Operators and Matrices
Volume 19, Number 3 (2025), 393–410 doi:10.7153/oam-2025-19-25

THE HILBERT MATRIX OPERATOR ACTING ON SPACES OF BOUNDED ANALYTIC FUNCTIONS

YUTING GUO AND PENGCHENG TANG*

(*Communicated by F. Gesztesy*)

Abstract. It is well known that the Hilbert matrix operator $\mathscr{H}$ is bounded from H^∞ to the mean Lipschitz spaces $\Lambda^p_{1/p}$ for all $1<p<\infty$. In this paper, we prove that the range of $\mathscr{H}$ acting on H^∞ is contained in a certain Zygmund-type space, denoted by $\Lambda^{1,*}_1$. We also provide explicit upper and lower bounds for the norm of $\mathscr{H}$ as an operator from H^∞ to $\Lambda^{1,*}_1$. Moreover, we characterize the positive Borel measures μ for which the generalized Hilbert matrix operator $\mathscr{H}_\mu$ is bounded from H^∞ to the Hardy space H^q.

1. Introduction

Throughout the paper, the letter C will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation "$P \lesssim Q$" if there exists a constant $C=C(\cdot)$ such that "$P \leqslant CQ$", and "$P \gtrsim Q$" is understood in an analogous manner. In particular, if "$P \lesssim Q$" and "$P \gtrsim Q$", then we will write "$P \asymp Q$". For two normed spaces X and Y, if there exists a bijective linear operator $T: X \to Y$ such that both T and its inverse T^{-1} are continuous, then we say that X and Y have equivalent normed structures (or topologically isomorphic) and we will write "$X \cong Y$".

Let $\mathbb{D}=\{z\in\mathbb{C}: |z|<1\}$ denote the open unit disk of the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$.

The Bloch space $\mathscr{B}$ consists of those functions $f\in H(\mathbb{D})$ for which

$$||f||_{\mathscr{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

Let $0<p\leqslant\infty$, the classical Hardy space H^p consists of those functions $g\in H(\mathbb{D})$ for which

$$||g||_p = \sup_{0\leqslant r<1} M_p(r,g) < \infty,$$

where

$$M_p(r,g) = \left(\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^p d\theta\right)^{1/p}, \quad 0<p<\infty,$$

Mathematics subject classification (2020): 47B35, 30H05, 30H10.
Keywords and phrases: Hilbert matrix operator, bounded analytic function, norm, Carleson measure.
* Corresponding author.

$$M_\infty(r,g) = \sup_{|z|=r} |g(z)|.$$

The BMOA space consists of all the functions $h \in H^2$ such that

$$||h||_{BMOA} = |h(0)| + \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |h'(z)|^2 \log\left|\frac{1-\overline{a}z}{a-z}\right| dA(z)\right)^{\frac{1}{2}} < \infty.$$

For $0 < p < \infty$, the Q_p space consists of all the functions $f \in H(\mathbb{D})$ such that

$$||f||_{Q_p}^2 = |f(0)| + \left(\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1-|\sigma_a(z)|^2)^p dA(z)\right)^{\frac{1}{2}} < \infty,$$

where σ_a stands for the Möbius transformation $\sigma_a(z) = \frac{a-z}{1-\overline{a}z}$. It is known (see [31]) that $Q_p = \mathscr{B}$ for $1 < p < \infty$, and that $Q_1 = \text{BMOA}$, with equivalent norms. The reader is referred to [31] for more on the Q_p spaces.

The mixed norm space $H^{p,q,\alpha}$, $0 < p,q \leqslant \infty$, $0 < \alpha < \infty$, is the space of all functions $f \in H(\mathbb{D})$ for which

$$||f||_{p,q,\alpha} = \left(\int_0^1 M_p^q(r,f)(1-r)^{q\alpha-1} dr\right)^{\frac{1}{q}} < \infty, \text{ for } 0 < q < \infty,$$

and

$$||f||_{p,\infty,\alpha} = \sup_{0\leqslant r<1} (1-r)^\alpha M_p(r,f) < \infty.$$

For $t \in \mathbb{R}$, the fractional derivative of order t of $f \in H(\mathbb{D})$ is defined by $D^t f(z) = \sum_{n=0}^\infty (n+1)^t \widehat{f}(n) z^n$. If $0 < p,q \leqslant \infty$, $0 < \alpha < \infty$, then $H_t^{p,q,\alpha}$ is the space of all analytic functions $f \in H(\mathbb{D})$ such that

$$||D^t f||_{p,q,\alpha} < \infty.$$

It is a well-known fact (see [27]) that if $f \in H(\mathbb{D})$, $0 < p,q \leqslant \infty$, $0 < \alpha,\beta < \infty$, and $s,t \in \mathbb{R}$ satisfy $s-t = \alpha-\beta$, then

$$||D^s f||_{p,q,\alpha} \asymp ||D^t f||_{p,q,\beta}.$$

Consequently, we get $H_s^{p,q,\alpha} \cong H_t^{p,q,\beta}$.

Let $1 \leqslant p < \infty$ and $0 < \alpha \leqslant 1$, the mean Lipschitz space Λ_α^p consists of those functions $f \in H(\mathbb{D})$ having a non-tangential limit almost everywhere such that $\omega_p(t,f) = O(t^\alpha)$ as $t \to 0$. Here $\omega_p(\cdot,f)$ is the integral modulus of continuity of order p of the function $f(e^{i\theta})$. It is known (see [15]) that Λ_α^p is a subset of H^p and

$$\Lambda_\alpha^p = \left\{ f \in H(\mathbb{D}) : M_p(r,f') = O\left(\frac{1}{(1-r^2)^{1-\alpha}}\right), \text{ as } r \to 1 \right\}.$$

The space Λ_α^p is a Banach space with the norm $||\cdot||_{\Lambda_\alpha^p}$ given by

$$||f||_{\Lambda_\alpha^p} = |f(0)| + \sup_{0\leqslant r<1} (1-r^2)^{1-\alpha} M_p(r,f').$$

Moreover, it is known (see e.g. [6, Theorem 2.5]) that

$$\Lambda^p_{\frac{1}{p}} \subsetneq \Lambda^q_{\frac{1}{q}} \subsetneq BMOA \subsetneq \mathscr{B}, \quad 1 < p < q < \infty.$$

For $0 < p \leqslant \infty$, the Zygmund type space $\mathscr{Z}_p$ is the space of $f \in H(\mathbb{D})$ such that

$$||f||_{\mathscr{Z}_p} = |f(0)| + |f'(0)| + \sup_{0<r<1} (1-r^2) M_p(r, f'') < \infty.$$

The space $\mathscr{Z}_1$ is closely related to the mean Lipschitz space $\Lambda^p_{\frac{1}{p}}$. For $1 < p < \infty$, with the notations above, we see that $\Lambda^p_{\frac{1}{p}} \cong H^{p,\infty,1}_{1+\frac{1}{p}}$. On the other hand, the inclusions between mixed norm spaces (see [1]) show that

$$\mathscr{Z}_1 \cong H^{1,\infty,1}_2 \cong H^{1,\infty,\frac{1}{p}}_{1+\frac{1}{p}} \subsetneq H^{p,\infty,1}_{1+\frac{1}{p}} \cong \Lambda^p_{\frac{1}{p}}.$$

Therefore, the space $\mathscr{Z}_1$ can be regarded as the limit case of $H^{p,\infty,1}_{1+\frac{1}{p}} \cong \Lambda^p_{\frac{1}{p}}$ as $p \to 1$. In view of this point, we will use the symbol $\Lambda^{1,*}_1$ instead of $\mathscr{Z}_1$ in the sequel. Note that

$$\Lambda^{1,*}_1 \subsetneq \Lambda^p_{\frac{1}{p}} \subsetneq BMOA \subsetneq \mathscr{B} \text{ for all } 1 < p < \infty.$$

Let μ be a finite positive Borel measure on $[0,1)$ and $n \in \mathbb{N}$. We use μ_n to denote the n-th moment of μ, that is, $\mu_n = \int_{[0,1)} t^n d\mu(t)$. Let $\mathscr{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\geqslant 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix $\mathscr{H}_\mu$ induces an operator on $H(\mathbb{D})$ by its action on the Taylor coefficients: $a_n \to \sum_{k=0}^{\infty} \mu_{n,k} a_k$, $n \in \mathbb{N} \cup \{0\}$. The generalized Hilbert operator $\mathscr{H}_\mu$ is defined on the spaces $H(\mathbb{D})$ of analytic functions in the unit disc $\mathbb{D}$ as follows:

If $f \in H(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\mathscr{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n, \quad z \in \mathbb{D},$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. If μ is the Lebesgue measure on $[0,1)$, then the matrix $\mathscr{H}_\mu$ reduces to the classical Hilbert matrix $\mathscr{H} = (\frac{1}{n+k+1})_{n,k\geqslant 0}$, which induces the classical Hilbert operator $\mathscr{H}$.

Carleson measures play a key role when we study the generalized Hilbert operators. Recall that if μ is a positive Borel measure on $[0,1)$ and $0 < s < \infty$, then μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu([t,1)) \leqslant C(1-t)^s, \quad \text{for all } \ 0 \leqslant t < 1.$$

The study of the Hilbert matrix operator $\mathscr{H}$ on analytic function spaces was initiated by Diamantopoulos and Siskakis in [11], where they proved that $\mathscr{H}$ is bounded on the Hardy space $H^p(1<p<\infty)$, and provided an upper bound estimate for its norm. Subsequently, Diamantopoulus [12] considered the boundedness of $\mathscr{H}$ on the Bergman spaces $A^p(2<p<\infty)$ and obtained an upper bound estimate for the norm of $\mathscr{H}$. Dostanic, Jevtić and Vukotić extended this work in [13], where they provided the exact value of the norm of $\mathscr{H}$ on the Hardy space $H^p(1<p<\infty)$, and determined the precise value of the norm of $\mathscr{H}$ on the Bergman space A^p for $4<p<\infty$. However, they left an open problem for the case $2<p<4$ which has been solved by Božin and Karapetrović [7]. Following these developments, significant research has been devoted to investigating the boundedness of $\mathscr{H}$ and its norm on various analytic function spaces, such as weighted Bergman spaces, mixed norm spaces, Korenblum spaces, and Lipschitz spaces (see [3, 17, 18, 20, 22, 25, 32, 33] and references therein).

In 2012, Łanucha, Nowak and Pavlovic [23] observed the boundedness of $\mathscr{H}$ from H^∞ into $BMOA$. In fact, it is also true that

$$\mathscr{H}(H^\infty)\subset \bigcap_{1<p<\infty}\Lambda^p_{\frac{1}{p}}\subset BMOA\subset \mathscr{B}.$$

Recently, Bellavita and Stylogiannis [4] investigated the norm of $\mathscr{H}$ from H^∞ to Q_p spaces, to the mean Lipschitz spaces $\Lambda^p_{\frac{1}{p}}$ and to certain conformally invariant Dirichlet spaces. In this note, we will prove that the range of $\mathscr{H}$ acting on H^∞ is contained in the Zygmund type space $\Lambda^{1,*}_1$. The space $\Lambda^{1,*}_1$ is strictly smaller than $\Lambda^p_{\frac{1}{p}}$ for any $1<p<\infty$. We also provide both upper and lower bounds for the norm of $\mathscr{H}$ from H^∞ into $\Lambda^{1,*}_1$.

THEOREM 1.1. *Let μ be a finite positive Borel measure on $[0,1)$, then the generalized Hilbert operator $\mathscr{H}_\mu$ is bounded from H^∞ to $\Lambda^{1,*}_1$ if and only if μ is a Carleson measure.*

THEOREM 1.2. *The norm of $\mathscr{H}$ acting from H^∞ to $\Lambda^{1,*}_1$ satisfies*

$$\frac{3}{2}+\frac{2}{\pi}\leqslant \|\mathscr{H}\|_{H^\infty\to\Lambda^{1,*}_1}\leqslant \frac{3}{2}+\frac{4}{\pi}.$$

In [4], the authors proved that $||\mathscr{H}||_{H^\infty\to BMOA}=1+\frac{\pi}{\sqrt{2}}=1+||\log(1-z)||_{BMOA}$. They also point out that $||\mathscr{H}||_{H^\infty\to Q_p}=1+||\log(1-z)||_{Q_p}$. We know that $Q_p=\mathscr{B}$ when $p>1$. However, it is difficult to calculate the exact value of $||\log(1-z)||_{Q_p}$ even for $p>1$. Here, we shall prove that the exact norm of $\mathscr{H}$ from H^∞ into $\mathscr{B}$ is actually equal to $1+||\log(1-z)||_{\mathscr{B}}=3$.

THEOREM 1.3. *The Hilbert matrix operator $\mathscr{H}$ is bounded from H^∞ to $\mathscr{B}$ and*

$$\|\mathscr{H}\|_{H^\infty\to\mathscr{B}}=1+||\log(1-z)||_{\mathscr{B}}=3.$$

Widom [30, Theorem 3.1] proved that $\mathscr{H}_\mu$ is a bounded operator on H^2 if and only if μ is a Carleson measure. In 2010, Galanopoulos and Peláez [19] characterized the positive and finite Borel measures μ on $[0,1)$ for which the generalized Hilbert operator $\mathscr{H}_\mu$ is well-defined and bounded on H^1. These measures are classified as Carleson-type measures. In 2014, Chatzifountas, Girela and Peláez [9] described the measures μ for which $\mathscr{H}_\mu$ is a bounded operator from H^p to H^q for $0<p,q<\infty$. The extreme case $p=q=\infty$ was considered by Girela and Merchán [21] (see also [5]). However, there are two extreme cases that have not yet been considered: namely, $0<q<p=\infty$ and $0<p<q=\infty$. Another purpose of this paper is to deal with the extreme case $0<q<p=\infty$. To present our results regarding this question, we will first provide some definitions and notions.

For $0<p<\infty$, the Dirichlet-type space D^p_{p-1} is the space of $h\in H(\mathbb{D})$ such that

$$||h||^p_{D^p_{p-1}}=|h(0)|^p+\int_{\mathbb{D}}|h'(z)|^p(1-|z|)^{p-1}dA(z)<\infty.$$

When $p=2$, the space D^2_1 is just the Hardy space H^2.

The Hardy-Littlewood space $HL(p)$ consists of those functions $h\in H(\mathbb{D})$ for which

$$||h||^p_{HL(p)}=\sum_{n=0}^{\infty}(n+1)^{p-2}|\widehat{h}(n)|^p<\infty.$$

It is well known (see [15, 16]) that

$$D^p_{p-1}\subset H^p\subset HL(p),\ \ 0<p\leqslant 2, \tag{1}$$

$$HL(p)\subset H^p\subset D^p_{p-1},\ \ 2\leqslant p<\infty. \tag{2}$$

For $0<q<1$, let B_q denote the space consisting of those functions $g\in H(\mathbb{D})$ for which

$$||g||_{B_q}=\int_0^1(1-r)^{\frac{1}{q}-2}M_1(r,g)dr<\infty.$$

The space B_q is the mixed norm space $H^{1,1,\frac{1}{q}-1}$. The Hardy space H^q is a dense subspace of B_q and the two spaces have the same continuous linear functionals [14]. In [9], Chatzifountas, Girela and Peláez showed that $\mathscr{H}_\mu$ is bounded from H^p to B_q for all $0<p<\infty$ and $0<q<1$, whenever μ satisfies certain necessary conditions. Nevertheless, we can know more for $p=\infty$ and $0<q<1$. That is, the operator $\mathscr{H}_\mu$ is compact from H^∞ to B_q for every finite positive Borel measure μ on $[0,1)$.

Our main results are stated as follows.

THEOREM 1.4. *Let $1\leqslant q<\infty$ and let μ be a finite positive Borel measure on $[0,1)$. Let $Y_q\in\{D^q_{q-1},H^q,HL(q)\}$. Then the following statements are equivalent.*

(1) *$\mathscr{H}_\mu$ is bounded from H^∞ to Y_q.*

(2) *$\mathscr{H}_\mu$ is compact from H^∞ to Y_q.*

(3) *The measure satisfies $\{(n+1)^{1-\frac{2}{q}}\mu_n\}_{n=0}^\infty\in\ell^q$.*

THEOREM 1.5. *Let $0<q<1$ and let μ be a finite positive Borel measure on $[0,1)$. Then $\mathscr{H}_\mu$ is compact from H^∞ to B_q.*

The rest of the paper is organized as follows. Section 2 is devoted to proving Theorems 1.1–1.3, while Section 3 focuses on proving Theorems 1.4 and 1.5.

2. The range of the Hilbert operator acting on H^∞

The integral representation of $\mathscr{H}_\mu$ plays a basic role in this work. If μ is a finite positive Borel measure on $[0,1)$ and $f \in H(\mathbb{D})$, we shall write throughout the paper

$$\mathscr{I}_\mu(f)(z) = \int_0^1 \frac{f(t)}{(1-tz)} d\mu(t),$$

whenever the right hand side makes sense and defines an analytic function on $\mathbb{D}$. It turns out that the operators $\mathscr{H}_\mu$ and $\mathscr{I}_\mu$ are closely related. For instance, if μ is a Carleson measure, then $\mathscr{H}_\mu(f) = \mathscr{I}_\mu(f)$ for all $f \in H^1$ [19]. Since $H^\infty \subset H^1$, this is also valid for $f \in H^\infty$.

The following characterization of Carleson measures on $[0,1)$ is due to Bao et al. [2].

LEMMA 2.1. *Suppose $\beta > 0$, $0 \leqslant q < s < \infty$ and μ is a finite positive Borel measure on $[0,1)$. Then the following conditions are equivalent:*

(1) *μ is an s-Carleson measure;*

(2)
$$S_1 := \sup_{w \in \mathbb{D}} \int_0^1 \frac{(1-|w|)^\beta}{(1-t)^q (1-|w|t)^{s+\beta-q}} d\mu(t) < \infty;$$

(3)
$$S_2 := \sup_{w \in \mathbb{D}} \int_0^1 \frac{(1-|w|)^\beta}{(1-t)^q |1-wt|^{s+\beta-q}} d\mu(t) < \infty.$$

We also need the following estimates (see Theorem 1.3 in [24]).

LEMMA 2.2. *For $z \in \mathbb{D}$ and $c \in \mathbb{R}$, define*

$$I_c(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|1-ze^{-i\theta}|^{1+c}} d\theta.$$

Then the following statements hold.

(1) *If $c < 0$, then*

$$1 \leqslant I_c(z) \leqslant \frac{\Gamma(-c)}{\Gamma^2(\frac{1-c}{2})}.$$

(2) *If $c > 0$, then*

$$1 \leqslant (1-|z|^2)^c I_c(z) \leqslant \frac{\Gamma(c)}{\Gamma^2(\frac{1+c}{2})}.$$

(3) *If $c = 0$, then*

$$\frac{1}{\pi} \leqslant |z|^2 \left(\log \frac{1}{1-|z|^2} \right)^{-1} I_0(z) \leqslant 1.$$

Furthermore, all these inequalities are sharp.

Proof of Theorem 1.1. If μ is a Carleson measure, then $\mathscr{H}_\mu(f) = \mathscr{I}_\mu(f)$ for all $f \in H^\infty$. By a simple calculation, we have that

$$\mathscr{H}_\mu(f)''(z) = \int_0^1 \frac{2f(t)t^2}{(1-tz)^3} d\mu(t). \tag{3}$$

By (3), Fubini's theorem and Lemmas 2.1–2.2, we obtain

$$\begin{aligned}
\sup_{0<r<1}(1-r^2)M_1(r,\mathscr{H}_\mu(f)'') &= \sup_{0<r<1}(1-r^2)\frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 \frac{2t^2 f(t)d\mu(t)}{(1-tre^{i\theta})^3}\right| d\theta \\
&\leqslant \sup_{0<r<1}(1-r^2)\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 \frac{2t^2|f(t)|}{|1-tre^{i\theta}|^3} d\mu(t)d\theta \\
&= \sup_{0<r<1}(1-r^2)\int_0^1 2t^2|f(t)|\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-tre^{i\theta}|^3} d\mu(t) \\
&\leqslant \frac{2\Gamma(2)}{\Gamma^2(\frac{3}{2})}\sup_{0<r<1}(1-r^2)\int_0^1 |f(t)|t^2 \frac{d\mu(t)}{(1-t^2r^2)^2} \\
&\lesssim \|f\|_{H^\infty}\sup_{0<r<1}\int_0^1 \frac{(1-r^2)}{(1-tr)^2} d\mu(t) \\
&\lesssim \|f\|_\infty.
\end{aligned}$$

Therefore, $\mathscr{H}_\mu : H^\infty \to \Lambda_1^{1,*}$ is bounded.

On the other hand, if $\mathscr{H}_\mu : H^\infty \to \Lambda_1^{1,*}$ is bounded, then $\mathscr{H}_\mu(1)(z) = F_\mu(z) = \sum_{n=1}^\infty \mu_n z^n \in \Lambda_1^{1,*}$. This implies that

$$\sup_{0<r<1}(1-r^2)M_1(r,F_\mu'') < \infty. \tag{4}$$

By Fejér-Riesz inequality and Fubini's theorem, we have

$$\begin{aligned}
M_1(r,F_\mu'') &= \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 \frac{2t^2 d\mu(t)}{(1-tre^{i\theta})^2}\right| d\theta \\
&\geqslant \frac{1}{\pi}\int_0^1\int_0^1 \frac{2t^2 d\mu(t)}{(1-trx)^3} dx \\
&= \frac{1}{\pi}\int_0^1 2t^2 \int_0^1 \frac{dx}{(1-trx)^3} d\mu(t) \\
&\asymp \int_0^1 \frac{2t^2}{(1-tr)^2} d\mu(t).
\end{aligned}$$

Using (4) and inequalities above, we have that

$$\begin{aligned}
1 &\gtrsim \sup_{0<r<1}(1-r^2)M_1(r,F_\mu'') \\
&\gtrsim \sup_{0<r<1}(1-r^2)\int_0^1 \frac{2t^2}{(1-tr)^2} d\mu(t)
\end{aligned}$$

$$\geqslant \sup_{0<r<1}(1-r^2)\int_r^1 \frac{2t^2}{(1-tr)^2}d\mu(t)$$
$$\geqslant \sup_{\frac{1}{2}<r<1}\frac{(1-r^2)2r^2}{(1-r^2)^2}\mu([r,1))$$
$$\gtrsim \sup_{\frac{1}{2}<r<1}\frac{\mu([r,1))}{1-r}.$$

This implies that μ is a Carleson measure. □

Proof of Theorem 1.2. Let $f(z)=1$, then $\mathscr{H}(f)(0)=\int_0^1 f(t)dt=1$ and $\mathscr{H}(f)'(0)=\int_0^1 tf(t)dt=\frac{1}{2}$. As shown previously, we have

$$\begin{aligned}
M_1\left(r,\mathscr{H}(f)''(z)\right) &\geqslant \frac{1}{\pi}\int_0^1\int_0^1 \frac{2t^2}{(1-trx)^3}dtdx\\
&= \frac{1}{\pi}\int_0^1\int_0^1 2\sum_{n=0}^{\infty}\frac{\Gamma(3+n)}{\Gamma(n+1)\Gamma(3)}t^{n+2}r^nx^n dtdx\\
&= \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{n+2}{n+3}r^n.
\end{aligned}$$

For $0<r<1$, is is easy to compute that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{n+2}{n+3}r^n &= \sum_{n=0}^{\infty}r^n-\frac{1}{r^3}\sum_{n=0}^{\infty}\frac{r^{n+3}}{n+3}\\
&= \frac{1}{1-r}-\frac{1}{r^3}\left(\log\frac{1}{1-r}-r-\frac{r^2}{2}\right).
\end{aligned}$$

This yields that

$$\begin{aligned}
\|\mathscr{H}(f)\|_{\Lambda_1^{1,*}} &= |\mathscr{H}(f)(0)|+|(\mathscr{H}(f))'(0)|+\sup_{0<r<1}(1-r^2)M_1(r,\mathscr{H}(f)'')\\
&= \frac{3}{2}+\sup_{0<r<1}(1-r^2)M_1(r,\mathscr{H}(f)'')\\
&\geqslant \frac{3}{2}+\frac{1}{\pi}\sup_{0<r<1}(1-r^2)\left[\frac{1}{1-r}-\frac{1}{r^3}\left(\log\frac{1}{1-r}-r-\frac{r^2}{2}\right)\right]\\
&= \frac{3}{2}+\frac{1}{\pi}\sup_{0<r<1}(1+r)\left[1-\frac{(1-r)}{r^3}\left(\log\frac{1}{1-r}-r-\frac{r^2}{2}\right)\right].
\end{aligned}$$

Let

$$F(r)=1-\frac{(1-r)}{r^3}\left(\log\frac{1}{1-r}-r-\frac{r^2}{2}\right),\ 0<r<1.$$

After careful calculations, we obtain

$$F'(r)=\frac{\frac{r^2}{2}-3r-2r\log\frac{1}{1-r}+3\log\frac{1}{1-r}}{r^4}.$$

To show that $F(r)$ is increasing on the interval $(0,1)$, it suffices to prove that $\psi(r) = \frac{r^2}{2} - 3r - 2r\log\frac{1}{1-r} + 3\log\frac{1}{1-r} > 0$ on $(0,1)$. Now, it is easy to check that

$$\psi'(r) = r - 1 + \frac{1}{1-r} - 2\log\frac{1}{1-r} \quad \text{and} \quad \psi''(r) = \frac{r^2}{(1-r)^2}.$$

Since $\psi(0) = \psi'(0) = 0$ and $\psi''(r) > 0$ for all $0 < r < 1$, this means that $\psi(r) > 0$ for all $r \in (0,1)$. So we conclude that $F(r)$ is monotonically increasing on the interval $(0,1)$. This also implies that $(1+r)\left[1 - \frac{(1-r)}{r^3}\left(\log\frac{1}{1-r} - r - \frac{r^2}{2}\right)\right]$ is increasing on $(0,1)$.

By L'Höpital's rule we have that

$$\sup_{0<r<1}(1+r)F(r) = 2\lim_{r\to 1^-}\left[1 - \frac{1-r}{r^3}\left(\log\frac{1}{1-r} - r - \frac{r^2}{2}\right)\right] = 2.$$

Therefore, we get $\|\mathscr{H}(f)\|_{\Lambda_1^{1,\star}} \geqslant \frac{3}{2} + \frac{2}{\pi}$.

On the other hand, for any $f \in H^\infty$, we have

$$|\mathscr{H}(f)(0)| = \left|\int_0^1 f(t)dt\right| \leqslant \|f\|_{H^\infty}\int_0^1 dt = \|f\|_\infty,$$

and

$$|\mathscr{H}(f)'(0)| = \left|\int_0^1 tf(t)dt\right| \leqslant \|f\|_\infty \int_0^1 tdt = \frac{1}{2}\|f\|_\infty.$$

By the definition of $\Lambda_1^{1,*}$, we get

$$\begin{aligned}||\mathscr{H}(f)||_{\Lambda_1^{1,*}} &= |\mathscr{H}f(0)| + |\mathscr{H}(f)'(0)| + \sup_{0<r<1}(1-r^2)M_1(r,\mathscr{H}(f)'') \\ &\leqslant \frac{3}{2}||f||_\infty + \sup_{0<r<1}(1-r^2)M_1\left(r,\mathscr{H}(f)''\right).\end{aligned}$$

As the proof of Theorem 1.1 shows, we have

$$\begin{aligned}M_1\left(r,\mathscr{H}(f)''\right) &\leqslant ||f||_{H^\infty}\int_0^1 2t^2\frac{\Gamma(2)}{\Gamma^2(\frac{3}{2})}\frac{1}{(1-t^2r^2)^2}dt \\ &= ||f||_{H^\infty}\frac{1}{\pi}\int_0^1\frac{8t^2}{(1-t^2r^2)^2}dt \\ &= ||f||_\infty\frac{1}{\pi}\int_0^1\frac{4\rho^{\frac{1}{2}}}{(1-\rho r^2)^2}d\rho.\end{aligned}$$

Using the above inequalities, we obtain that

$$\begin{aligned}&\sup_{0<r<1}(1-r^2)M_1\left(r,\mathscr{H}(f)''\right) \\ \leqslant&||f||_{H^\infty}\frac{1}{\pi}\sup_{0<r<1}(1-r^2)\int_0^1\frac{4\rho^{\frac{1}{2}}}{(1-\rho r^2)^2}d\rho \\ =&||f||_{H^\infty}\frac{4}{\pi}\sup_{0<r<1}(1-r^2)\int_0^1\sum_{n=0}^\infty(n+1)\rho^{n+\frac{1}{2}}r^{2n}d\rho\end{aligned}$$

$$
\begin{aligned}
&=||f||_{H^\infty}\frac{4}{\pi}\sup_{0<r<1}(1-r^2)\sum_{n=0}^{\infty}\frac{n+1}{n+3/2}r^{2n}\\
&=||f||_{H^\infty}\frac{4}{\pi}\sup_{0<r<1}(1-r^2)\left[\sum_{n=0}^{\infty}r^{2n}-\sum_{n=0}^{\infty}\frac{r^{2n}}{2n+3}\right]\\
&=||f||_{H^\infty}\frac{4}{\pi}\sup_{0<r<1}(1-r^2)\left[\frac{1}{1-r^2}-\frac{\frac{1}{2}\log\frac{1+r}{1-r}-r}{r^3}\right]\\
&=\frac{4}{\pi}||f||_{H^\infty}\left(1-\inf_{0<r<1}\frac{(1-r^2)(\frac{1}{2}\log\frac{1+r}{1-r}-r)}{r^3}\right)\\
&=\frac{4}{\pi}||f||_{H^\infty}.
\end{aligned}
$$

Therefore, $\|\mathscr{H}\|_{H^\infty\to\Lambda_1^{1,*}}\leqslant\frac{3}{2}+\frac{4}{\pi}$. □

REMARK 2.3. In [29], the author proved that

$$C=\sup_{z\in\mathbb{D}}\int_{\mathbb{D}}\frac{2(1-|z|^2)|w|}{|1-z\overline{w}|^3}dA(w)=\sup_{0<r<1}\int_{\mathbb{D}}\frac{2(1-r^2)|w|}{|1-r\overline{w}|^3}dA(w)=\frac{8}{\pi}.$$

Using this result, we may easily obtain an upper bound estimate for the norm of $\mathscr{H}$ from H^∞ to $\Lambda_1^{1,*}$. As shown above,

$$
\begin{aligned}
M_1\left(r,\mathscr{H}(f)''\right)&\leqslant||f||_{H^\infty}\int_0^1 2t^2\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{d\theta}{|1-tre^{i\theta}|^3}\right)dt\\
&=||f||_{H^\infty}\int_{\mathbb{D}}\frac{2|w|}{|1-r\overline{w}|^3}dA(w).
\end{aligned}
$$

This implies that

$$\sup_{0<r<1}(1-r^2)M_1\left(r,\mathscr{H}(f)''\right)\leqslant||f||_{H^\infty}\sup_{0<r<1}\int_{\mathbb{D}}\frac{2(1-r^2)|w|}{|1-r\overline{w}|^3}dA(w)=\frac{8}{\pi}||f||_{H^\infty}.$$

Proof of Theorem 1.3. Let $f\in H^\infty$, by the integral form of $\mathscr{H}(f)$ we have that

$$\mathscr{H}(f)'(z)=\int_0^1\frac{tf(t)}{(1-tz)^2}dt.$$

The convergence of the integral and the analyticity of the function f guarantee that we can change the path of integration to

$$t=\gamma(s)=\frac{s}{1-(1-s)z},\quad 0\leqslant s\leqslant 1.$$

Therefore, we have that

$$
\begin{aligned}
\mathscr{H}(f)'(z)&=\frac{1}{1-z}\int_0^1\frac{s}{1-(1-s)z}f\left(\frac{s}{1-(1-s)z}\right)ds\\
&=:-\frac{1}{1-z}g(z).
\end{aligned}
$$

Since $\psi_s(z) := \frac{s}{1-(1-s)z}$ maps the unit disc into itself for each $0 \leqslant s < 1$, it follows that

$$|\psi_s(z) f(\psi_s(z))| \leqslant ||f||_\infty.$$

So we can rewrite $\mathscr{H}(f)'$ as

$$\mathscr{H}(f)'(z) = g(z)(\log(1-z))', \tag{5}$$

where $g \in H^\infty$ and $||g||_\infty \leqslant ||f||_\infty$.

Now, using (5) we get

$$\begin{aligned}
\|\mathscr{H}(f)\|_{\mathscr{B}} &= |\mathscr{H} f(0)| + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left|\mathscr{H}'(f)(z)\right| \\
&\leqslant ||f||_\infty + ||g||_\infty || \log(1-z)||_{\mathscr{B}} \\
&\leqslant ||f||_\infty + ||f||_\infty || \log(1-z)||_{\mathscr{B}} \\
&= ||f||_\infty (1 + || \log(1-z)||_{\mathscr{B}}) = 3||f||_\infty.
\end{aligned}$$

On the other hand, we choose the test function $h(z) = 1$. Then, $h \in H^\infty$ and $||h||_\infty = 1$. Thus,

$$\begin{aligned}
\|\mathscr{H}\|_{\mathscr{B}} \geqslant ||\mathscr{H}(h)||_{\mathscr{B}} &= 1 + \sup_{z\in\mathbb{D}} \left(1-|z|^2\right) \left| \int_0^1 \frac{t}{(1-tz)^2} dt \right| \\
&\geqslant 1 + \sup_{0\leqslant x<1} (1-x^2) \int_0^1 \frac{t}{(1-tx)^2} dt.
\end{aligned}$$

Therefore,

$$\|\mathscr{H}\|_{H^\infty \to \mathscr{B}} \geqslant 1 + \sup_{0\leqslant x<1} \left(1-x^2\right) \int_0^1 \frac{t}{(1-tx)^2} dt.$$

By making a change of variables $t = \frac{1-s}{1-xs}$, we have

$$\begin{aligned}
&\sup_{0\leqslant x<1} (1-x^2) \int_0^1 \frac{t}{(1-tx)^2} dt \\
&= \sup_{0\leqslant x<1} \left(1-x^2\right) \int_0^1 \frac{1-s}{1-xs} \left(\frac{1-xs}{1-x}\right)^2 \frac{1-x}{(1-xs)^2} ds \\
&= \sup_{0\leqslant x<1} (1+x) \int_0^1 \frac{1-s}{1-xs} ds.
\end{aligned}$$

Let

$$G(x) = (1+x) \int_0^1 \frac{1-s}{1-xs} ds, \quad x \in [0,1).$$

For fixed $s \in [0,1]$, $(1-xs)^{-1}$ is monotonically increasing with respect to x on the interval $[0,1)$. It follows that $G(x)$ is monotonically increasing in $[0,1)$ and hence

$$\sup_{0\leqslant x<1} G(x) = \lim_{x\to 1^-} G(x) = 2.$$

Hence, we conclude that

$$\|\mathscr{H}\|_{H^\infty\to\mathscr{B}} = 3. \quad \square$$

REMARK 2.4. Recall that the Cesàro operator $\mathscr{C}$ is defined in $H(\mathbb{D})$ as follows: If $f \in H(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then

$$\mathscr{C}(f)(z) = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n} a_k\right) z^n = \int_0^1 \frac{f(tz)}{1-tz}dt,\ z\in\mathbb{D}.$$

The Cesàro operator $\mathscr{C}$ has been extensively studied on various spaces of analytic functions, and its integral form is closely related to the Hilbert operator $\mathscr{H}$. In [10], Danikas and Siskakis proved that $\mathscr{C}$ is bounded from H^∞ to $BMOA$, and $\|\mathscr{C}\|_{H^\infty\to BMOA} = 1+\frac{\pi}{\sqrt{2}}$. Since $BMOA \subsetneq \mathscr{B}$, the Cesàro operator $\mathscr{C}$ is also bounded from H^∞ to $\mathscr{B}$. Following the above arguments, it is easy to obtain that

$$\|\mathscr{C}\|_{H^\infty\to\mathscr{B}} = 3.$$

3. The Hilbert operator acting from H^∞ to Hardy spaces

We begin with some preliminary results that will be used repeatedly throughout the rest of the paper. The first lemma provides a characterization of L^p-integrability of power series with nonnegative coefficients. See [28, Theorem 1] for the proof.

LEMMA 3.1. *Let* $0<\beta, p<\infty$, $\{\lambda_n\}_{n=0}^{\infty}$ *be a sequence of non-negative numbers. Then*

$$\int_0^1 (1-r)^{p\beta-1}\left(\sum_{n=0}^{\infty}\lambda_n r^n\right)^p dr \asymp \sum_{n=0}^{\infty} 2^{-np\beta}\left(\sum_{k\in I_n}\lambda_k\right)^p,$$

where $I_0=\{0\}$, $I_n = [2^{n-1}, 2^n)\cap\mathbb{N}$ *for* $n\in\mathbb{N}$.

The following result can be found in [26] and hence its proof is omitted.

LEMMA 3.2. *Let* $1<q<\infty$ *and* $Y_q \in \{D_{q-1}^q, H^q, HL(q)\}$. *Suppose* $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ *and the sequence* $\{a_n\}_{n=0}^{\infty}$ *is non-negative decreasing, then* $f\in Y_q$ *if and only if*

$$\{(n+1)^{1-\frac{2}{q}} a_n\}_{n=0}^{\infty} \in \ell^q.$$

The following lemma is a consequence of the Lebesgue Dominated Convergence Theorem.

LEMMA 3.3. *Let* μ *be a finite positive Borel measure on* $[0,1)$. *Let* $\{f_k\}_{k=1}^{\infty} \subset H^\infty$ *such that* $\sup_{k\geqslant 1}\|f_k\|_\infty < \infty$ *and* $f_k\to 0$ *uniformly on compact subsets of* $\mathbb{D}$. *Then*

$$\lim_{k\to\infty}\int_0^1 |f_k(t)|d\mu(t) = 0.$$

Proof of Theorem 1.4. It suffices to prove that (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (2).

(1) $\Rightarrow$ (3). Let $f(z) \equiv 1 \in H^\infty$, then $\mathscr{H}(1)(z) = \sum_{n=0}^{\infty} \mu_n z^n \in Y_p$. If $1 < q < \infty$, the desired result follows from Lemma 3.2. If $q = 1$, then (1) shows that $D_0^1 \subset H^1 \subset HL(1)$. This means that $Y_1 \subset HL(1)$, so we have that $(n+1)^{-1}\mu_n \in \ell^1$.

(3) $\Rightarrow$ (2). Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in H^∞ which converges to 0 uniformly on every compact subset of $\mathbb{D}$. Without loss of generality, we may assume that $f_k(0) = 0$ for all $k \geqslant 1$ and $\sup_{k\geqslant 1} ||f||_\infty \leqslant 1$.

Case $1 \leqslant q \leqslant 2$. Since $D_{q-1}^q \subset H^q \subset HL(q)$, it suffices to prove that

$$\lim_{k\to\infty} ||\mathscr{H}_\mu(f_k)||_{D_{q-1}^q} = 0.$$

Assume that $\sum_{n=1}^{\infty}(n+1)^{q-2}\mu_n^q < \infty$. Then,

$$\begin{aligned}\sum_{n=1}^{\infty}(n+1)^{q-2}\mu_n^q &= \sum_{n=1}^{\infty}\left(\sum_{k=2^{n-1}}^{2^n-1}(k+1)^{q-2}\mu_k^q)\right)\\ &\asymp \sum_{n=1}^{\infty} 2^{n(q-1)}\mu_{2^n}^q\\ &\asymp \sum_{n=1}^{\infty} 2^{-nq}\left(\sum_{k=2^n}^{2^{n+1}-1}(k+1)^{1-\frac{1}{q}}\mu_k\right)^q.\end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} 2^{-nq}\left(\sum_{k=2^n}^{2^{n+1}-1}(k+1)^{1-\frac{1}{q}}\mu_k\right)^q < \infty.$$

By Lemma 3.1 we have that

$$\begin{aligned}&\int_0^1 (1-r)^{q-1}\left(\sum_{n=0}^{\infty}(n+1)^{1-\frac{1}{q}}\mu_n r^n\right)^q dr\\ &\asymp \sum_{n=0}^{\infty} 2^{-nq}\left(\sum_{k=2^n}^{2^{n+1}-1}(k+1)^{1-\frac{1}{q}}\mu_k\right)^q < \infty.\end{aligned}$$

Therefore, for any $\varepsilon > 0$ there exists a $0 < r_0 < 1$ such that

$$\int_{r_0}^1 (1-r)^{q-1}\left(\sum_{n=0}^{\infty}(n+1)^{1-\frac{1}{q}}\mu_k r^n\right)^q dr < \varepsilon. \tag{6}$$

It is clear that

$$\begin{aligned}||\mathscr{H}_\mu(f_k)||_{D_{q-1}^q}^q &= \int_{|z|\leqslant r_0} |\mathscr{H}_\mu(f_k)'(z)|^q(1-|z|)^{q-1}dA(z)\\ &\quad + \int_{r_0<|z|<1} |\mathscr{H}_\mu(f_k)'(z)|^q(1-|z|)^{q-1}dA(z)\\ &:= J_{1,k} + J_{2,k}.\end{aligned}$$

By the integral representation of $\mathscr{H}_\mu$, we get

$$\mathscr{H}_\mu(f_k)'(z) = \int_0^1 \frac{t f_k(t)}{(1-tz)^2} d\mu(t). \tag{7}$$

Since $\{f_k\}_{k=1}^\infty$ is converge to 0 uniformly on every compact subset of $\mathbb{D}$, for $|z| \leqslant r_0$ we have that

$$\begin{aligned} |\mathscr{H}_\mu(f_k)'(z)| &\leqslant \int_0^1 \frac{|f_k(t)|}{|1-tz|^2} d\mu(t) \\ &\lesssim \int_0^1 |f_k(t)| d\mu(t). \end{aligned}$$

It follows Lemma 3.3 that

$$J_{1,k} \to 0, \quad \text{as } k \to \infty.$$

By Minkowski's inequality and Lemma 2.2, we have that

$$\begin{aligned} M_q(r, \mathscr{H}_\mu(f_k)') &= \left\{ \int_0^{2\pi} \left| \int_0^1 \frac{t f_k(t)}{(1-tre^{i\theta})^2} d\mu(t) \right|^q d\theta \right\}^{\frac{1}{q}} \\ &\lesssim \left\{ \int_0^{2\pi} \left(\int_0^1 \frac{1}{|1-tre^{i\theta}|^2} d\mu(t) \right)^q d\theta \right\}^{\frac{1}{q}} \\ &\lesssim \int_0^1 \left(\int_0^{2\pi} \frac{d\theta}{|1-tre^{i\theta}|^{2q}} \right)^{\frac{1}{q}} d\mu(t) \\ &\lesssim \int_0^1 \frac{1}{(1-tr)^{2-\frac{1}{q}}} d\mu(t) \\ &\asymp \sum_{n=0}^\infty (n+1)^{1-\frac{1}{q}} \mu_n r^n. \end{aligned}$$

Thus, by the polar coordinate formula and (6), we obtain

$$\begin{aligned} J_{2,k} &= \int_{r_0<|z|<1} |\mathscr{H}_\mu(f_k)'(z)|^q (1-|z|)^{q-1} dA(z) \\ &\lesssim \int_{r_0}^1 (1-r)^{q-1} M_q^q(r, \mathscr{H}_\mu(f_k)') dr \\ &\lesssim \int_{r_0}^1 (1-r)^{q-1} \left(\sum_{n=0}^\infty (n+1)^{1-\frac{1}{q}} \mu_n r^n \right)^q dr \\ &\lesssim \varepsilon. \end{aligned}$$

Consequently,

$$\lim_{k\to\infty} ||\mathscr{H}_\mu(f_k)||_{D^q_{q-1}} = 0.$$

Case $q > 2$. By (2) we see that $HL(q) \subset Y_q$. To complete the proof, we have to prove that $\lim_{k\to\infty} ||\mathscr{H}_\mu(f_k)||_{HL(q)} = 0$.

It is clear that the integral $\int_0^1 t^n f_k(t)d\mu(t)$ converges absolutely for all $n,k\in\mathbb{N}$. It follows that

$$\begin{aligned}\mathscr{H}_\mu(f_k)(z) &= \int_0^1 \frac{f_k(t)}{1-tz}d\mu(t)\\ &= \int_0^1 \sum_{n=0}^{\infty} t^n f_k(t)z^n d\mu(t)\\ &= \sum_{n=0}^{\infty}\left(\int_0^1 t^n f_k(t)d\mu(t)\right)z^n.\end{aligned}$$

Since $\sum_{n=1}^{\infty}(n+1)^{q-2}\mu_n^q<\infty$, we see that for any $\varepsilon>0$, there exists a positive integer N such that

$$\sum_{n=N+1}^{\infty}(n+1)^{q-2}\mu_n^q<\varepsilon. \tag{8}$$

For each $k\geqslant 1$, we have

$$\begin{aligned}&\sum_{n=0}^{N}(n+1)^{q-2}\left|\int_0^1 t^n f_k(t)d\mu(t)\right|^q\\ &\leqslant \sum_{n=0}^{N}(n+1)^{q-2}\left(\int_0^1 |f_k(t)|d\mu(t)\right)^q\\ &\lesssim \left(\int_0^1 |f_k(t)|d\mu(t)\right)^q.\end{aligned}$$

By Lemma 3.3, there there exists $k_0\in\mathbb{N}$ such that

$$\left(\int_0^1 |f_k(t)|d\mu(t)\right)^q<\varepsilon \text{ for all } k>k_0. \tag{9}$$

Hence, for $k>k_0$, by (8) and (9) we have that

$$\begin{aligned}||\mathscr{H}_\mu(f_k)||_{HL(q)}^q &= \left(\sum_{n=0}^{N}+\sum_{n=N+1}^{\infty}\right)(n+1)^{q-2}\left|\int_0^1 t^n f_k(t)d\mu(t)\right|^q\\ &\lesssim \left(\int_0^1 |f_k(t)|d\mu(t)\right)^q+\sup_{k\geqslant 1}||f_k||_\infty \sum_{n=N+1}^{\infty}(n+1)^{q-2}\mu_n^q\\ &\lesssim \varepsilon+\sum_{n=N+1}^{\infty}(n+1)^{q-2}\mu_n^q\\ &\lesssim \varepsilon.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}||\mathscr{H}_\mu(f_k)||_{HL(q)}=0. \quad \square$$

Proof of Theorem 1.5. Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in H^∞ which converges to 0 uniformly on every compact subset of $\mathbb{D}$. For $0 < r < 1$, by Fubini's theorem and Lemma 2.2 we have that

$$\begin{aligned} M_1(r, \mathscr{H}_\mu(f_k)) &= \frac{1}{2\pi}\int_0^{2\pi}\left|\int_0^1 \frac{f_k(t)}{(1-tre^{i\theta})}d\mu(t)\right|d\theta \\ &\leqslant \int_0^1 |f_k(t)|\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|1-tre^{i\theta}|}d\theta d\mu(t) \\ &\lesssim \log\frac{e}{1-r}\int_0^1 |f_k(t)|d\mu(t). \end{aligned}$$

Since the integral $\int_0^1 (1-r)^{\frac{1}{q}-2}\log\frac{e}{1-r}dr$ converges, by Lemma 3.3 we have that

$$\lim_{k\to\infty} ||\mathscr{H}_\mu(f_k)||_{B_q} = 0.$$

This implies that $\mathscr{H}_\mu$ is compact from H^∞ to B_q. □

Conflicts of interest. The authors declare that there is no conflict of interest.

Funding. The first author was supported by the Scientific Research Foundation of Hunan Provincial Education Department (No. 24B0861) and Natural Science Foundation of Hunan Province (No. 2025JJ60064), the second author was supported by the Scientific Research Foundation of Hunan Provincial Education Department (No. 24C0222).

Availability of data and materials. Data sharing not applicable to this article as no datasets were generated or analysed during the current study: the article describes entirely theoretical research.

Acknowledgements. The authors sincerely thank the anonymous reviewers for their insightful comments and constructive suggestions, which have significantly improved the quality of this manuscript.

REFERENCES

[1] I. ARÉVALO, *A characterization of the inclusions between mixed norm spaces*, J. Math. Anal. Appl. **429** (2015) 942–955.

[2] G. BAO, F. SUN, H. WULAN, *Carleson measure and the range of Cesàro-like operator acting on H^∞*, Anal. Math. Phys. **12** (2022) no. 142.

[3] G. BAO, H. WULAN, *Hankel matrices acting on Dirichlet spaces*, J. Math. Anal. Appl. **409** (2014) 228–235.

[4] C. BELLAVITA, G. STYLOGIANNIS, *Hilbert matrix operator acting between conformally invariant spaces*, Canad. Math. Bull. **68** (2025) 550–567.

[5] M. BELTRÁN-MENEU, J. BONET, E. JORDÁ, *Generalized Hilbert operators acting on weighted spaces of holomorphic functions with sup-norms*, Banach J. Math. Anal. **19** (2025) no. 10.

[6] P. BOURDON, J. SHAPIRO, AND W. SLEDD, *Fourier series, mean Lipschitz spaces, and bounded mean oscillation*, Analysis at Urbana, vol. I (Urbana, IL, 1986–1987), 81–110, London Math. Soc. Lecture Note Ser. **137**, Cambridge Univ. Press, Cambridge, 1989.

[7] V. BOŽIN, B. KARAPETROVIĆ, *Norm of the Hilbert matrix on Bergman spaces*, J. Funct. Anal. **274** (2018) 525–543.

[8] S. BUCKLEY, P. KOSKELA, D. VUKOTIĆ, *Fractional integration, differentiation, and weighted Bergman spaces*, Math. Proc. Camb. Philos. Soc. **126** (2) (1999) 369–385.

[9] C. CHATZIFOUNTAS, D. GIRELA, J. PELÁEZ, *A generalized Hilbert matrix acting on Hardy spaces*, J. Math. Anal. Appl. **413** (2014) 154–168.

[10] N. DANIKAS, A. SISKAKIS, *The Cesàro operator on bounded analytic functions*, Analysis **13** (1993) 295–299.

[11] E. DIAMANTOPOULOS, A. SISKAKIS, *Composition operators and the Hilbert matrix*, Studia Math. **140** (2000) 191–198.

[12] E. DIAMANTOPOULOS, *Hilbert matrix on Bergman spaces*, Illinois J. Math. **48** (3) (2004) 1067–1078.

[13] M. DOSTANIĆ, M. JEVTIĆ, D. VUKOTIĆ, *Norm of the Hilbert matrix on Bergman and Hardy spaces and theorem of Nehari type*, J. Funct. Anal. **254** (2008) 2800–2815.

[14] P. DUREN, B. ROMBERG, A. SHIELDS, *Linear functionals on H^p spaces with $0 < p < 1$*, J. Reine Angew. Math. **238** (1969) 32–60.

[15] P. DUREN, *Theory of H^p spaces*, Academic Press, New York, 1970.

[16] T. FLETT, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. **38** (1972) 746–765.

[17] P. GALANOPOULOS, D. GIRELA, A. MAS, N. MERCHÁN, *Operators induced by radial measures acting on the Dirichlet space*, Results Math. **78** (2023) no. 106.

[18] P. GALANOPOULOS, D. GIRELA, J. PELÁEZ, A. SISKAKIS, *Generalized Hilbert operators*, Ann. Acad. Sci. Fenn. Math. **39** (1) (2014) 231–258.

[19] P. GALANOPOULOS, J. PELÁEZ, *A Hankel matrix acting on Hardy and Bergman spaces*, Studia Math. **200** (3) (2010) 201–220.

[20] D. GIRELA, N. MERCHÁN, *A generalized Hilbert operator acting on conformally invariant spaces*, Banach J. Math. Anal. Appl. **12** (2018) 374–398.

[21] D. GIRELA, N. MERCHÁN, *A Hankel matrix acting on spaces of analytic functions*, Integral Equ. Oper. Theory **89** (4) (2017) 581–594.

[22] M. JEVTIĆ, B. KARAPETROVIĆ, *Hilbert matrix on spaces of Bergman-type*, J. Math. Anal. Appl. **453** (2017) 241–254.

[23] B. ŁANUCHA, M. NOWAK, M. PAVLOVIĆ, *Hilbert matrix operator on spaces of analytic functions*, Ann. Acad. Sci. Fenn. Math. **37** (2012) 161–174.

[24] C. LIU, *Sharp Forelli-Rudin estimates and the norm of the Bergman projection*, J. Funct. Anal. **268** (2) (2015) 255–277.

[25] N. MERCHÁN, *Mean Lipschitz spaces and a generalized Hilbert operator*, Collect. Math. **70** (2019) 59–69.

[26] M. PAVLOVIĆ, *Analytic functions with decreasing coefficients and Hardy and Bloch spaces*, Proc. Edinb. Math. Soc. **56** (2013) 623–635.

[27] M. PAVLOVIĆ, *Function classes on the unit disc, An Introduction*, De Gruyter Studies in Mathematics, vol. 52, De Gruyter, Berlin, 2014.

[28] M. PAVLOVIĆ, M. MATELJEVIĆ, *L^p-behavior of power series with positive coefficients and Hardy spaces*, Proc. Amer. Math. Soc. **87** (2) (1983) 309–316.

[29] A. PERÄLÄ, *On the optimal constant for the Bergman projection onto the Bloch space*, Ann Acad. Sci. Fenn. Math. **37** (2012) 245–249.

[30] H. WIDOM, *Hankel matrices*, Trans. Amer. Math. Soc. **121** (1966) 1–35.

[31] J. XIAO, *Holomorphic Q Classes*, Springer, Lecture Notes in Math, Berlin, 2001.

[32] S. YE, Z. ZHOU, *A Derivative-Hilbert operator acting on Bergman spaces*, J. Math. Anal. Appl. **506** (2021) no. 125553.

[33] S. YE, G. FENG, *Generalized Hilbert operators acting on weighted Bergman spaces and Dirichlet spaces*, Banach J. Math. Anal. **17** (2023) no. 38.
[34] A. ZYGMUND, *Trigonometric Series*, vol. I, II. Cambridge University Press, London, 1959.

(Received February 20, 2025)

Yuting Guo
School of Mathematics and Statistics
Hunan First Normal University
Changsha, Hunan 410205, China
e-mail: rainting2770@163.com

Pengcheng Tang
School of Mathematics and Statistics
Hunan University of Science and Technology
Xiangtan, Hunan 411201, China
e-mail: www.tang-tpc.com@foxmail.com