARY 5. *Suppose that* $0<p\leqslant q<\infty$, $0<\frac{q}{p}\lambda\leqslant\eta<2$ *or* $\eta+\frac{q(2-\lambda)}{p}\geqslant2$, $\eta+\frac{q(2-\lambda)}{p}-2<q\leqslant p<\infty$ *and* $0<\lambda,\eta<2$. *Then* $V_g:\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,\eta}$ *is compact* $g\in\mathscr{B}_0^{\frac{q+2-(\eta+\frac{q(2-\lambda)}{p})}{q}}$.

From Corollary 5, we easy to have the following results.

COROLLARY 6. *Suppose that* $0<q\leqslant p<\infty$ *and* $0<\lambda<2$. *Then* $V_g:\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}$ *is compact* $g\in\mathscr{B}_0$.

COROLLARY 7. *Suppose that* $0<p<\infty$ *and* $0<\lambda<2$. *Then* $V_g:\mathscr{A}^{p,\lambda}\to\mathscr{A}^{p,\lambda}$ *is compact* $g\in\mathscr{B}_0$.

THEOREM 8. *Let* $0<q\leqslant p<\infty$ *and* $0<\lambda<2$. *Suppose that* $g\in H(\mathbb{D})$ *such that* $S_g:\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}$ *is bounded, then*

$$\|S_g\|_{e,\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}\approx\|g\|_{H^\infty}.$$

Proof. Let $\{a_n\}$ and T be defined as in the proof of Theorem 3. Let g_{a_n} be defined as in Lemma 3. Since $T:\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}$ is compact, by Lemma 6 we get $\lim_{n\to\infty}\|Tg_{a_n}\|_{\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}=0$. Hence,

$$\begin{aligned}\|S_g-T\|&\gtrsim\limsup_{n\to\infty}\|(S_g-T)g_{a_n}\|_{\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}\\&\gtrsim\limsup_{n\to\infty}\left(\|S_gg_{a_n}\|_{\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}-\|TF_{w_n}\|_{\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}\right)\\&=\limsup_{n\to\infty}\|S_gg_{a_n}\|_{\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}.\end{aligned}$$

Therefore,

$$\|S_g\|_{e,\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}\gtrsim\limsup_{n\to\infty}\|S_gg_{a_n}\|_{\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}.$$

Similar argument as in the proof of Theorem 2 shows that

$$\|S_gg_{a_n}\|_{\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}\gtrsim|g(a_n)|,$$

which implies that $\|S_g\|_{e,\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}\gtrsim\|g\|_{H^\infty}$.

On the other hand, Theorem 4 gives

$$\|S_g\|_{e,\mathscr{A}^{p,\lambda}\to\mathscr{A}^{q,2-\frac{q(2-\lambda)}{p}}}=\inf_T\|S_g-T\|\leqslant\|S_g\|\lesssim\|g\|_{H^\infty}.\quad\square$$

From Theorem 8, we get the following result.

COROLLARY 8. *Let* $0<q\leqslant p<\infty$ *and* $0<\lambda<2$. *Suppose that* $g\in H(\mathbb{D})$, *then* $S_g:\mathscr{A}^{p,\lambda}\to A^{q,2-\frac{q(2-\lambda)}{p}}$ *is compact if and only if* $g=0$.

5. Closed range of Volterra integral operators

If X and Y are norm spaces, the operator $T:X\to Y$ is bounded below if there exists $C>0$ such that

$$\|Tx\|_Y\geqslant C\|x\|_X$$

for all $x\in X$.

If X and Y are Banach spaces and $T: X \to Y$ is bounded operator. By [1], we known that T is bounded below if and only if T is one to one and has closed range. It is easily to see that V_g is one to one. Thus, V_g is bounded below if and only if V_g has closed range.

The problem of closed range is a fundamental issue in operator theory, and there are many publications studying this issue. However, there has been relatively little research on the closed range of Volterra integral operator, we refer to [2, 17]. In this section, we studying the closed range of Volterra integral operator on Bergman-Morrey Spaces $\mathscr{A}^{p,\lambda}$.

Let us recall the following useful result.

LEMMA 8. ([15]) *Let G be a measurable subset of $\mathbb{D}$, $0<p<\infty$ and $\alpha>-1$. There are $\eta>0$ and $0<r<1$, such that*

$$A(G\cap\mathbb{D}(a,r)) \geqslant \eta A(\mathbb{D}(a,r))$$

if and only if there exist $C>0$ such that

$$\int_G |f(z)|^p(1-|z|^2)^\alpha dA(z) \geqslant C\int_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha dA(z).$$

THEOREM 9. *Let $0<p<\infty$ and $0<\lambda<1$. Suppose that $g\in\mathscr{B}$, then $V_g: \mathscr{A}^{p,\lambda}\to\mathscr{A}^{p,\lambda}$ has closed range if and only if there is an $r\in(0,1)$ and $a,c>0$ such that*

$$A(G_c\cap\mathbb{D}(a,r)) \geqslant \delta A(\mathbb{D}(a,r)),$$

where $G_c=\{z\in\mathbb{D}:(1-|z|^2)|g'(z)|>c\}$.

Proof. Sufficiency. Since V_g is one to one, we only need to prove V_g is bounded below. Suppose that $f\in\mathscr{A}^{p,\lambda}=F(p,p-\lambda,\lambda)=\mathscr{N}(p,-\lambda,\lambda)$. Let

$$h_a(z)=\frac{f(z)(1-|a|^2)^{\frac{\lambda}{p}}}{(1-\overline{a}z)^{\frac{2\lambda}{p}}},\quad a,z\in\mathbb{D}.$$

Then $h_a\in A^p$. Thus,

$$\begin{aligned}
&\sup_{w\in\mathbb{D}}\int_{\mathbb{D}}|(V_gf)'(z)|^p(1-|z|^2)^{p-\lambda}(1-|\sigma_w(z)|^2)^\lambda dA(z)\\
&\geqslant \int_{\mathbb{D}}|f(z)|^p|g'(z)|^p(1-|z|^2)^{p-\lambda}(1-|\sigma_a(z)|^2)^\lambda dA(z)\\
&\geqslant c^p\int_{G_c}|f(z)|^p(1-|z|^2)^{-\lambda}(1-|\sigma_a(z)|^2)^\lambda dA(z)\\
&= c^p\int_{G_c}|h_a(z)|^p dA(z)\\
&\geqslant C\int_{\mathbb{D}}|h_a(z)|^p dA(z)\ \text{ (by Lemma 8)}\\
&\gtrsim \int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^{-\lambda}(1-|\sigma_a(z)|^2)^\lambda dA(z).
\end{aligned}$$

Hence, we deduced that

$$\|V_g f\|^p_{\mathscr{A}^{p,\lambda}} \gtrsim \|f\|^p_{\mathscr{A}^{p,\lambda}}.$$

Necessity. Suppose that $V_g : \mathscr{A}^{p,\lambda} \to \mathscr{A}^{p,\lambda}$ has closed range, then for $f \in \mathscr{A}^{p,\lambda}$, we have

$$\|V_g f\|^p_{\mathscr{A}^{p,\lambda}} \gtrsim \|f\|^p_{\mathscr{A}^{p,\lambda}}.$$

For $a,z \in \mathbb{D}$, let $f_a(z) = \frac{1-|a|^2}{(1-\overline{a}z)^{1+\frac{2-\lambda}{p}}}$. By Lemma 3, we have $f_a \in \mathscr{A}^{p,\lambda}$ and $\|f_a\|_{\mathscr{A}^{p,\lambda}} \approx 1$. Therefore,

$$\begin{aligned}
C &\leqslant \|V_g f_a\|^p_{\mathscr{A}^{p,\lambda}} \\
&= \sup_{w\in\mathbb{D}} \int_{\mathbb{D}} |(V_g f_a)'(z)|^p (1-|z|^2)^{p-\lambda}(1-|\sigma_w(z)|^2)^{\lambda} dA(z) \\
&\leqslant \int_{\mathbb{D}} |f_a(z)|^p |g'(z)|^p (1-|z|^2)^{p-\lambda} dA(z) \\
&= M_1 + M_2 + M_3,
\end{aligned}$$

where

$$M_1 = \int_{G_c \cap \mathbb{D}(a,r)} |f_a(z)|^p |g'(z)|^p (1-|z|^2)^{p-\lambda} dA(z),$$

$$M_2 = \int_{\mathbb{D}(a,r)\setminus G_c} |f_a(z)|^p |g'(z)|^p (1-|z|^2)^{p-\lambda} dA(z)$$

and

$$M_3 = \int_{\mathbb{D}\setminus\mathbb{D}(a,r)} |f_a(z)|^p |g'(z)|^p (1-|z|^2)^{p-\lambda} dA(z).$$

Noted that

$$1-|a|^2 \approx 1-|z|^2$$

and

$$(1-|a|^2)^2 \approx A(\mathbb{D}(a,r)), \quad z \in \mathbb{D}(a,r).$$

An easy computation gives

$$\begin{aligned}
M_1 &\lesssim \frac{1}{(1-|a|^2)^{2-\lambda}} \int_{G_c\cap\mathbb{D}(a,r)} |g'(z)|^p (1-|z|^2)^{p-\lambda} dA(z) \\
&\lesssim \frac{\|g\|^p_{\mathscr{B}}}{(1-|a|^2)^{2-\lambda}} \int_{G_c\cap\mathbb{D}(a,r)} (1-|z|^2)^{-\lambda} dA(z) \leqslant C_1 \|g\|^p_{\mathscr{B}} \frac{A(G_c\cap\mathbb{D}(a,r))}{A(\mathbb{D}(a,r))}.
\end{aligned}$$

We can also easily to deduced that

$$\begin{aligned}
M_2 &\leqslant c^p \int_{\mathbb{D}(a,r)\setminus G_c} \frac{(1-|a|^2)^p}{|1-\overline{a}z|^{p+2-\lambda}} (1-|z|^2)^{-\lambda} dA(z) \\
&\leqslant c^p \int_{\mathbb{D}} \frac{(1-|a|^2)^p}{|1-\overline{a}z|^{p+2-\lambda}} (1-|z|^2)^{-\lambda} dA(z) \leqslant C_2 \cdot c^p.
\end{aligned}$$

Now, we are going to estimated M_3. Making change of variables $z = \varphi_a(w)$, we have

$$\begin{aligned} M_3 &\leqslant \|g\|_{\mathscr{B}}^p \int_{\mathbb{D}\setminus\mathbb{D}(a,r)} |f_a(z)|^p (1-|z|^2)^{-\lambda} dA(z) \\ &= \|g\|_{\mathscr{B}}^p \int_{\mathbb{D}\setminus\mathbb{D}(0,r)} |f_a(\varphi_a(w))|^p (1-|\varphi_a(w)|^2)^{-\lambda} |(\varphi_a(w))'|^2 A(w) \\ &= \|g\|_{\mathscr{B}}^p \int_{\mathbb{D}\setminus\mathbb{D}(0,r)} \frac{(1-|w|^2)^{-\lambda}}{|1-\overline{a}w|^{2-p-\lambda}} A(w). \end{aligned}$$

Noted that

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{-\lambda}}{|1-\overline{a}w|^{2-p-\lambda}} A(w) < \infty.$$

Thus, for any $\varepsilon > 0$, there exists $0 < r < 1$ such that

$$\int_{\mathbb{D}\setminus\mathbb{D}(0,r)} \frac{(1-|w|^2)^{-\lambda}}{|1-\overline{a}w|^{2-p-\lambda}} A(w) < \varepsilon.$$

Therefore, let r close enough to 1, so that ε is small enough such that $\varepsilon\|g\|_{\mathscr{B}}^p < \frac{C}{3}$. And let c small enough so that $C_2 \cdot c^p < \frac{C}{3}$. Hence, we have

$$\frac{C}{3} \leqslant C_1 \|g\|_{\mathscr{B}}^p \frac{A(G_c \cap \mathbb{D}(a,r))}{A(\mathbb{D}(a,r))}.$$

The proof is completed. □

It is easy to see that S_g is not one to one when S_g acting on differential constant. Thus, we only consider the spaces $\mathscr{A}^{p,\lambda} \setminus C = \{f \in \mathscr{A}^{p,\lambda} : \ f(0) = 0\}$. Following the proof of [31], we easily to have $S_g : \mathscr{A}^{p,\lambda} \setminus C \to \mathscr{A}^{p,\lambda} \setminus C$ is bounded if and only if $g \in H^\infty$.

THEOREM 10. *Let $0 < p < \infty$ and $0 < \lambda < 1$ (or $1 < p < \infty$ and $0 < \lambda < 2$). Suppose that $g \in H^\infty$, then $S_g : \mathscr{A}^{p,\lambda} \setminus C \to \mathscr{A}^{p,\lambda} \setminus C$ has closed range if and only if there is an $\eta \in (0,1)$ and $a, c > 0$ such that*

$$A(G_c \cap \Delta(a,\delta)) \geqslant \delta A(\Delta(a,\delta)),$$

where $G_c = \{z \in \mathbb{D} : |g(z)| > c\}$.

Proof. We using the same idea as Theorem 9.

Sufficiency. We only need to prove S_g is bounded below. Let $f \in \mathscr{A}^{p,\lambda} = F(p, p-\lambda, \lambda)$ and

$$J_a(z) = \frac{f'(z)(1-|a|^2)^{\frac{\lambda}{p}}}{(1-\overline{a}z)^{\frac{2\lambda}{p}}}, \ \ a, z \in \mathbb{D}.$$

Thus $J_a \in A_p^p$ and we have

$$\begin{aligned}
&\sup_{w\in\mathbb{D}}\int_{\mathbb{D}} |(S_g f)'(z)|^p(1-|z|^2)^{p-\lambda}(1-|\sigma_w(z)|^2)^{\lambda}dA(z)\\
&\geqslant \int_{\mathbb{D}} |f'(z)|^p|g(z)|^p(1-|z|^2)^{p-\lambda}(1-|\sigma_a(z)|^2)^{\lambda}dA(z)\\
&\geqslant c^p\int_{G_c} |f'(z)|^p(1-|z|^2)^{p-\lambda}(1-|\sigma_a(z)|^2)^{\lambda}dA(z)\\
&= c^p\int_{G_c} |J_a(z)|^p(1-|z|^2)^{p}dA(z)\\
&\geqslant C\int_{\mathbb{D}} |J_a(z)|^p(1-|z|^2)^{p}dA(z) \;\; \text{(by Lemma 8)}\\
&\gtrsim \int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-\lambda}(1-|\sigma_a(z)|^2)^{\lambda}dA(z).
\end{aligned}$$

That is

$$\|S_g f\|^p_{\mathscr{A}^{p,\lambda}} \gtrsim \|f\|^p_{\mathscr{A}^{p,\lambda}}.$$

Necessity. Suppose that $S_g : \mathscr{A}^{p,\lambda} \to \mathscr{A}^{p,\lambda}$ has closed range, then for $f \in \mathscr{A}^{p,\lambda}$, we have

$$\|S_g f\|^p_{\mathscr{A}^{p,\lambda}} \gtrsim \|f\|^p_{\mathscr{A}^{p,\lambda}}.$$

For $a,z\in\mathbb{D}$, let $g_a(z) = \frac{1-|a|^2}{\overline{a}(1-\overline{a}z)^{1+\frac{2-\lambda}{p}}}$. Thus, $g_a \in \mathscr{A}^{p,\lambda}$ and $\|g_a\|_{\mathscr{A}^{p,\lambda}} \approx 1$ by Lemma 3. Therefore,

$$\begin{aligned}
C &\leqslant \|S_g g_a\|^p_{\mathscr{A}^{p,\lambda}}\\
&= \sup_{w\in\mathbb{D}}\int_{\mathbb{D}} |(S_g g_a)'(z)|^p(1-|z|^2)^{p-\lambda}(1-|\sigma_w(z)|^2)^{\lambda}dA(z)\\
&\leqslant \int_{\mathbb{D}} |g_a'(z)|^p|g(z)|^p(1-|z|^2)^{p-\lambda}dA(z)\\
&= L_1+L_2+L_3,
\end{aligned}$$

where

$$L_1 = \int_{G_c\cap\mathbb{D}(a,r)} |g_a'(z)|^p|g(z)|^p(1-|z|^2)^{p-\lambda}dA(z),$$

$$L_2 = \int_{\mathbb{D}(a,r)\setminus G_c} |g_a'(z)|^p|g(z)|^p(1-|z|^2)^{p-\lambda}dA(z)$$

and

$$L_3 = \int_{\mathbb{D}\setminus\mathbb{D}(a,r)} |g_a'(z)|^p|g(z)|^p(1-|z|^2)^{p-\lambda}dA(z).$$

An easy computation gives

$$\begin{aligned} L_1 &\leqslant \|g\|_{H^\infty}^p \int_{G_c\cap\mathbb{D}(a,r)} |g_a'(z)|^p (1-|z|^2)^{p-\lambda} dA(z) \\ &\leqslant \|g\|_{H^\infty}^p \int_{G_c\cap\mathbb{D}(a,r)} \frac{(1-|a|^2)^p}{|1-\overline{a}z|^{2p+2-\lambda}} (1-|z|^2)^{p-\lambda} dA(z) \\ &\leqslant c_1 \|g\|_{H^\infty}^p \frac{A(G_c\cap\mathbb{D}(a,r))}{A(\mathbb{D}(a,r))}. \end{aligned}$$

We can also easily to deduced that

$$\begin{aligned} L_2 &\leqslant c^p \int_{\mathbb{D}(a,r)\setminus G_c} \frac{(1-|a|^2)^p}{|1-\overline{a}z|^{2p+2-\lambda}} (1-|z|^2)^{p-\lambda} dA(z) \\ &\leqslant c^p \int_{\mathbb{D}} \frac{(1-|a|^2)^p}{|1-\overline{a}z|^{2p+2-\lambda}} (1-|z|^2)^{p-\lambda} dA(z) \leqslant c_2\cdot c^p. \end{aligned}$$

Now, we are going to estimated L_3. Making change of variables $z=\varphi_a(w)$, we have

$$\begin{aligned} L_3 &\leqslant \|g\|_{H^\infty}^p \int_{\mathbb{D}\setminus\mathbb{D}(a,r)} |g_a'(z)|^p (1-|z|^2)^{p-\lambda} dA(z) \\ &= \|g\|_{H^\infty}^p \int_{\mathbb{D}\setminus\mathbb{D}(0,r)} |g_a'(\varphi_a(w))|^p (1-|\varphi_a(w)|^2)^{p-\lambda} |(\varphi_a(w))'|^2 A(w) \\ &\leqslant \|g\|_{H^\infty}^p \int_{\mathbb{D}\setminus\mathbb{D}(0,r)} \frac{(1-|w|^2)^{p-\lambda}}{|1-\overline{a}w|^{2-\lambda}} A(w). \end{aligned}$$

Noted that

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{p-\lambda}}{|1-\overline{a}w|^{2-\lambda}} A(w) < \infty.$$

Thus, for any $\varepsilon>0$, there exists $0<r<1$ such that

$$\int_{\mathbb{D}\setminus\mathbb{D}(0,r)} \frac{(1-|w|^2)^{-\lambda}}{|1-\overline{a}w|^{2-p-\lambda}} A(w) < \varepsilon.$$

Therefore, let r close enough to 1, so that ε is small enough such that $\varepsilon\|g\|_{H^\infty}^p < \frac{C}{3}$. And let c small enough so that $c_2\cdot c^p < \frac{C}{3}$. Hence, we have

$$\frac{C}{3} \leqslant c_1 \|g\|_{H^\infty}^p \frac{A(G_c\cap\mathbb{D}(a,r))}{A(\mathbb{D}(a,r))}.$$

The proof is completed. □

REFERENCES

[1] Y. ABRAMOVICH AND C. ALIPRANTIS, *An Invitation to Operator Theory*, Graduate Studies in Mathematics, AMS, vol. 50, (2002).

[2] A. ANDERSON, *Some Closed Range Integral Operators on Spaces of Analytic Functions*, Integr. Equ. Oper. Theory **69** (2011), 87–99.

[3] A. ALEMAN AND A. SISKAKIS, *An integral operator on H^p*, Complex Var. Theory Appl. **28** (1955), 149–158.

[4] A. ALEMAN AND A. SISKAKIS, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), 337–356.

[5] P. DUREN, *Theory of H^p Spaces*, Academic Press, New York, 1970.

[6] M. ESSEN, H. WULAN AND J. XIAO, *Several function-theoretic characterizations of Möbius invariant $\mathcal{Q}_K$ spaces*, J. Funct. Anal. **230** (2006), 78-115.

[7] P. GALANOPOULOS, N. MERCHAN AND A. G. SISKAKIS, *A family of Dirichlet-Morrey spaces*, Complex Var. Ellt. Equ. **64** (2019), 1686–1702.

[8] J. GARNETT, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[9] D. GIRELA, *Analytic functions of bounded mean oscillation*, Complex Function Spaces (Mekrijärvi, 1999), 61–170, Univ. Joensuu Dept. Math. Rep. Ser. **4**, Univ. Joensuu, Joensuu, 2001.

[10] D. GIRELA AND J. PELÁEZ, *Carleson measures for spaces of Dirichlet type*, Integral Equations Operator Theory **55** (2006), 415–427.

[11] D. GIRELA AND J. PELÁEZ, *Carleson measures, multipliers and integration operators for spaces of Dirichlet type*, J. Funct. Anal. **241** (2006), 334–358.

[12] B. HU AND S. LI, *$\mathcal{N}(p,q,s)$-type spaces in the unit ball of $\mathbb{C}^n$ (III): Various characterizations*, Publ. Math. Debrecen. **97** (2020), 41–61.

[13] P. LI, J. LIU AND Z. LOU, *Integral operators on analytic Morrey spaces*, Sci. China Math. **57** (2014), 1961–1974.

[14] S. LI, J. LIU AND C. YUAN, *Embedding theorems for Dirichlet type spaces*, Canad. Math. Bull. **63** (2020), 106–117.

[15] D. LUECKING, *Inequalities on Bergman spaces*, Illinois J. Math. 25 (1981), 1–11.

[16] C. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938), 126–166.

[17] K. PANTERIS, *Closed range integral operators on Hardy, BMOA and Besov spaces*, Complex Var. Ellt. Equ. **67** (2021), 2011–2029.

[18] J. PAU AND R. ZHAO, *Carleson measures, Riemann-Stieltjes and multiplication operators on a general family of function spaces*, Integr. Equ. Oper. Theory **78** (2014), 483–514.

[19] C. POMMERENKE, *Schlichte funktionen und analytische funktionen von beschränkten mittlerer Oszillation*, Comm. Math. Helv. **52** (1977), 591–602.

[20] R. QIAN AND S. LI, *Volterra type operators on Morrey type spaces*, Math. Inequal. Appl. **18** (2015), 1589–1599.

[21] R. QIAN AND X. ZHU, *Embedding Hardy spaces H^p into tent spaces and generalized inte*, Annales Polonici Mathematici. **128** (2022), 143–157.

[22] R. QIAN AND X. ZHU, *Integral operators from Morrey type spaces to weighted BMOA spaces*, Bull. Korean Math. Soc. **58** (2021), 815–828.

[23] Y. SHI AND S. LI, *Essential norm of integral operators on Morrey type spaces*, Math. Inequal. Appl. **19** (2016), 385–393.

[24] A. SISKAKIS AND R. ZHAO, *A Volterra type operator on spaces of analytic functions*, (Edwardsville, IL, 1998), Contemp. Math. American Mathematical Society. Rhode Island. **232** (1999), 299–311.

[25] M. TJANI, *Compact Composition Operators on Some Möbius Invariant Banach Spaces*, PhD dissertation, Michigan State University, 1996.

[26] Z. WU AND C. XIE, *$\mathcal{Q}_p$ spaces and Morrey spaces*, J. Funct. Anal. **201** (2003), 282–297.

[27] H. WULAN AND J. ZHOU, *$\mathcal{Q}_K$ and Morrey type spaces*, Ann. Acad. Sci. Fenn. Math. **38** (2013), 193–207.

[28] J. XIAO, *Holomorphic $\mathcal{Q}$ Classes*, Springer, LNM 1767, Berlin, 2001.

[29] J. XIAO, *The Q_p Carleson measure problem*, Adv. Math. **217** (2008), 2075–2088.

[30] J. XIAO AND C. YUAN, *Analytic Campanato spaces and their compositions*, Indian. Univ. Math. J. **64** (2015), 1001–1025.

[31] Y. YANG, J. LIU, *Integral operators on Bergman-Morrey spaces*, J. Geom. Anal. **32** (2022), Paper No. 181, 23 p.

[32] R. ZHAO, *On a general family of function spaces*, Ann. Acad. Sci. Fenn. Math. Diss. **105** (1996), 56 pp.

[33] J. ZHOU AND X. ZHU, *Essential norm of a Volterra-type integral operator from Hardy spaces to some analytic function spaces*, J. Integral Equations Applications. **28** (2016), 581–593.

[34] K. ZHU, *Bloch type spaces of analytic functions*, Rocky Mountain J. Math. **23** (1993), 1143–1177.

[35] K. ZHU, *Operator Theory in Function Spaces*, Mathematical Society, Providence, RI, 2007.

[36] X. ZHU, L. HU AND D. QU, *Dirichlet-Morrey type spaces and Volterra integral operators*, Complex Var. Ellt. Equ. **69** (2024), 301–316.

(Received October 31, 2024)

Ruishen Qian
School of Mathematics and Computation Science
Lingnan normal University
524048, Zhanjiang, Guangdong, P. R. China
e-mail: qianruishen@sina.cn

Zhonghua He
School of Financial Mathematics and Statistics
Guangdong University of Finance
510521, Guangzhou, Guangdong, P. R. China
e-mail: zhonghuahe2010@163.com