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ON K–FRAMES GENERATED BY OPERATORS ON HILBERT SPACES

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Abstract. The aim of this paper is to analyze K-frames generated by a bounded linear operator on a separable Hilbert space $\mathscr{H}$. First, we establish some lower bounds for the norm of an operator T when the sequence $\{T^n g\}_{g\in G, n\geqslant 0}$ satisfies the lower K-frame bound for some set $G\subset\mathscr{H}$. Furthermore, we derive a necessary condition for the sequence $\{T^n g\}_{g\in G, n\geqslant 0}$ to be a K-frame. As a consequence, we prove that the hypercyclic operator T with a hypercyclic vector in the range of K cannot generate a K-frame. Additionally, under certain conditions, we construct a Parseval iterative K-frame using an operator. Finally, we determine the form of the K-dual for K-frames generated by an operator.

1. Introduction

The concept of frames was first introduced in Hilbert spaces by Duffin and Schaeffer [13] in 1952, in their research on nonharmonic Fourier series. After almost 3 decades, in 1986, Daubechies et al. [12] gave new life to frameworks. It is well known that frames can be used as redundant bases to cover the whole of Hilbert space. Frames are becoming increasingly significant not only in theory but in a wide range of applications, and have found many applications in sampling theory, signal processing, coding and communications, filter bank theory, and so on [4, 14–16, 25]. With the expansion of frames theory, several specific types of frames are proposed, including fusion frames, weaving frames, g-frames, and K-frames [5, 6, 17, 26]. This paper focuses on K-frames, which were recently introduced by L. Găvruţa to study atomic systems in terms of a bounded linear operator K in Hilbert spaces [17]. K-frames have been investigated by many researchers. Liang et al. explored the relationship between K-frames and operator K [22]. Xiao et al. discussed the interchangeability of two Bessel sequences for K-frames and the stability of a general perturbation for K-frames in [27]. K-frames are a generalization of frames; in fact, a K-frame is precisely a frame when $K = I_{\mathscr{H}}$.

The first formulation of a frame with the structure $\{T^n f\}_{n=0}^{\infty}$ connected with dynamical sampling was presented by Aldroubi et al. [2], and further studied in [1,3]. This type of frame was also characterized by Christensen et al. [9]. For more information on these developments, readers are referred to [8, 10, 11]. In addition to recent developments, some foundational contributions that underline the interplay between frames and

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operator theory include the work of Han and Larson [19], who investigated frame structures arising from group representations, and P. Găvruţa [18], who explored wavelet systems and spectral measures through operator-theoretic lenses. Leng et al. in [20] studied a K-frame $\{f_n\}_{n\in\mathbb{Z}}$ for a Hilbert space $\mathscr{H}$, which has the form $\{T^n f_0\}_{n\in\mathbb{Z}}$ for some operator T. They investigated conditions under which a K-frame can be represented by an operator and discussed the properties of that operator.

A K-frame for a Hilbert space $\mathscr{H}$, generated by an operator, is the main topic of this article. This study is motivated by the desire to better understand the interplay between operator theory and frame theory in the generalized setting of K-frames. While classical frame theory focuses on reconstruction of elements in $\mathscr{H}$ itself, K-frames allow for reconstruction in the range of a given bounded linear operator K, thus providing a more flexible and potentially more efficient framework for signal representation, especially in contexts where exact reconstruction is not required or not possible in the full space. This framework is particularly relevant in inverse problems, sampling in subspaces, and signal recovery under system constraints [2, 3, 17].

Throughout this paper, we assume that $\mathscr{H}$ is a separable Hilbert space, I and J countable index sets and $I_{\mathscr{H}}$ is the identity operator on $\mathscr{H}$. For two Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ we denote the collection of all bounded linear operators between $\mathscr{H}_1$ and $\mathscr{H}_2$ by $B(\mathscr{H}_1,\mathscr{H}_2)$. Moreover, $GL(\mathscr{H}_1,\mathscr{H}_2)$ will denote the set of all bijective operators in $B(\mathscr{H}_1,\mathscr{H}_2)$. As usual, we set $B(\mathscr{H}) := B(\mathscr{H},\mathscr{H})$ and $GL(\mathscr{H}) := GL(\mathscr{H},\mathscr{H})$. Also, we denote the range and the null space of $K \in B(\mathscr{H})$ by $\mathscr{R}(K)$ and $\mathscr{N}(K)$, respectively. The orthogonal projection of $\mathscr{H}$ onto a closed subspace $M \subseteq \mathscr{H}$ is denoted by π_M.

In Section 2, we present definitions and basic properties of the concepts used throughout the paper. In Section 3, we first provide a bound for the norm of the operator T when it generates a sequence satisfying the lower K-frame bound. Then, we establish a necessary condition for the sequence $\{T^n g\}_{g\in G, n\geqslant 0}$, where $G \subset \mathscr{H}$, to be a K-frame. Additionally, for a given contraction operator T and under certain conditions, we determine a set G for which the sequence $\{T^n g\}_{g\in G, n\geqslant 0}$ constitutes a Parseval K-frame. We also prove that hypercyclic operators with a hypercyclic vector in $\mathscr{R}(K)$ cannot generate a K-frame. Furthermore, we discuss the lower bound of a K-frame whose frame operator generates a K-frame. Section 4 focuses on the properties of K-dual frames corresponding to a K-frame of the form $\{f_n\}_{n\in\mathbb{Z}} = \{T^n f_0\}_{n\in\mathbb{Z}}$. Under specific conditions, we characterize the canonical K-dual frames represented by a bounded bijective operator.

Overall, our findings generalize several results from the classical iterative frame setting to the more general K-frame framework, and lay the foundation for further investigations into potential applications.

2. Preliminaries

In this section, we provide some preliminary information and background on the theory of the concepts used in this note.

2.1. Frames

A sequence $\{f_n\}_{n\in I}$ of elements of the Hilbert space $\mathscr{H}$ is called a *frame* for $\mathscr{H}$ if there are constants $A,B>0$ such that

$$A\|f\|^2 \leqslant \sum_{n\in I}|\langle f,f_n\rangle|^2 \leqslant B\|f\|^2, \qquad (f\in\mathscr{H}).$$

The numbers A and B are referred to as the lower and upper frame bounds, respectively. The frame is considered a tight frame if $A=B$ and a Parseval frame if $A=B=1$. If only the right inequality is satisfied, then $\{f_n\}_{n\in I}$ is called a Bessel sequence for $\mathscr{H}$. It is well-known that for any frame $\{f_n\}_{n\in I}$, there exists at least one alternate dual frame, meaning a Bessel sequence $\{g_n\}_{n\in I}$ such that

$$f=\sum_{n\in I}\langle f,g_n\rangle f_n, \qquad (f\in\mathscr{H}).$$

It is well known that in this case $\{g_n\}_{n\in I}$ is also a frame and $\{f_n\}_{n\in I}$ is one of its alternate duals. If $\{f_n\}_{n\in I}$ is a Bessel sequence the *synthesis operator* can be defined as

$$U:\ell^2(I)\to\mathscr{H}, \qquad U\big(\{c_n\}_{n\in I}\big):=\sum_{n\in I}c_nf_n.$$

It is well-defined and a bounded operator and its adjoint, called the *analysis operator*, is given by $U^*(f)=\{\langle f,f_n\rangle\}_{n\in I}$. The *frame operator* is given by

$$S:\mathscr{H}\to\mathscr{H}, \qquad Sf:=UU^*f=\sum_{n\in I}\langle f,f_n\rangle f_n, \qquad (f\in\mathscr{H}).$$

For a frame $\{f_n\}_{n\in I}$, the *frame operator* is bounded, positive, invertible and for any $f\in\mathscr{H}$, $f=\sum_{n\in I}\langle f,S^{-1}f_n\rangle f_n$. The sequence $\{S^{-1}f_n\}_{n\in I}$ is also a frame, which is called the *canonical dual frame*.

Let $I=\mathbb{N}\cup\{0\}$ or $I=\mathbb{Z}$. We consider frames $\{f_n\}_{n\in I}$ in a Hilbert space $\mathscr{H}$ arising via iterated action of a linear operator $T:\operatorname{span}\{f_n\}_{n\in I}\to\operatorname{span}\{f_n\}_{n\in I}$, i.e., frames of the form $\{f_n\}_{n\in I}=\{T^nf_0\}_{n\in I}$. In this case, we say that the frame $\{f_n\}_{n\in I}$ is generated by the operator T.

2.2. K-frames and its dual

Let $K\in B(\mathscr{H})$. A sequence $\{f_n\}_{n\in I}\subseteq\mathscr{H}$ is called a K*-frame* for $\mathscr{H}$, if there exist constants $A,B>0$ such that

$$A\|K^*f\|^2 \leqslant \sum_{n\in I}|\langle f,f_n\rangle|^2 \leqslant B\|f\|^2, \qquad (f\in\mathscr{H}). \tag{2.1}$$

The constants A and B in (2.1) are called the *lower* and the *upper* K-frame bounds, respectively. A K-frame is called a *tight* K-frame if there exists $A>0$ such that

$$A\|K^*f\|^2=\sum_{n\in I}|\langle f,f_n\rangle|^2, \qquad (f\in\mathscr{H}),$$

and a *Parseval* K-frame if this holds with $A = 1$. Every K-frame is obviously a Bessel sequence; therefore, similar to ordinary frames, we can define its synthesis operator, analysis operator and frame operator.

Many properties that hold for ordinary frames do not apply to K-frames. For example, the corresponding synthesis operator for K-frames is not surjective, so the frame operator for K-frames is not generally invertible. It is important to note that if K has a closed range, then the operator S from $\mathscr{R}(K)$ onto $S(\mathscr{R}(K))$ is invertible.

Let $\{f_n\}_{n\in I}$ be a K-frame. A Bessel sequence $\{g_n\}_{n\in I} \subseteq \mathscr{H}$ is called a K-dual of $\{f_n\}_{n\in I}$ if

$$Kf = \sum_{n\in I}\langle f, g_n\rangle f_n, \qquad (f \in \mathscr{H}). \tag{2.2}$$

It was proven that for every K-frame of $\mathscr{H}$, there exists at least one Bessel sequence $\{g_n\}_{n\in I}$ which satisfies (2.2). The sequences $\{f_n\}_{n\in I}$ and $\{g_n\}_{n\in I}$ in (2.2) are not interchangeable in general. More specifically, from (2.2), it follows that

$$K^* f = \sum_{n\in I}\langle f, f_n\rangle g_n, \qquad (f \in \mathscr{H}).$$

Therefore, $\{f_n\}_{n\in I}$ and $\{g_n\}_{n\in I}$ in (2.2) are interchangeable if and only if K is self-adjoint. For more details, refer to [17, 27].

The following theorem will be used later on. We apply this result to the unital C^*-algebra $B(\mathscr{H})$. Recall that in a C^*-algebra $\mathscr{A}$, we denote $\mathscr{A}_{sa}$ as the set of all self-adjoint elements and $\mathscr{A}^+$ is used for positive elements of $\mathscr{A}$.

THEOREM 2.1. ([23]) *Let $\mathscr{A}$ be a C^*-algebra.*

(i) *If $a, b \in \mathscr{A}_{sa}$ and $c \in \mathscr{A}$, then $a \leqslant b$ implies $c^*ac \leqslant c^*bc$.*

(ii) *If for $a, b \in \mathscr{A}^+$ with $ab = ba$, $a \leqslant b$ then $a^2 \leqslant b^2$.*

(iii) *If $a, b \in \mathscr{A}^+$ and $ab = ba$ then $ab \geqslant 0$.*

3. K-frames generated by an operator

In this section, we obtain bounds for the norm of the operator T when the sequence $\{T^n g\}_{g\in G, n\geqslant 0}$ satisfies the lower K-frame bound for some subset G of $\mathscr{H}$. Moreover, we provide a necessary condition for the sequence $\{T^n g\}_{g\in G, n\geqslant 0}$ to be a K-frame and we construct a Parseval K-frame using an operator $T \in B(\mathscr{H})$. Furthermore, the relationship between two K-frames is presented.

Let us start by providing a concrete example of an iterative K-frame.

EXAMPLE 3.1. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for $\mathscr{H}$. Let $K \in B(\mathscr{H})$ be defined by

$$Ke_n = \begin{cases} 0 & \text{if } n = 1, 2, 3; \\ \frac{1}{n} e_n & \text{if } n \geqslant 4. \end{cases}$$

By [21, Example 3.7], $\{f_n\}_{n=1}^{\infty} = \{\frac{1}{n} e_n\}_{n=1}^{\infty}$ is a K-frame. One can see that $\{f_n\}_{n=1}^{\infty} = \{T^n f_1\}_{n=0}^{\infty}$, where $Te_n = \frac{n}{n+1} e_{n+1}$ for any $n \in \mathbb{N}$.

Consider $I=\mathbb{N}\cup\{0\}$ or $I=\mathbb{Z}$ and $\{T^n f_0\}_{n\in I}$ is a frame and $K\in B(\mathscr{H})$ with the property that $KT^n f_0 = T^n K f_0$ for all $n\in I$. We also assume that T is invertible when $I=\mathbb{Z}$. Then T generates a K-frame with the vector Kf_0. Indeed, for some $A>0$ and any $f\in\mathscr{H}$, we have

$$\begin{aligned}A\|K^* f\|^2 &\leqslant \sum_{n\in I}|\langle K^* f, T^n f_0\rangle|^2\\ &= \sum_{n\in I}|\langle f, KT^n f_0\rangle|^2\\ &= \sum_{n\in I}|\langle f, T^n K f_0\rangle|^2,\end{aligned}$$

it follows that $\{T^n K f_0\}_{n\in I}$ is a K-frame with the upper K-frame bound $B\|K\|^2$, where B is the upper bound of $\{T^n f_0\}_{n\in I}$. In particular if $KT=TK$, then $KT^n f_0 = T^n K f_0$ for all $n\in I$, so the above assertions are valid. Trivially, if $\{T^n f_0\}_{n\in\mathbb{Z}}$ is a frame, then it is a K-frame with $K=T$.

Note that if $\{T^n f_0\}_{n\in\mathbb{Z}}$ is a K-frame for $K=T$, then according to [21, Remark 1.1], the synthesis operator U of $\{T^n f_0\}_{n\in\mathbb{Z}}$ satisfies $\mathscr{H}=\mathscr{R}(T)\subseteq\mathscr{R}(U)$. This implies that U is onto, confirming that $\{T^n f_0\}_{n\in\mathbb{Z}}$ is indeed a frame.

Our first result concerns the bound for the norm of the operator T when the sequence $\{T^n g\}_{g\in G, n\geqslant 0}$ satisfies the lower K-frame bound. We require the following lemma.

LEMMA 3.2. *Let $\mathscr{H}$ be an infinite dimensional Hilbert space, H_0 and H_1 be two subspace of $\mathscr{H}$ with* $\dim H_0<\infty$ *and* $\dim H_1=\infty$. *Then* $\dim(H_0^\perp\cap H_1)=\infty$.

Proof. Let $H_0=\operatorname{span}\{f_1,\dots,f_n\}$ and define $\psi:\mathscr{H}\to\mathbb{C}^n$ by $\psi(f)=(\langle f,f_1\rangle,\dots,\langle f,f_n\rangle)$. Trivially, $\psi|_{H_1}$ is a linear operator. Also, by the definition of ψ, $\mathscr{N}(\psi|_{H_1})\subseteq H_0^\perp$ hence $\mathscr{N}(\psi|_{H_1})$ is a subset of $H_0^\perp\cap H_1$. But $\dim\mathscr{N}(\psi|_{H_1})=\infty$ which implies that $\dim(H_0^\perp\cap H_1)=\infty$. $\square$

THEOREM 3.3. *If $\mathscr{R}(K)$ is closed and infinite dimensional, $|G|<\infty$ and $\{T^n g\}_{g\in G,n\geqslant 0}$ satisfies the lower K-frame bound, then $\|T\|\geqslant 1$.*

Proof. Assume that $\|T\|<1$. The set G is finite and $\mathscr{R}(K)$ is infinite dimensional, so by putting $H_N:=\{T^n g;\, 0\leqslant n\leqslant N, g\in G\}$, for some fixed N and using Lemma 3.2, we have $H_N^\perp\cap\mathscr{R}(K)\neq\{0\}$. Consider a nonzero vector $f_N\in H_N^\perp\cap\mathscr{R}(K)$. Trivially $f_N\notin\mathscr{R}(K)^\perp=\mathscr{N}(K^*)$. Put $f_N':=\frac{f_N}{\|K^* f_N\|}$. Since $\mathscr{R}(K)$ is closed and the mapping $K:\mathscr{H}\to\mathscr{R}(K)$ is surjective, by [7, Lemma 2.4.1], $K^*:\mathscr{R}(K)\to\mathscr{H}$ is bounded below. Thus there exists $M>0$ such that $\|K^* f\|\geqslant M\|f\|$ for each $f\in\mathscr{R}(K)$, which implies that $\|f_N'\|=\frac{\|f_N\|}{\|K^* f_N\|}\leqslant\frac{1}{M}$. On the other hand, since $f_N\in H_N^\perp$, we get

$\langle T^n g, f'_N\rangle = 0$, for all $g \in G$ and $0 \leqslant n \leqslant N$. Therefore for some $A > 0$,

$$\begin{aligned} A = A\|K^* f'_N\|^2 &\leqslant \sum_{g\in G}\sum_{n\geqslant 0} |\langle T^n g, f'_N\rangle|^2 \\ &= \sum_{g\in G}\sum_{n=N+1}^{\infty} |\langle T^n g, f'_N\rangle|^2 \\ &\leqslant \sum_{g\in G} \|f'_N\|^2\|g\|^2 \sum_{n=N+1}^{\infty} \|T\|^{2n} \\ &\leqslant \frac{1}{M^2}\sum_{g\in G}\|g\|^2 \sum_{n=N+1}^{\infty} \|T\|^{2n}. \end{aligned}$$

The right hand side tends to zero as $N \to \infty$. Therefore, $A = 0$, which leads to a contradiction. $\square$

In the next example, we demonstrate that the assumption $\dim \mathscr{R}(K) = \infty$ cannot be removed in Theorem 3.3.

EXAMPLE 3.4. Let $\mathscr{H} = \mathbb{C}^m$, $\theta = \frac{2\pi}{m}$ and $T : \mathscr{H} \to \mathscr{H}$ be defined by

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & & & 0 \\ & e^{i\theta} & & \\ & & \ddots & \\ 0 & & & e^{i(m-1)\theta} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}.$$

Then for $x_0 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$, $\{T^n x_0\}_{n=0}^{m-1}$ generates the $m \times m$ matrix given by

$$\begin{pmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{m-1} \\ 1 & \alpha z & \alpha^2 z^2 & \cdots & \alpha^{m-1} z^{m-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \alpha z^{m-1} & \alpha^2 z^{2(m-1)} & \cdots & \alpha^{m-1} z^{(m-1)(m-1)} \end{pmatrix},$$

where $z = e^{\frac{2\pi i}{m}}$ and $\alpha = \frac{1}{\sqrt{m}}$. Indeed the jth columns of the matrix is equal to $T^{j-1}x_0$. Therefore $\{T^n x_0\}_{n=0}^{m-1}$ is a basis, and hence forms a frame for $\mathbb{C}^m$ by [7, Theorem 1.5.1]. Now for any $K \in M_m(\mathbb{C})$ with $KT = TK$ and $x \in \mathbb{C}^m$ we have

$$\|K^* x\|^2 \leqslant \sum_{n=0}^{m-1} |\langle K^* x, T^n x_0\rangle|^2 = \sum_{n=0}^{m-1} |\langle x, T^n K x_0\rangle|^2,$$

which implies that $\{T^n K x_0\}_{n=0}^{m-1}$ satisfies the lower K-frame bound. Now $\mathscr{R}(K)$ is closed since it is finite dimensional, but $\|T\| = \dfrac{1}{\sqrt{m}} < 1$ for $m \geqslant 2$.

We recall that a subspace M of $\mathscr{H}$ is called coinvariant under T if $T(M^{\perp}) \subseteq M^{\perp}$. Note that, according to [23, page 50], M is coinvariant under T if and only if

$$\pi_M T = \pi_M T \pi_M. \tag{3.1}$$

In the next corollary, we will provide a lower bound for the norm of the operator T which is better than the bound obtained in Theorem 3.3.

COROLLARY 3.5. *Let K has a closed range and $\dim \mathscr{R}(K) = \infty$. Also $\{T^n g\}_{g\in G, n\geqslant 0}$ with $|G| < \infty$ satisfies the lower K-frame bound. Then for each coinvariant closed subspace M of T containing $\mathscr{R}(K)$, $\|\pi_M T\| \geqslant 1$.*

Proof. According to the fact that M is coinvariant under T, equality (3.1) follows that

$$\pi_M T^2 = \pi_M TT = \pi_M T \pi_M T = \pi_M T \pi_M T \pi_M = \pi_M T \pi_M \pi_M T \pi_M = (\pi_M T \pi_M)^2.$$

Thus by induction for any $n > 0$, $\pi_M T^n = (\pi_M T \pi_M)^n$. Hence, if $\{T^n g\}_{g\in G, n\geqslant 0}$ satisfies the lower K-frame bound inequality in $\mathscr{H}$, then $\{(\pi_M T \pi_M)^n g\}_{g\in G, n\geqslant 0}$ also satisfies the lower K-frame bound inequality for M, indeed for some $A > 0$ and for any $f \in M$ we have

$$\begin{aligned} A\|K^* f\|^2 &\leqslant \sum_{g\in G}\sum_{n=0}^{\infty} |\langle f, T^n g\rangle|^2 \\ &= \sum_{g\in G}\sum_{n=0}^{\infty} |\langle \pi_M f, T^n g\rangle|^2 \\ &= \sum_{g\in G}\sum_{n=0}^{\infty} |\langle f, \pi_M T^n g\rangle|^2 \\ &= \sum_{g\in G}\sum_{n=0}^{\infty} |\langle f, (\pi_M T \pi_M)^n g\rangle|^2. \end{aligned}$$

Therefore from Theorem 3.3, $\|\pi_M T \pi_M\| \geqslant 1$. Now by (3.1) $\|\pi_M T\| \geqslant 1$. □

According to the assumptions of this result and using Theorem 3.3, we have $1 \leqslant \|\pi_M T\| \leqslant \|T\|$. Therefore $\|\pi_M T\|$ is a better bound than 1 for the norm of the operator T. This is important because if $1 < \|\pi_M T\|$ then we obtained $1 < \|T\|$.

In the next theorem, we provide a necessary condition for $\{T^n g\}_{g\in G, n\geqslant 0}$ to be a K-frame for some at most countable set G. The proof is based on [3, Theorem 7]. But, we state it here for the convenience of the reader.

THEOREM 3.6. *If for an operator $T \in B(\mathscr{H})$ there exists a set of vector G in $\mathscr{H}$ such that $\{T^n g\}_{g\in G, n\geqslant 0}$ is a K-frame in $\mathscr{H}$, then for each $f \in \mathscr{H}$, $K^*(T^*)^n f \to 0$ as $n \to \infty$.*

Proof. Let $\{T^n g\}_{g\in G, n\geqslant 0}$ be a K-frame with K-frame bounds A and B. Then for any $f \in \mathscr{H}$ and $m \in \mathbb{N}$ we have

$$\sum_{g\in G}\sum_{n=0}^{\infty} |\langle (T^*)^m f, T^n g\rangle|^2 = \sum_{g\in G}\sum_{n=0}^{\infty} |\langle f, T^{n+m} g\rangle|^2$$
$$= \sum_{g\in G}\sum_{n=m}^{\infty} |\langle f, T^n g\rangle|^2.$$

The inequality $\sum_{g\in G}\sum_{n=0}^{\infty}|\langle f, T^n g\rangle|^2 \leqslant B\|f\|^2$ implies that $\sum_{n=m}^{\infty}\sum_{g\in G}|\langle f, T^n g\rangle|^2$ tends to zero as $m \to \infty$. Thus

$$\sum_{g\in G}\sum_{n=0}^{\infty} |\langle (T^*)^m f, T^n g\rangle|^2 \to 0 \qquad \text{as} \qquad m \to \infty. \tag{3.2}$$

Now by using the lower K-frame inequality, we get

$$A\|K^*(T^*)^m f\|^2 \leqslant \sum_{g\in G}\sum_{n=0}^{\infty} |\langle (T^*)^m f, T^n g\rangle|^2.$$

By equation (3.2), we can conclude that $K^*(T^*)^m f \to 0$ as $m \to \infty$. □

COROLLARY 3.7. *Let $T \in B(\mathscr{H})$ be a unitary operator and $0 \neq K \in B(\mathscr{H})$ such that $KT = TK$. Then for any set of vectors $G \subset \mathscr{H}$, the sequence $\{T^n g\}_{g\in G, n\geqslant 0}$ cannot be a K-frame.*

Proof. According to $KT = TK$ and the fact that T is unitary, it is clear that $K^*T = TK^*$. Therefore, for any $f \in \mathscr{H}$,

$$\|K^* f\| = \|K^* T^n (T^*)^n f\| = \|T^n K^* (T^*)^n f\| \leqslant \|T\|^n \|K^*(T^*)^n f\| = \|K^*(T^*)^n f\|.$$

If $\{T^n g\}_{g\in G, n\geqslant 0}$ is a K-frame, then by the previous theorem, $\|K^*(T^*)^n f\| \to 0$ as $n \to \infty$ hence $K^* f = 0$ for all $f \in \mathscr{H}$, which is a contradiction. □

By the proof of Corollary 3.7, if the operator T is unitary and $K \neq 0$ with $KT = TK$ then there exists $f \in \mathscr{H}$ such that $\{K^*(T^*)^n f\}_{n=1}^{\infty}$ does not converge to zero.

In the following theorem, we show that a contraction operator T generates a Parseval K-frame under certain conditions.

THEOREM 3.8. *Let $0 \neq K \in B(\mathscr{H})$ and T be a contraction (i.e., $\|T\| \leqslant 1$) with the property $KT = TK$ and for every $f \in \mathscr{H}$, $K^*(T^*)^n f \to 0$ as $n \to \infty$. Then we can choose $G \subseteq \mathscr{H}$ such that $\{T^n g\}_{g\in G, n\geqslant 0}$ is a Parseval K-frame.*

Proof. Suppose that for any $f \in \mathscr{H}$, $K^*(T^*)^n f \to 0$ as $n \to \infty$ and $\|T\| \leqslant 1$. By the inequalities $\|T^* K^* f\| \leqslant \|T\|\|K^* f\| \leqslant \|K^* f\|$ for any $f \in \mathscr{H}$, we get

$$\langle KTK^*T^* f, f\rangle \leqslant \langle KK^* f, f\rangle,$$

which implies that $0 \leqslant KK^* - KTK^*T^*$. Assume that $D = (KK^* - KTK^*T^*)^{\frac{1}{2}}$ and $M = cl(D\mathscr{H})$, where $cl(D\mathscr{H})$ denotes the norm-closure of the set $\{Df : f \in \mathscr{H}\}$ in $\mathscr{H}$. Let $\{h_\alpha\}_{\alpha \in J}$ be an orthonormal basis for M and define $G = \{Dh_\alpha; \alpha \in J\}$. Thus for any $m \in \mathbb{N}$,

$$\begin{aligned}
\sum_{n=0}^{m}\sum_{\alpha\in J}|\langle f, T^nDh_\alpha\rangle|^2 &= \sum_{n=0}^{m}\sum_{\alpha\in J}|\langle D(T^*)^n f, h_\alpha\rangle|^2 \\
&= \sum_{n=0}^{m}\|D(T^*)^n f\|^2 \\
&= \sum_{n=0}^{m}\langle D^2(T^*)^n f, (T^*)^n f\rangle \\
&= \sum_{n=0}^{m}\langle (KK^* - KTK^*T^*)(T^*)^n f, (T^*)^n f\rangle \\
&= \sum_{n=0}^{m}\langle KK^*(T^*)^n f, (T^*)^n f\rangle - \sum_{n=0}^{m}\langle KTK^*T^*(T^*)^n f, (T^*)^n f\rangle \\
&= \|K^*f\|^2 - \|K^*(T^*)^{m+1}f\|^2.
\end{aligned}$$

Now, due to the fact that $K^*(T^*)^m f \to 0$ as $m \to \infty$, we obtain

$$\sum_{n=0}^{\infty}\sum_{\alpha\in J}|\langle f, T^nDh_\alpha\rangle|^2 = \|K^*f\|^2,$$

which completes the proof. □

Note that if $\dim \mathscr{R}(K) = \infty$ then by Theorem 3.3, $\|T\| \geqslant 1$ and hence in the previous theorem we are required to choose $\|T\| = 1$.

As a consequence of Theorem 3.6, the next proposition demonstrates that hypercyclic operators with a hypercyclic vector in $\mathscr{R}(K)$ cannot generate a K-frame. Recall that a linear operator $T \in B(\mathscr{H})$ is hypercyclic if there exists $\phi \in \mathscr{H}$ such that $\{T^n\phi\}_{n=0}^{\infty}$ is dense in $\mathscr{H}$. In this case, the vector ϕ is called a hypercyclic vector.

PROPOSITION 3.9. *Let $T, K \in B(\mathscr{H})$ and T be a hypercyclic operator for some hypercyclic vector $\phi \in \mathscr{R}(K)$. Then $\{T^n f\}_{n=0}^{\infty}$ and $\{(T^*)^n f\}_{n=0}^{\infty}$ are not K-frames for any choice of $f \in \mathscr{H}$.*

Proof. Suppose that $\phi \in \mathscr{R}(K)$ is a hypercyclic vector for T, so $\phi = K\eta$ for some $\eta \in \mathscr{H}$. Hence for any nonzero vector $f \in \mathscr{H}$ and $m \in \mathbb{N}$, $N_1(mf) \cap \{T^n\phi\}_{n=0}^{\infty} \neq \emptyset$, where $N_1(mf)$ is a neighbourhood with the center mf and radius 1. Therefore, there exists n_m such that $\|T^{n_m}\phi - mf\| < 1$, so

$$\langle T^{n_m}\phi, T^{n_m}\phi\rangle - m\langle T^{n_m}\phi, f\rangle - m\langle f, T^{n_m}\phi\rangle + m^2\langle f, f\rangle < 1,$$

thus

$$\begin{aligned} m^2\|f\|^2 - 1 &\leqslant \|T^{n_m}\phi\|^2 + m^2\|f\|^2 - 1 \\ &< 2mRe\langle f, T^{n_m}\phi\rangle \\ &\leqslant 2m|\langle f, T^{n_m}\phi\rangle|, \end{aligned}$$

hence

$$\frac{m}{2}\|f\|^2 - \frac{1}{2m} < |\langle f, T^{n_m}\phi\rangle|.$$

Since the left side of the inequality tends to infinity as $m \to \infty$, we can see that the sequence $\{\langle f, T^n K\eta\rangle\}_{n=0}^{\infty} = \{\langle (T^*)^n f, K\eta\rangle\}_{n=0}^{\infty}$ is unbounded. This implies that $\{\|K^*(T^*)^n f\|\}_{n=0}^{\infty}$ is unbounded. Therefore, by Theorem 3.6 the sequence $\{T^n f\}_{n=0}^{\infty}$ is not a K-frame. Furthermore, for each $f \neq 0$,

$$\sum_{n=0}^{\infty} |\langle (T^*)^n f, K\eta\rangle|^2 = \sum_{n=0}^{\infty} |\langle f, T^n K\eta\rangle|^2 = \infty,$$

from this, we can conclude that $\{(T^*)^n f\}_{n=0}^{\infty}$ is not a K-frame. □

Our next result concerns the lower bound of a K-frame generated by the frame operator of a K-frame.

PROPOSITION 3.10. *Let $K^*K \geqslant I$ and $\{f_n\}_{n=1}^{\infty}$ be a K-frame with the lower K-frame bound A and the K-frame operator S. Furthermore suppose that G is an at most countable subset of $\mathscr{H}$ and $\{S^n g\}_{g\in G, n\geqslant 0}$ is a K-frame for $\mathscr{H}$, then $A < 1$ if one of the following statements holds:*

(*i*) *The sequence $\{f_n\}_{n=1}^{\infty}$ is a tight K-frame.*

(*ii*) *For the K-frame operator S, $SK = KS$.*

Proof. First note that for each $f \in \mathscr{H}$,

$$A\langle f, f\rangle \leqslant A\langle K^*Kf, K^*Kf\rangle \leqslant \langle SKf, Kf\rangle = \langle K^*SKf, f\rangle \leqslant \|K^*SKf\|\|f\|,$$

so $A\|f\| \leqslant \|K^*SKf\|$. Thus

$$A^2\|f\| \leqslant A\|K^*SKf\| \leqslant \|(K^*SK)^2 f\|,$$

for all $f \in \mathscr{H}$. Hence by induction for any $m \geqslant 0$ and $f \in \mathscr{H}$,

$$A^m\|f\| \leqslant \|(K^*SK)^m f\|. \tag{3.3}$$

If (i) holds, then $KK^* = \frac{1}{A}S$ so for each $m > 0$, $(K^*SK)^m = \frac{1}{A^{m-1}}K^*S^{2m-1}K$. Therefore, by using (3.3)

$$A^{2m-1}\|f\| \leqslant \|K^*S^{2m-1}Kf\|, \qquad (f \in \mathscr{H}, m > 0).$$

On the other hand $\{S^n g\}_{g\in G, n\geqslant 0}$ is a K-frame, thus by Theorem 3.6, for any $f \in \mathscr{H}$, $\|K^*S^{2m-1}Kf\| \to 0$ as $m \to \infty$. Therefore $A^{2m-1} \to 0$ as $m \to \infty$ which implies that $A < 1$.

Now assume that (ii) is valid. By the fact that $KK^* \leqslant \frac{1}{A}S$ and using Theorem 2.1, we have

$$SKK^*S \leqslant \frac{1}{A}S^3.$$

Hence by Theorem 2.1,

$$(K^*SK)^2 \leqslant \frac{1}{A}K^*S^3K. \tag{3.4}$$

Again by using Theorem 2.1, relation (3.4) and the equality $SK = KS$, we obtain

$$(K^*SK)^3 \leqslant \frac{1}{A}K^*SKK^*S^3K = \frac{1}{A}K^*S^2KK^*S^2K \leqslant \frac{1}{A^2}K^*S^5K.$$

Thus by induction, for any natural number m,

$$(K^*SK)^m \leqslant \frac{1}{A^{m-1}}K^*S^{2m-1}K.$$

Hence for any $m > 0$ and $f \in \mathscr{H}$,

$$\begin{aligned}\|(K^*SK)^m f\|^2 = \langle (K^*SK)^{2m}f, f\rangle &\leqslant \frac{1}{A^{2m-2}}\langle (K^*S^{2m-1}K)^2 f, f\rangle \\ &= \frac{1}{A^{2m-2}}\|K^*S^{2m-1}K\|^2.\end{aligned}$$

Therefore by (3.3),

$$A^{2m-1}\|f\| \leqslant \|K^*S^{2m-1}Kf\|, \qquad (f \in \mathscr{H}, m > 0)$$

which completes the proof as same as proof of part (i). □

The following proposition shows that K-frames properties are preserved under quasiconjugacy.

PROPOSITION 3.11. *Let $T_i, K_i \in B(\mathscr{H}_i)$, $i = 1,2$, and $V : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator with closed range such that $VT_1 = T_2V$, $VK_1 = K_2V$ and $\mathscr{R}(K_2^*) \subseteq \mathscr{R}(V)$. If $\{T_1^n f\}_{n=0}^{\infty}$ is a K_1-frame for $\mathscr{H}_1$, then $\{T_2^n Vf\}_{n=0}^{\infty}$ is a K_2-frame for $\mathscr{R}(V)$.*

Proof. Suppose that $g = Vf_1$, for some $f_1 \in \mathscr{H}_1$ and $V^\dagger$ is the pseudo inverse of

V. Then for some $A>0$,

$$\begin{aligned}\|K_2^* g\|^2 &= \|K_2^* V f_1\|^2\\ &= \|(VV^\dagger)^* K_2^* V f_1\|^2\\ &\leqslant \|(V^\dagger)^*\|^2 \|V^* K_2^* V f_1\|^2\\ &= \|(V^\dagger)^*\|^2 \|K_1^* V^* V f_1\|^2\\ &\leqslant \frac{\|(V^\dagger)^*\|^2}{A}\sum_{n=0}^{\infty}|\langle V^* V f_1, T_1^n f\rangle|^2\\ &= \frac{\|(V^\dagger)^*\|^2}{A}\sum_{n=0}^{\infty}|\langle V f_1, V T_1^n f\rangle|^2\\ &= \frac{\|(V^\dagger)^*\|^2}{A}\sum_{n=0}^{\infty}|\langle g, T_2^n V f\rangle|^2.\end{aligned}$$

Moreover for some $B>0$,

$$\begin{aligned}\sum_{n=0}^{\infty}|\langle g, T_2^n V f\rangle|^2 &= \sum_{n=0}^{\infty}|\langle V f_1, V T_1^n f\rangle|^2\\ &= \sum_{n=0}^{\infty}|\langle V^* V f_1, T_1^n f\rangle|^2\\ &\leqslant B\|V^* V f\|^2\\ &\leqslant \|V\|^2\|g\|^2,\end{aligned}$$

which completes the proof. □

COROLLARY 3.12. *Let* $T_i, K_i \in B(\mathscr{H}_i)$, $i=1,\dots,n$, $T := \bigoplus_{i=1}^n T_i$ *and* $K := \bigoplus_{i=1}^n K_i$. *If* $\{T^n f\}_{n=0}^{\infty}$ *is a* K*-frame for* $\mathscr{H} = \bigoplus_{i=1}^n \mathscr{H}_i$, *then* $\{T_i^n V f\}_{n=0}^{\infty}$ *is a* K_i*-frame for* $\mathscr{H}_i$ *for some operator* V.

Proof. Let $\{T^n f\}_{n=0}^{\infty}$ be a K-frame for $\mathscr{H} = \bigoplus_{i=1}^n \mathscr{H}_i$ and let $\pi_i : \bigoplus_{j=1}^n \mathscr{H}_j \to \mathscr{H}_i$ be the orthogonal projection onto $\mathscr{H}_i$. Clearly, $\pi_i T = T_i \pi_i$ and $\pi_i K = K_i \pi_i$. Moreover $\mathscr{R}(K_i) \subseteq \mathscr{H}_i = \mathscr{R}(\pi_i)$, hence by Proposition 3.11, $\{T_i^n \pi_i f\}_{n=0}^{\infty}$ is a K_i-frame for $\mathscr{H}_i$. □

DEFINITION 3.13. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces, with $T_i, K_i \in B(\mathscr{H}_i)$ and $f_i \in \mathscr{H}_i$, for $i=1,2$. We say that the triple (T_1, K_1, f_1) and (T_2, K_2, f_2) are equivalent (or similar) via $V \in GL(\mathscr{H}_1, \mathscr{H}_2)$ if

$$T_2 = V T_1 V^{-1}, \qquad K_2 = V K_1 V^{-1} \qquad \text{and} \qquad f_2 = V f_1 \tag{3.5}$$

In this case, we write $(T_1, K_1, f_1) \cong (T_2, K_2, f_2)$.

The properties of a K-frame of similar triple are given in the following propositions.

PROPOSITION 3.14. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $T_i, K_i \in B(\mathscr{H}_i)$ and $f_i \in \mathscr{H}_i$, $i = 1,2$, such that $(T_1, K_1, f_1) \cong (T_2, K_2, f_2)$. Then $\{T_1^n f_1\}_{n=0}^{\infty}$ is a K_1-frame for $\mathscr{H}_1$ if and only if $\{T_2^n f_2\}_{n=0}^{\infty}$ is a K_2-frame for $\mathscr{H}_2$.*

In the affirmative case, the operator V in (3.5) is unique on $\mathscr{R}(K_1)$.

Proof. Let $V \in GL(\mathscr{H}_1, \mathscr{H}_2)$ be an operator as in (3.5). According to Proposition 3.11 we only need to show that V is unique. For $f \in \mathscr{H}_1$, let $K_1 f = \sum_{n=0}^{\infty} c_n T_1^n f_1$, where $\{c_n\}_{n=0}^{\infty} \in \ell^2(\mathbb{N} \cup \{0\})$. If W is an another operator with the property (3.5), then

$$\begin{aligned} VK_1 f &= \sum_{n=0}^{\infty} c_n V T_1^n f_1 \\ &= \sum_{n=0}^{\infty} c_n T_2^n f_2 \\ &= \sum_{n=0}^{\infty} c_n (W T_1^n W^{-1}) W f_1 \\ &= \sum_{n=0}^{\infty} c_n W T_1^n f_1 \\ &= W K_1 f. \end{aligned}$$

Thus V is unique on $\mathscr{R}(K_1)$. □

COROLLARY 3.15. *Let $T_i, K \in B(\mathscr{H})$, $i = 1,2$. If T_2 is unitary and $T_2 K = K T_2$, then $\{(T_2 T_1)^n f\}_{n=0}^{\infty}$ is a K-frame for $\mathscr{H}$ if and only if $\{(T_1 T_2)^n V f\}_{n=0}^{\infty}$ is a K-frame for $\mathscr{H}$ for some V.*

Proof. Let T_2 be unitary and $T_2 K = K T_2$. Consequently, $K = T_2^* K T_2$. Furthermore $T_1 T_2 = T_2^*(T_2 T_1) T_2$ so by Proposition 3.14, the result can be deduced. □

4. K-dual for iterative K-frames

In this section, we study the form of the K-dual for a K-frames which has the form $\{f_n\}_{n\in\mathbb{Z}} = \{T^n f_0\}_{n\in\mathbb{Z}}$ for some bounded bijective operator $T : \operatorname{span}\{f_n\}_{n\in\mathbb{Z}} \to \operatorname{span}\{f_n\}_{n\in\mathbb{Z}}$. We identify the canonical K-dual frames represented by a bounded bijective operator. In the sequel, we suppose that K has a closed range and we denote the operator S from $\mathscr{R}(K)$ onto $S(\mathscr{R}(K))$ by S_F.

PROPOSITION 4.1. *Let $\{f_n\}_{n\in\mathbb{Z}} = \{T^n f_0\}_{n\in\mathbb{Z}}$ be a K-frame, where $T \in GL(\mathscr{H})$. Assume that $\{g_n\}_{n\in\mathbb{Z}} = \{V^n g_0\}_{n\in\mathbb{Z}}$ is a K-dual frame of $\{f_n\}_{n\in\mathbb{Z}}$ and that $V \in GL(\mathscr{H})$. The following statements hold:*

(*i*) *If $TK = KT$ and K is injective, then $V = (T^*)^{-1}$.*

(*ii*) *If $VK^* = K^*V$ and K^* is injective, then $V = (T^*)^{-1}$.*

Proof. (i) For any $f \in \mathscr{H}$,

$$\begin{aligned}
Kf &= \sum_{n\in\mathbb{Z}} \langle f, V^n g_0\rangle T^n f_0 \\
&= T\sum_{n\in\mathbb{Z}} \langle f, V^n g_0\rangle T^{n-1} f_0 \\
&= T\sum_{n\in\mathbb{Z}} \langle V^* f, V^{n-1} g_0\rangle T^{n-1} f_0 \\
&= T\sum_{n\in\mathbb{Z}} \langle V^* f, V^n g_0\rangle T^n f_0 \\
&= TKV^* f \\
&= KTV^* f.
\end{aligned}$$

Therefore $TV^*f - f \in \mathscr{N}(K)$ but K is injective, so $TV^* = I$ and hence $V = (T^*)^{-1}$.
(ii) This proof is similar to the proof of (i). $\square$

Our next theorem concerns the form of the canonical K-dual of an iterative K-frame. We need the following results. Recall that the right shift operator is defined by

$$\mathscr{T} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \qquad \mathscr{T}\{c_n\}_{n\in\mathbb{Z}} = \{c_{n-1}\}_{n\in\mathbb{Z}}.$$

Also, denote by c_{00} the set of all finitely supported sequences in $\ell^2(\mathbb{Z})$, i.e.,

$$c_{00} := \{\{c_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) : c_n = 0 \text{ for all but finitely many } n\}.$$

LEMMA 4.2. [9] *Consider a Bessel sequence having the form* $\{f_n\}_{n\in\mathbb{Z}} = \{T^n f_0\}_{n\in\mathbb{Z}}$ *for a linear operator* $T : \operatorname{span}\{f_n\}_{n\in\mathbb{Z}} \to \operatorname{span}\{f_n\}_{n\in\mathbb{Z}}$. *Then* $TU = U\mathscr{T}$ *on* c_{00}. *Assuming that* T *has an extension to a bounded operator* $\widetilde{T} : \mathscr{H} \to \mathscr{H}$, *the following hold:*

(i) $\widetilde{T}U = U\mathscr{T}$ *on* $\ell^2(\mathbb{Z})$.

(ii) *If* $\{T^n f_0\}_{n\in\mathbb{Z}}$ *is* K*-frame and* $\widetilde{T}$ *is invertible, then* $\widetilde{T}S = S(\widetilde{T^*})^{-1}$.

PROPOSITION 4.3. [24] *Let* K *be a bounded operator on* $\mathscr{H}$ *with closed range, and let* $F := \{f_n\}_{n\in I}$ *be a* K*-frame with* A *and* B *bounds, respectively. Then,* $\{K^* S_F^{-1} \pi_{S_F(\mathscr{R}(K))} f_n\}_{n\in I}$ *is a* K*-dual of* $\pi_{\mathscr{R}(K)}F$ *with* B^{-1} *and* $A^{-1}\|K\|^2\|K^\dagger\|^2$ *bounds, respectively.*

THEOREM 4.4. *Let* $T, K \in B(\mathscr{H})$ *and* $F = \{f_n\}_{n\in\mathbb{Z}} = \{T^n f_0\}_{n\in\mathbb{Z}}$ *be a* K*-frame where* $T \in B(\mathscr{H})$ *is invertible,* $TK^* = K^*T$ *and* $S_F(\mathscr{R}(K))$ *is invariant under* T^*. *Then* $\{K^*(T^*)^{-n} S_F^{-1} \pi_{S_F(\mathscr{R}(K))} f_0\}_{n\in\mathbb{Z}}$ *is a* K*-dual of* $\{\pi_{\mathscr{R}(K)} f_n\}_{n\in\mathbb{Z}}$.

Furthermore if $TK = KT$, *then* $\{K^*(T^*)^{-n} S_F^{-1} \pi_{S_F(\mathscr{R}(K))} f_0\}_{n\in\mathbb{Z}}$ *can be expressed as* $\{(T^*)^{-n}\widetilde{f_0}\}_{n\in\mathbb{Z}}$, *where* $\widetilde{f_0} = K^* S_F^{-1} \pi_{S_F(\mathscr{R}(K))} f_0$ *and* $\{\pi_{\mathscr{R}(K)} f_n\}_{n\in\mathbb{Z}} = \{(T^*)^{-n}\widehat{f_0}\}_{n\in\mathbb{Z}}$ *such that* $\widehat{f_0} = \pi_{\mathscr{R}(K)} f_0$.

Proof. Lemma 4.2 implies that $TS = S(T^*)^{-1}$, so

$$TS\pi_{\mathscr{R}(K)} = S(T^*)^{-1}\pi_{\mathscr{R}(K)}. \tag{4.1}$$

Now from $TK^* = K^*T$, we get $(T^*)^{-1}K = K(T^*)^{-1}$. Thus $(T^*)^{-1}(\mathscr{R}(K)) \subseteq \mathscr{R}(K)$ and so $S(T^*)^{-1}\pi_{\mathscr{R}(K)} \subseteq S(\mathscr{R}(K))$. Therefore by (4.1), $TS\pi_{\mathscr{R}(K)} \subseteq S(\mathscr{R}(K))$, which implies that

$$T(S(\mathscr{R}(K))) \subseteq S(\mathscr{R}(K)).$$

By the fact that $S(\mathscr{R}(K)) = S_F(\mathscr{R}(K))$, we can conclude that

$$T(S_F(\mathscr{R}(K))) \subseteq S_F(\mathscr{R}(K)).$$

On the other hand, the inclusion $T^*(S_F(RK))) \subseteq S_F(\mathscr{R}(K))$ by [23, page 50] implies that

$$T\pi_{S_F(\mathscr{R}(K))} = \pi_{S_F(\mathscr{R}(K))}T. \tag{4.2}$$

Hence

$$T^{-1}\pi_{S_F(\mathscr{R}(K))} = \pi_{S_F(\mathscr{R}(K))}T^{-1}. \tag{4.3}$$

Now by using $S(\mathscr{R}(K)) = S_F(\mathscr{R}(K))$ and (4.1),

$$TS_F\pi_{\mathscr{R}(K)} = S_F(T^*)^{-1}\pi_{\mathscr{R}(K)},$$

so

$$(T^*)^{-1}\pi_{\mathscr{R}(K)} = S_F^{-1}S_F(T^*)^{-1}\pi_{\mathscr{R}(K)} = S_F^{-1}TS_F\pi_{\mathscr{R}(K)}.$$

Hence

$$(T^*)^{-1}\pi_{\mathscr{R}(K)}S_F^{-1}\pi_{S_F(\mathscr{R}(K))} = S_F^{-1}TS_F\pi_{\mathscr{R}(K)}S_F^{-1}\pi_{S_F(\mathscr{R}(K))},$$

and therefore

$$(T^*)^{-1}S_F^{-1}\pi_{S_F(\mathscr{R}(K))} = S_F^{-1}T\pi_{S_F(\mathscr{R}(K))}. \tag{4.4}$$

So we have

$$\begin{aligned}
(T^*)^{-2}S_F^{-1}\pi_{S_F(\mathscr{R}(K))} &= (T^*)^{-1}(T^*)^{-1}S_F^{-1}\pi_{S_F(\mathscr{R}(K))} \\
&= (T^*)^{-1}S_F^{-1}T\pi_{S_F(\mathscr{R}(K))} && \text{(by (4.4))} \\
&= (T^*)^{-1}S_F^{-1}\pi_{S_F(\mathscr{R}(K))}T && \text{(by (4.2))} \\
&= S_F^{-1}T\pi_{S_F(\mathscr{R}(K))}T && \text{(by (4.4))} \\
&= S_F^{-1}T^2\pi_{S_F(\mathscr{R}(K))}, && \text{(by (4.2))}.
\end{aligned}$$

Therefore by induction for $n \in \mathbb{N}$,

$$(T^*)^{-n}S_F^{-1}\pi_{S_F(\mathscr{R}(K))} = S_F^{-1}T^n\pi_{S_F(\mathscr{R}(K))}. \tag{4.5}$$

Now by (4.4) we get,

$$(T^*)^{-1}S_F^{-1}\pi_{S_F(\mathscr{R}(K))}T^{-1} = S_F^{-1}T\pi_{S_F(\mathscr{R}(K))}T^{-1}.$$

Hence by (4.3)

$$(T^*)^{-1}S_F^{-1}T^{-1}\pi_{S_F(\mathscr{R}(K))} = S_F^{-1}\pi_{S_F(\mathscr{R}(K))},$$

and so

$$S_F^{-1}T^{-1}\pi_{S_F(\mathscr{R}(K))} = T^*S_F^{-1}\pi_{S_F(\mathscr{R}(K))}. \tag{4.6}$$

Therefore

$$\begin{aligned}(T^*)^2S_F^{-1}\pi_{S_F(\mathscr{R}(K))} &= T^*T^*S_F^{-1}\pi_{S_F(\mathscr{R}(K))}\\ &= T^*S_F^{-1}T^{-1}\pi_{S_F(\mathscr{R}(K))} && \text{(by (4.6))}\\ &= T^*S_F^{-1}\pi_{S_F(\mathscr{R}(K))}T^{-1} && \text{(by (4.3))}\\ &= S_F^{-1}T^{-1}\pi_{S_F(\mathscr{R}(K))}T^{-1} && \text{(by (4.6))}\\ &= S_F^{-1}T^{-2}\pi_{S_F(\mathscr{R}(K))}, && \text{(by (4.3)).}\end{aligned}$$

By induction for any $n \in \mathbb{N}$, we deduce that

$$(T^*)^nS_F^{-1}\pi_{S_F(\mathscr{R}(K))} = S_F^{-1}T^{-n}\pi_{S_F(\mathscr{R}(K))}. \tag{4.7}$$

Now by (4.5) and (4.7) for any $n \in \mathbb{Z}$, we have

$$(T^*)^{-n}S_F^{-1}\pi_{S_F(\mathscr{R}(K))} = S_F^{-1}T^n\pi_{S_F(\mathscr{R}(K))}. \tag{4.8}$$

This together with Proposition 4.3 implies that

$$\begin{aligned}\{K^*S_F^{-1}\pi_{S_F(\mathscr{R}(K))}f_n\}_{n\in\mathbb{Z}} &= \{K^*S_F^{-1}\pi_{S_F(\mathscr{R}(K))}T^nf_0\}_{n\in\mathbb{Z}}\\ &= \{K^*S_F^{-1}T^n\pi_{S_F(\mathscr{R}(K))}f_0\}_{n\in\mathbb{Z}} && \text{(by (4.2))}\\ &= \{K^*(T^*)^{-n}S_F^{-1}\pi_{S_F(\mathscr{R}(K))}f_0\}_{n\in\mathbb{Z}}, && \text{(by (4.8)).}\end{aligned}$$

Moreover, if $TK = KT$, then $K^*T^* = T^*K^*$. Therefore $(T^*)^{-1}K^* = K^*(T^*)^{-1}$ and consequently, $K^*(T^*)^{-n} = (T^*)^{-n}K^*$, which implies that

$$\{K^*(T^*)^{-n}S_F^{-1}\pi_{S_F(\mathscr{R}(K))}f_0\}_{n\in\mathbb{Z}} = \{(T^*)^{-n}K^*S_F^{-1}\pi_{S_F(\mathscr{R}(K))}f_0\}_{n\in\mathbb{Z}} = \{(T^*)^{-n}\widetilde{f_0}\}_{n\in\mathbb{Z}},$$

where $\widetilde{f_0} = K^*S_F^{-1}\pi_{S_F(\mathscr{R}(K))}f_0$, thus the result follows.

Finally, according to the equalities $TK = KT$ and $TK^* = K^*T$, $R(K)$ is invariant under T and T^*, which implies that $T\pi_{\mathscr{R}(K)} = \pi_{\mathscr{R}(K)}T$ so

$$\{\pi_{\mathscr{R}(K)}f_n\}_{n\in\mathbb{Z}} = \{\pi_{\mathscr{R}(K)}T^nf_0\}_{n\in\mathbb{Z}} = \{T^n\pi_{\mathscr{R}(K)}f_0\}_{n\in\mathbb{Z}} = \{T^n\widehat{f_0}\}_{n\in\mathbb{Z}},$$

where $\widehat{f_0} = \pi_{\mathscr{R}(K)}f_0$. □

In the following example using Theorem 4.4, we find the canonical K-dual of a K-frame.

EXAMPLE 4.5. Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for $\mathscr{H}$ and $F=\{f_n\}_{n\in\mathbb{Z}}=\{\frac{1}{(2n)^2+1}e_{2n}\}_{n\in\mathbb{Z}}$. Then $\{f_n\}_{n\in\mathbb{Z}}=\{T^n f_0\}_{n\in\mathbb{Z}}$, where $Te_n=\frac{n^2+1}{(n+2)^2+1}e_{n+2}$ for any $n\in\mathbb{Z}$, also T is invertible and $T^{-1}e_n=\frac{n^2+1}{(n-2)^2+1}e_{n-2}$. Trivially, F is a Bessel sequence in $\mathscr{H}$. Furthermore if $M=\overline{\text{span}}\{f_n\}_{n\in\mathbb{Z}}$ and $K=\pi_M$, then $\mathscr{R}(K)=\mathscr{R}(U)$, where U is the synthesis operator of F. Hence by [21, Remark 1.1] F is a K-frame. On the other hand, the property $\mathscr{H}=M\oplus M^{\perp}$ implies that $TK=KT$; indeed if $f\in\text{span}\{f_n\}_{n\in\mathbb{Z}}$ then $f=\sum_{n\in\mathbb{Z}}c_n\frac{1}{(2n)^2+1}e_{2n}$, where $\{c_n\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$ so

$$TKf=Tf=\sum_{n\in\mathbb{Z}}c_n\frac{1}{(2n+2)^2+1}e_{2n+2}=K\sum_{n\in\mathbb{Z}}c_n\frac{1}{(2n+2)^2+1}e_{2n+2}=KTf.$$

Moreover, for any $f\in\text{span}^{\perp}\{f_n\}_{n\in\mathbb{Z}}$, $f=\sum_{n\in\mathbb{Z}}d_ne_{2n+1}$, where $\{d_n\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})$ thus

$$TKf=T(0)=0=K\Big(\sum_{n\in\mathbb{Z}}d_n\frac{(2n+1)^2+1}{(2n+3)^2+1}e_{2n+3}\Big)=KTf.$$

It follows that TK is equal to KT on M and $M^{\perp}$, so $TK=KT$ on $\mathscr{H}$. In addition, it is easy to check that $T^*e_n=\frac{(n-2)^2+1}{n^2+1}e_{n-2}$ and $S_F(M)\subseteq M$, so $S_F(M)$ is invariant under T^*. Now according to $S_Ff_0=f_0$, $S_F^{-1}f_0=f_0$, thus $\pi_MS_F^{-1}\pi_{S_F(\mathscr{R}(M))}f_0=f_0$. Therefore, by Theorem 4.4, the canonical K-dual of $\{T^nf_0\}_{n\in\mathbb{Z}}$ is $\{(T^*)^{-n}f_0\}_{n\in\mathbb{Z}}$.

In [20], the authors present an equivalent condition for a K-frame to be iterative with respect to its K-dual which is presented here. This result is true if K is surjective. Additionally, if K is co-isometry then this result is true. The proof can be obtained by a similar argument to the proof of this proposition by using K^* instead of $K^{\dagger}$. In the sequel, a counterexample is provided for the non-surjective case.

PROPOSITION 4.6. [20] *Consider a K-frame $\{f_n\}_{n\in\mathbb{Z}}$ with K-frame bounds A, B, respectively. Let T be a bounded linear operator. Then the following conclusions are equivalent:*

(i) *The K-frame has a representation $\{f_n\}_{n\in\mathbb{Z}}=\{T^nf_0\}_{n\in\mathbb{Z}}$ for the bounded operator T.*

(ii) *For K-dual frame $\{g_n\}_{n\in\mathbb{Z}}$, we obtain*

$$f_{j+1}=\sum_{n\in\mathbb{Z}}\langle K^{\dagger}f_j,g_n\rangle f_{n+1}, \tag{4.9}$$

where $K^{\dagger}$ is the pseudo inverse of K.

EXAMPLE 4.7. Let $\{e_n\}_{n\in\mathbb{Z}}$ be an orthonormal basis for $\mathscr{H}$ and $m\in\mathbb{N}$ be given. Define $\{f_n\}_{n\in\mathbb{Z}}=\{\frac{1}{n^2+1}e_n\}_{n\in\mathbb{Z}}$ and let $K\in B(\mathscr{H})$ be defined by

$$Ke_n=\begin{cases}0 & \text{if } |n|<m;\\ \frac{1}{n}e_n & \text{if } |n|\geqslant m.\end{cases}$$

Obviously, $\mathscr{R}(K) \subset \mathscr{R}(U)$ where U is the synthesis operator of $\{f_n\}_{n\in\mathbb{Z}}$. Hence by [21, Remark 1.1], $\{f_n\}_{n\in\mathbb{Z}}$ is a K-frame. According to [7, Lemma 2.5.2], $K^\dagger = K^*(KK^*)^{-1}$ on $\mathscr{R}(K)$, so $K^\dagger e_n = ne_n$ for $|n| \geqslant m$. Now we see that $\{f_n\}_{n\in\mathbb{Z}} = \{T^n f_0\}_{n\in\mathbb{Z}}$ where $Te_n = \frac{n^2+1}{(n+1)^2+1}e_{n+1}$ for any $n \in \mathbb{Z}$, but according to $K^\dagger$ is zero on the orthogonal complement of $\mathscr{R}(K)$, for any K-dual frame $\{g_n\}_{n\in\mathbb{Z}}$, we have

$$\begin{aligned} e_1 &\neq \sum_{n\in\mathbb{Z}} \langle K^\dagger e_0, g_n\rangle f_{n+1} \\ &= \sum_{n\in\mathbb{Z}} \langle 0, g_n\rangle f_{n+1} \\ &= 0. \end{aligned}$$

Declarations

Data availability. Not applicable to this article as no datasets were generated or analysed during the current study.

Conflict of interest. The authors have no conflicts of interest to declare that are relevant to the content of this article.

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