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OPERATOR RADIUS INEQUALITIES FOR SEVERAL OPERATORS ON HILBERT SPACES

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Abstract. Let $\omega_\rho(X)$ denote the ρ-operator radius of a bounded linear operator X on a finite dimensional Hilbert space $\mathscr{H}$, where $0<\rho\leqslant 2$. In this article, we present ρ-operator radii generalizations of various numerical radius commutator inequalities, including

$$\omega(SX+XS)\leqslant 2\sqrt{2}\,\omega(S)\cdot\|X\|,$$

$$\omega(SX^*+X^*S)\leqslant 2\,\omega(S)\cdot\|X\|,$$

and the arithmetic-geometric mean inequality:

$$\omega(XSY^*)\leqslant \frac{1}{2}\,\omega\left(|X|^2S+S|Y|^2\right),$$

under various conditions on X and Y.

1. Introduction

Let $\mathscr{H}$ be a finite dimensional complex Hilbert space equipped with the inner product $\langle\cdot,\cdot\rangle$, and let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. For clarity of exposition, the discussion is restricted to the finite-dimensional setting, although the results extend naturally to compact trace class operators on separable complex Hilbert spaces. The bounded linear operator $S\in\mathscr{B}(\mathscr{H})$ is said to admit a unitary ρ-dilation if there exists a unitary operator V on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ as a subspace, such that for every positive integer n the following relation holds:

$$S^n=\rho P_{\mathscr{H}}\circ V^n\big|_{\mathscr{H}},$$

where $\rho>0$ and $P_{\mathscr{H}}$ represents the projection from $\mathscr{K}$ onto $\mathscr{H}$ [18]. The set of all such operators S is denoted by $\mathscr{E}_\rho$. Halbrook [8] and Williams [19] observed that $\mathscr{E}_\rho$ is an absorbing subset of $\mathscr{B}(\mathscr{H})$. This observation led them to define the ρ-operator radius $\omega_\rho(S)$ of an operator S as the Minkowski functional associated with the absorbing set $\mathscr{E}_\rho$. Specifically,

$$\omega_\rho(X)=\inf\left\{\mu>0\mid \mu^{-1}X\in\mathscr{E}_\rho\right\}.$$

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ρ-operator radii have been a subject of particular interest in the literature. It has been shown that $\omega_\rho(\cdot)$ generally acts as a quasi-norm and satisfies the triangle inequality only for $0 < \rho \leqslant 2$. The study of ρ-operator radii becomes more important given that the notion of ρ-radii unifies three key measures in operator theory. Notably, $\omega_\rho(X)$ equals the operator norm $\|X\|$ when $\rho = 1$, the numerical radius $\omega(X)$ when $\rho = 2$, and converges to the spectral radius $\nu(X)$ as $\rho \to \infty$.

Throughout this article, we consider ρ to be in the range $0 < \rho \leqslant 2$. Some basic properties of $\omega_\rho(\cdot)$, where $0 < \rho \leqslant 2$, that will be used in this paper are the following: For $X, Y \in \mathscr{B}(\mathscr{H})$, we have

1. $$\omega_\rho\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\left(\omega_\rho(X), \omega_\rho(Y)\right)$$

2. $\omega_\rho(\cdot)$ is a self adjoint norm,
$$\omega_\rho(X^*) = \omega_\rho(X)$$

3. $\omega_\rho(\cdot)$ is weakly unitarily invariant, i.e., for any unitary operator $U \in \mathscr{B}(\mathscr{H})$,
$$\omega_\rho(U^*XU) = \omega_\rho(X)$$

4. For all $\rho > 0$ and integer $n \geqslant 1$,
$$\omega_\rho(X^n) \leqslant \left(\omega_\rho(X)\right)^n.$$

For proofs of all these properties and more information about operator radii we refer the reader to [3, 5, 8–10, 18, 19].

In [5], Fong and Holbrook established two remarkable numerical radius inequalities for commutators of bounded linear operators. Specifically, for any operators $S, X, T \in \mathscr{B}(\mathscr{H})$, they proved that

$$\omega(X^*S + SX^*) \leqslant 2\|X\|\omega(S) \tag{1.1}$$

and

$$\omega(SX + XS) \leqslant 2\sqrt{2}\|X\|\omega(S). \tag{1.2}$$

Some generalization of these inequalities have been given in [6, 7, 16].

We now present some recent known inequalities involving the norm and numerical radius for products of bounded operators on finite-dimensional Hilbert spaces. For $S, X, Y \in \mathscr{B}(\mathscr{H})$, it is well known that

$$\|XSY^*\| \leqslant \frac{1}{2}\left\||X|^2S + S|Y|^2\right\|, \tag{1.3}$$

where $|X| = (X^*X)^{1/2}$. This inequality is commonly referred as the arithmetic-geometric mean inequality. For further discussion, we refer the reader to [4]. A natural question arises as to whether an analogous inequality holds for the numerical radius. Specifically, one may inquire whether

$$\omega(XSY^*) \leqslant \frac{1}{2}\,\omega\big(|X|^2S+S|Y|^2\big) \tag{1.4}$$

holds for arbitrary operators $S,X,Y \in \mathscr{B}(\mathscr{H})$. It has been established that this inequality holds in the particular cases when either $X=Y$ or $X^*Y=0$ [13]. However, it does not hold in general, as demonstrated by a counterexample presented in [17].

To illustrate this, consider the 2×2 matrices

$$X=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad Y=\begin{bmatrix}2+\sqrt{3} & 0\\ 0 & 1\end{bmatrix}, \quad S=\begin{bmatrix}\frac{1}{4+\sqrt{3}} & 3\\ 0 & -2\end{bmatrix}.$$

Note that X and Y are positive definite matrices. By direct computation, we obtain

$$XSY^*=\begin{bmatrix}0.5 & 3\\ 0 & -2\end{bmatrix} \quad \text{and} \quad |X|^2S+S|Y|^2=\begin{bmatrix}2 & 6\\ 0 & -4\end{bmatrix}.$$

Using Theorem 1 in [11], it follows that

$$\omega(XSY^*)=2.7025 \quad \text{and} \quad \omega(|X|^2S+S|Y|^2)=2.6213,$$

which furnishes a counterexample to inequality (1.4). Throughout this article, for any bounded operator S on a finite-dimensional Hilbert space $\mathscr{H}$, the Frobenius norm of S is defined and denoted by

$$\|S\|_F=\sqrt{\operatorname{tr}(S^*S)},$$

where $\operatorname{tr}(S^*S)$ denotes the trace of the positive operator S^*S.

The remainder of this paper is organized as follows.

In Section 2, we present ρ-operator radii generalizations of the generalized commutator various inequalities including (1.1) and (1.2).

In Section 3, we give refinements and ρ-operator radii generalizations of inequality (1.4) for several specific cases.

2. Operator radius commutator inequalities

The motivation for our first main result in this section stems from a query raised in the work of Fong and Holbrook [5] on unitarily invariant operator norms. In their study, they establish the following key result:

LEMMA 2.1. *Let $x_k \in \mathscr{H}$ with $\|x_k\| \leqslant 1$ for all $1 \leqslant k \leqslant n$, and let $S \in \mathscr{B}(\mathscr{H})$. Then,*

$$|\langle Sx_1,x_2\rangle+\langle Sx_2,x_3\rangle+\ldots+\langle Sx_{n-1},x_n\rangle| \leqslant n\,\omega(S),$$

where $\omega(S)$ denotes the numerical radius of S.

As an application of this lemma, they derive the following two commutator inequalities:

$$\omega(SX+XS) \leqslant 3\,\omega(S)\|X\|, \tag{2.1}$$

and

$$\omega(SX^*+X^*S) \leqslant 2\,\omega(S)\|X\|. \tag{2.2}$$

Further, they remarked that, "The bound 3 in inequality (2.1) is not optimal and can be improved; we have presented it separately because it is based on a particularly simple method (Lemma 2.1) that could have analogues for general operator radii ω_ρ". But, the analogues version of Lemma 2.1, for general operator radii does not appear to be a simple task, since the proof of Lemma 2.1 crucially relies on the specific definition of the numerical radius in terms of the inner product:

$$\omega(S) = \sup_{\|x\|=1} |\langle Sx, x\rangle|,$$

a characterization that does not have a direct analogue for general operator radii.

In [6], Hirzallah and Kittaneh, as an application of slightly improved version of Lemma 2.1, expressed the numerical radius of a bounded operator S in terms of the numerical radius of a block operator matrix. Using this formulation, they derived further generalizations of the numerical radius inequality (2.2).

In the following proposition, we present a generalization of their representation for operator radii, expressing the operator radius of a bounded operator S in terms of block operator matrices. This formulation will subsequently be used to derive generalizations of inequality (2.2) for general operator radii.

PROPOSITION 2.1. *Let $S \in \mathscr{B}(\mathscr{H})$, and let $\tilde{S}$ be a $n \times n$ operator matrix in $\mathscr{B}(\oplus_{k=1}^{n}\mathscr{H})$ that has the operator S in the sub diagonal and in the top right hand corner, in the position $(1,n)$ and zero O everywhere else. Then*

$$\omega_\rho(\tilde{S}) = \omega_\rho(S).$$

Proof. The proof depends on the the fact that ω_ρ is weakly unitarily invariant. Define a block version of unitary discrete transform matrix $\tilde{D}_n$ as,

$$\tilde{D}_n = \frac{1}{\sqrt{n}} \begin{bmatrix} I_n & I_n & I_n & \cdots & I_n \\ I_n & \theta I_n & \theta^2 I_n & \cdots & \theta^{n-1} I_n \\ I_n & \theta^2 I_n & \theta^4 I_n & \cdots & \theta^{2(n-1)} I_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ I_n & \theta^{n-1} I_n & \theta^{2(n-1)} I_n & \cdots & \theta^{(n-1)^2} I_n \end{bmatrix},$$

where θ denotes the $n-th$ primitive root of unity. Note that $\tilde{D}_n \in \mathscr{B}(\oplus_{k=1}^{n}\mathscr{H})$. For $1 \leqslant k \leqslant n$, consider the $n \times 1$ block matrix $\tilde{E}_k$ having I_n at $(k,1)$ and 0 everywhere else. For all $1 \leqslant k \leqslant n$, we have

$$
\begin{aligned}
E_k^*\tilde{D_n}\tilde{S}\tilde{D_n}^*E_k &= \frac{1}{n}\left[I_n\ \theta^{1(k-1)}I_n\ \ldots\ \theta^{(n-1)(k-1)}I_n\right]\begin{bmatrix} O & O & \ldots & O & S \\ S & O & \ldots & O & O \\ O & S & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & O & O \\ O & O & \ldots & S & O \end{bmatrix}\begin{bmatrix} I_n \\ \overline{\theta}^{1(k-1)}I_n \\ \overline{\theta}^{2(k-1)}I_n \\ \vdots \\ \overline{\theta}^{(n-1)(k-1)}I_n \end{bmatrix} \\
&= \frac{1}{n}\left[I_n\ \theta^{1(k-1)}I_n\ \ldots\ \theta^{(n-1)(k-1)}I_n\right]\begin{bmatrix} \overline{\theta}^{(n-1)(k-1)}S \\ S \\ \overline{\theta}^{1(k-1)}S \\ \vdots \\ \overline{\theta}^{(n-2)(k-1)}S \end{bmatrix} \\
&= \frac{1}{n}\overline{\theta}^{(n-1)(k-1)}\cdot X + \frac{1}{n}\sum_{j=2}^{j=n}\theta^{(j-1)\cdot(k-1)}\cdot\overline{\theta}^{(j-2)\cdot(k-1)}\cdot S \\
&\quad (\text{since } \theta^{(j-1)\cdot(k-1)}\cdot\overline{\theta}^{(j-2)\cdot(k-1)} = \theta^{k-1}, \quad \forall\, 2 \leqslant j \leqslant n) \\
&= \theta^{k-1}\cdot S.
\end{aligned}
\tag{2.3}
$$

From equation (2.3), we can conclude that

$$
\sum_{k=1}^{n}\|E_k^*\tilde{D_n}\tilde{S}\tilde{D_n}^*E_k\|_F^2 = \sum_{k=1}^{n} tr(S^*S) = \|\tilde{D_n}\tilde{S}\tilde{D_n}^*\|_F^2. \tag{2.4}
$$

We have,

$$
\|\tilde{S}\|_F^2 = tr(\tilde{S}^*\tilde{S}) = \sum_{k=1}^{n} tr(S^*S). \tag{2.5}
$$

Since $\tilde{D_n}$ is a unitary matrix and Frobenious norm is invariant under unitary transforms, from equations (2.4) and (2.5), we can conclude that $\tilde{D_n}\tilde{S}\tilde{D_n}^*$ is a block diagonal operator matrix, with $k-th$ diagonal block $\omega^{k-1}\cdot S$. Hence $\omega_\rho(\tilde{S}) = \omega_\rho(S)$. □

To establish our purpose we need to recall the following two results. The first result provides a bound on the numerical radius of product of operators and has been given in [5].

LEMMA 2.2. *Let $S, X \in \mathscr{B}(\mathscr{H})$. Then*

$$
\omega(X^*SX) \leqslant \|X\|^2\omega(S).
$$

The following lemma plays a key role in deriving operator radius versions of numerical radius inequalities. This lemma has been given in [12].

LEMMA 2.3. *Let $S \in (\mathscr{H})$ and $0 < \rho \leqslant 2$. Then*

$$
\omega_\rho(S) = \frac{2}{\rho}\omega\left(\begin{bmatrix} O & \sqrt{\rho(2-\rho)}S \\ O & |1-\rho|S \end{bmatrix}\right).
$$

The following lemma presents operator radii generalization of Lemma 2.2.

LEMMA 2.4. *Let* $S, X \in \mathscr{B}(\mathscr{H})$. *Then*

$$\omega_\rho(X^*SX) \leqslant \|X\|^2 \omega_\rho(S).$$

Proof. Let $S, X \in \mathscr{B}(\mathscr{H})$. By Lemma 2.3, we have

$$\begin{aligned}
\omega_\rho(X^*SX) &= \frac{2}{\rho}\omega\left(\begin{bmatrix} O & \sqrt{\rho(2-\rho)}X^*SX \\ O & |1-\rho|X^*SX \end{bmatrix}\right) \\
&= \frac{2}{\rho}\omega\left(\begin{bmatrix} X^* & O \\ O & X^* \end{bmatrix}\begin{bmatrix} O & \sqrt{\rho(2-\rho)}S \\ O & |1-\rho|S \end{bmatrix}\begin{bmatrix} X & O \\ O & X \end{bmatrix}\right) \\
&\leqslant \left\|\begin{bmatrix} X & O \\ O & X \end{bmatrix}\right\|^2 \cdot \frac{2}{\rho}\omega\left(\begin{bmatrix} O & \sqrt{\rho(2-\rho)}S \\ O & |1-\rho|S \end{bmatrix}\right) \\
&\quad \text{(by using Lemma 2.2)} \\
&= \|X\|^2 \cdot \omega_\rho(S). \quad \square
\end{aligned}$$

We are now ready to present another important result in this section, which will serve as a key step toward generalizing inequality (2.2) for operator radii.

PROPOSITION 2.2. *Let* $X_1, X_2, \ldots X_n, S \in \mathscr{B}(\mathscr{H})$. *Then*

$$\omega_\rho\left(\sum_{k=1}^{k=n} X_{k+1}^* S X_k + X_1^* S X_n\right) \leqslant \left(\sum_{k=1}^{k=n} \|X_k\|^2\right) \cdot \omega_\rho(S).$$

Proof. Let $\tilde{S}$ be a $n \times n$ operator matrix in $\mathscr{B}(\oplus_{k=1}^n \mathscr{H})$ that has the operator S in the sub diagonal and in the top right hand corner, in the position $(1,n)$ and zero 0 everywhere else. Also, consider the $n \times n$ block matrix

$$\tilde{X} = \begin{bmatrix} X_2 & O & \ldots & O \\ X_3 & O & \ldots & O \\ \vdots & \vdots & & \vdots \\ X_n & O & \ldots & O \\ X_1 & O & \ldots & O \end{bmatrix}.$$

Then, by Lemma 2.4 and Proposition 2.1,

$$\begin{aligned}
\omega_\rho(\tilde{X}^*\tilde{S}\tilde{X}) &\leqslant \|\tilde{X}\|^2 \omega_\rho(\tilde{S}) \\
&\leqslant \left(\sum_{k=1}^{k=n} \|X_k\|^2\right) \cdot \omega_\rho(S).
\end{aligned}$$

Now the observation

$$\tilde{X}^*\tilde{S}\tilde{X} = \sum_{k=1}^{k=n} X_{k+1}^* S X_k + X_1^* S X_n,$$

completes the proof. □

The inequality in Proposition 2.2 for the case $\rho = 2$ was presented in [6]. We now present some corollaries of Proposition 2.2.

COROLLARY 2.1. *Let* $X_1, X_2, \ldots, X_{2n}$, $S \in \mathscr{B}(\mathscr{H})$. *Then*

$$\omega_\rho \left(\sum_{k=1}^{2n-1} X_{k+1}^* S X_k + X_1^* S X_{2n} \right) \leqslant 2 \left(\sum_{k=1}^{k=n} \|X_{2k-1}\|^2 \right) \left(\sum_{k=1}^{k=n} \|X_{2k}\|^2 \right) \omega_\rho(S). \tag{2.6}$$

Proof. It can be proved straightforward, following the same steps as in proof of Theorem 2.6 [6]. □

COROLLARY 2.2. *Let* $X, Y, S \in \mathscr{B}(\mathscr{H})$. *Then*

$$\omega_\rho(X^* S Y + Y^* S X) \leqslant 2\|X\|\|Y\|\omega_\rho(S). \tag{2.7}$$

In particular,

$$\omega_\rho(X^* S + S X) \leqslant 2\|X\|\omega_\rho(S). \tag{2.8}$$

Proof. Inequality (2.7) is a special case of (2.6), obtained by substituting $n = 1$, $X_1 = X$, and $X_2 = Y$. □

The optimal bound in inequality (2.1) is $2\sqrt{2}$, and it can again be found in [5]. We now present a slight generalization of this improved version of inequality (2.1) for operator radii. Our result depends upon following sharp estimates, for ρ-operator radius contraction S and unit vector x,

$$\|Sx\|^2 + \|S^*x\|^2 \leqslant \frac{2\rho^2}{2-\rho} \quad \text{for } 0 < \rho \leqslant 1, \tag{2.9}$$

and

$$\|Sx\|^2 + \|S^*x\|^2 \leqslant 2\rho \quad \text{for } 1 \leqslant \rho \leqslant 2. \tag{2.10}$$

The proofs of inequalities (2.9), and (2.10) has been given in [14].

PROPOSITION 2.3. *Let* $X, S \in \mathscr{B}(\mathscr{H})$. *Then*

$$\omega(XS + SX) \leqslant \frac{2\rho}{\sqrt{2-\rho}}\, \omega_\rho(S) \cdot \|X\| \quad \textit{for } 0 < \rho \leqslant 1, \tag{2.11}$$

and

$$\omega(XS + SX) \leqslant 2\sqrt{\rho}\,\omega_\rho(S) \cdot \|X\| \quad \textit{for } 1 \leqslant \rho \leqslant 2. \tag{2.12}$$

Proof. Let $x \in \mathscr{H}$ be a unit vector, and let X and S be a contraction and ρ-operator contraction on Hilbert space $\mathscr{H}$, respectively. By Cauchy- Schwarz inequality, we have

$$\begin{aligned} |\langle (XS+SX)x,\, x\rangle| &= |\langle Sx,\, X^*x\rangle + \langle Xx,\, S^*x\rangle| \\ &\leqslant \|Sx\|\cdot\|X^*x\| + \|Xx\|\cdot\|S^*x\| \\ &\leqslant \|Sx\| + \|S^*x\| \qquad (2.13) \\ &\quad \text{(since } X \text{ is a contraction).} \end{aligned}$$

From inequalities (2.9) and (2.10), we can deduce that

$$\|Sx\| + \|S^*x\| \leqslant \frac{2\rho}{\sqrt{2-\rho}} \quad \text{for } 0<\rho\leqslant 1, \tag{2.14}$$

and

$$\|Sx\| + \|S^*x\| \leqslant 2\sqrt{\rho} \quad \text{for } 1\leqslant\rho\leqslant 2. \tag{2.15}$$

Combining inequalities (2.13)–(2.15) with the identity

$$\omega(SX+XS) = \sup_{\|x\|=1} |\langle (SX+XS)x,\, x\rangle|,$$

we obtain

$$\omega(XS+SX) \leqslant \frac{2\rho}{\sqrt{2-\rho}}, \quad \text{for } 0<\rho\leqslant 1, \tag{2.16}$$

and

$$\omega(XS+SX) \leqslant 2\sqrt{\rho}, \quad \text{for } 1\leqslant\rho\leqslant 2. \tag{2.17}$$

These estimates establish inequalities (2.11) and (2.12) in the case $\|X\| \leqslant 1$ and $\omega_\rho(S) \leqslant 1$. The general case follows by applying inequalities (2.16) and (2.17) to the normalized operators

$$\tilde{X} = \frac{X}{\|X\|}, \quad \tilde{Y} = \frac{Y}{\|Y\|}. \quad \square$$

Observe that for $\rho = 2$, inequality (2.12) becomes

$$\omega(XS+SX) \leqslant 2\sqrt{2}\,\omega(S)\|X\|. \tag{2.18}$$

Inequality (2.18) have been proved in [5].

3. Operator radii mean inequalities

Let $X, Y \in \mathscr{B}(\mathscr{H})$. As discussed in the introduction, the following inequality holds for the operator norm:

$$\|XSY^*\| \leqslant \frac{1}{2}\left\| |X|^2 S + S|Y|^2 \right\|.$$

A weaker version of this inequality holds for the numerical radius:

$$\omega(XSX^*) \leqslant \frac{1}{2}\omega(|X|^2S + S|X|^2). \tag{3.1}$$

Our next proposition extends inequality (3.1) to the more general setting of ρ-operator radii.

PROPOSITION 3.1. *Let X and S be $n\times n$ complex matrices. Then for $0<\rho\leqslant 2$,*

$$\omega_\rho(XSX^*) \leqslant \frac{1}{2}\omega_\rho(|X|^2S + S|X|^2). \tag{3.2}$$

Proof. By Lemma 2.3, we have

$$\begin{aligned}
\omega_\rho(XSX^*) &= \frac{2}{\rho}\omega\left(\begin{bmatrix} O & \sqrt{\rho(2-\rho)}XSX^* \\ O & |1-\rho|XSX^* \end{bmatrix}\right) \\
&= \frac{2}{\rho}\omega\left(\begin{bmatrix} X & O \\ O & X \end{bmatrix}\begin{bmatrix} O & \sqrt{\rho(2-\rho)}S \\ O & |1-\rho|S \end{bmatrix}\begin{bmatrix} X^* & O \\ O & X^* \end{bmatrix}\right) \\
&\leqslant \frac{1}{\rho}\omega\left(\begin{bmatrix} |X|^2 & O \\ O & |X|^2 \end{bmatrix}\begin{bmatrix} O & \sqrt{\rho(2-\rho)}S \\ O & |1-\rho|S \end{bmatrix} + \begin{bmatrix} O & \sqrt{\rho(2-\rho)}S \\ O & |1-\rho|S \end{bmatrix}\begin{bmatrix} |X|^2 & O \\ O & |X|^2 \end{bmatrix}\right) \\
&\quad \text{(by using inequality (3.1))} \\
&= \frac{1}{\rho}\omega\left(\begin{bmatrix} O & \sqrt{\rho(2-\rho)}(|X|^2S+S|X|^2) \\ O & |1-\rho|(|X|^2S+S|X|^2) \end{bmatrix}\right) \\
&= \frac{1}{2}\omega_\rho(|X|^2S+S|X|^2) \\
&\quad \text{(using Lemma 2.3).}
\end{aligned}$$

This completes the proof. □

REMARK 3.1. An alternative proof of inequality (3.2) can be obtained using the theory of Schur multiplier operators. It suffices to establish the inequality in the special case where X is positive semi-definite, since the general case then follows by applying the argument given in the proof of Theorem 2.1 in [13].

Let $X,S \in M_n(\mathbb{C})$, and assume that X is positive semi-definite. Without loss of generality, we may take $X = \mathrm{diag}(\lambda_i)$ with $\lambda_i \geqslant 0$. Then, we have:

$$XSX = T\circ(X^2S+SX^2), \quad \text{where } T = \left(\frac{\lambda_i\lambda_j}{\lambda_i^2+\lambda_j^2}\right)_{i,j}.$$

The matrix T is positive definite. By invoking Theorem 8 of [15] and Corollary 4 of [2], we deduce that

$$\omega_\rho(T\circ(X^2S+SX^2)) \leqslant \sup_{\lambda_i,\lambda_j\in\mathbb{R}} \frac{\lambda_i\lambda_j}{\lambda_i^2+\lambda_j^2}\cdot\omega_\rho(X^2S+SX^2).$$

Since the supremum above is equal to $1/2$, it follows that

$$\omega_\rho(T\circ(X^2S+SX^2)) \leqslant \frac{1}{2}\cdot\omega_\rho(X^2S+SX^2).$$

This completes the proof.

Another special case of inequality (1.3), which holds for numerical radius is the following:

$$\omega(XSY^*) \leqslant \frac{1}{2}\,\omega\big(|X|^2S+S|Y|^2\big) \quad \text{provided} \;\; X^*Y=O. \tag{3.3}$$

Next, we prove a refined version of this inequality for $\omega_\rho(\cdot)$. To this end we need to recall the following remarkable and highly useful result from [1].

LEMMA 3.1. *Let $S=[S_{i,j}]$ be $n\times n$ operator matrix with $S_{i,j}\in\mathscr{B}(\mathscr{H})$. Then*

$$\omega(S)\leqslant\omega([s_{i,j}]),$$

where

$$s_{i,j}=\begin{cases}\omega(S_{i,j}) & \text{if } i=j\\ \|S_{i,j}\| & \text{if } i\neq j.\end{cases}$$

LEMMA 3.2. *Let S be a $n\times n$ complex matrix. Then for $0<\rho\leqslant 2$,*

$$\omega_\rho\left(\begin{bmatrix}O & S\\ O & O\end{bmatrix}\right)\leqslant\frac{1}{\rho}\|S\|\leqslant\omega_\rho(S).$$

Proof. Using Lemma 3.1, we have

$$\begin{aligned}\omega_\rho\left(\begin{bmatrix}O & S\\ O & O\end{bmatrix}\right) &= \frac{2}{\rho}\omega\left(\begin{bmatrix}O & O & O & \sqrt{\rho(2-\rho)}S\\ O & O & O & O\\ O & O & O & |1-\rho|S\\ O & O & O & O\end{bmatrix}\right)\\ &\leqslant \frac{2}{\rho}\cdot\|S\|\cdot\omega\left(\begin{bmatrix}0 & 0 & 0 & \sqrt{\rho(2-\rho)}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & |1-\rho|\\ 0 & 0 & 0 & 0\end{bmatrix}\right)\\ &\quad\text{(by Lemma 3.1)}\\ &\leqslant\frac{2}{\rho}\cdot\|S\|\cdot\sup\{\sqrt{\rho(2-\rho)}\,xz+|1-\rho|\,yz:\; x^2+y^2+z^2=1\}\end{aligned} \tag{3.4}$$

Supremum of quadratic form in equation (3.4) can be equivalently expressed as,

$$\sup\{|u^TAu|:\; u\in\mathbb{R}^3,\; u^Tu=1\}, \tag{3.5}$$

where

$$A = \begin{bmatrix} 0 & 0 & \frac{\sqrt{\rho(2-\rho)}}{2} \\ 0 & 0 & \frac{|1-\rho|}{2} \\ \frac{\sqrt{\rho(2-\rho)}}{2} & \frac{|1-\rho|}{2} & 0 \end{bmatrix}.$$

Since A is a symmetric matrix, from equations (3.4) and (3.5) we can conclude that

$$\sup\{\sqrt{\rho(2-\rho)}\,xz + |1-\rho|\,yz : \; x^2+y^2+z^2=1\} = \max\{|\lambda| : \; Ax=\lambda x\}. \quad (3.6)$$

Since A is symmetric matrix with $rank(A)=2$ and $trace(A)=0$, set of eigenvalues of A have form $\{\lambda,-\lambda,0\}$. Therefore,

$$tr(A^*A) = 2\lambda^2 = 2\left(\frac{\rho(2-\rho)}{4} + \frac{(1-\rho)^2}{4}\right) = \frac{1}{2}.$$

so,

$$\lambda = \pm\frac{1}{2}.$$

Thus, $\|A\| = \max\{|\lambda| : \; Ax=\lambda x\} = 1/2$, and, in conjunction with inequalities (3.4) and (3.6), this implies that

$$\omega_\rho\left(\begin{bmatrix} O & S \\ O & O \end{bmatrix}\right) \leqslant \frac{1}{\rho}\cdot\|S\| \quad (3.7)$$

Now the fact $\|S\| \leqslant \rho\omega_\rho(S)$, together with equation (3.7) completes the proof (see [9]). □

REMARK 3.2. A weaker version of above inequality can be proved as follows, Note that

$$\omega_\rho\left(\begin{bmatrix} S & O \\ O & S \end{bmatrix}\right) = \omega_\rho(S) \quad \text{and} \quad \omega_\rho\left(\begin{bmatrix} S & O \\ O & -S \end{bmatrix}\right) = \omega_\rho(S). \quad (3.8)$$

Consider the unitary matrices $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I_n & I_n \\ -I_n & I_n \end{bmatrix}$ and $V = \begin{bmatrix} I_n & O \\ O & \imath I_n \end{bmatrix}$. Then

$$\omega_\rho\left(V^*\begin{bmatrix} O & S \\ -S & O \end{bmatrix}V\right) = \omega_\rho\left(\imath\begin{bmatrix} O & S \\ S & O \end{bmatrix}\right) = \omega_\rho\left(\begin{bmatrix} O & S \\ S & O \end{bmatrix}\right), \quad (3.9)$$

and

$$\omega_\rho(U^*(S\oplus -S)U) = \omega_\rho\left(\begin{bmatrix} O & S \\ S & O \end{bmatrix}\right). \quad (3.10)$$

Combining the equations (3.8), (3.9) and (3.10), we obtain

$$\begin{aligned} 2\omega_\rho\left(\begin{bmatrix} O & S \\ O & O \end{bmatrix}\right) &= \omega_\rho\left(\begin{bmatrix} O & S \\ S & O \end{bmatrix} + \begin{bmatrix} O & S \\ -S & O \end{bmatrix}\right) \\ &\leqslant \omega_\rho\left(\begin{bmatrix} O & S \\ S & O \end{bmatrix}\right) + \omega_\rho\left(\begin{bmatrix} O & S \\ -S & O \end{bmatrix}\right) \\ &\leqslant 2\omega_\rho(S). \end{aligned}$$

This implies

$$\omega_\rho\left(\begin{bmatrix} O & S \\ O & O \end{bmatrix}\right) \leqslant \omega_\rho(S).$$

Now, we are in a position to present aforementioned refinement of the inequality (3.3).

PROPOSITION 3.2. *Let X, Y and S be $n \times n$ matrices such that $X^*Y = O$. Then for $0 < \rho \leqslant 2$,*

$$\omega_\rho(XSY^*) \leqslant \frac{1}{2\rho}\||X|^2 S + S|Y|^2\| \leqslant \frac{1}{2}\omega_\rho(|X|^2 S + S|Y|^2).$$

Proof. Let $Z = \begin{bmatrix} X & Y \\ O & O \end{bmatrix}$ and $T = \begin{bmatrix} O & S \\ O & O \end{bmatrix}$. Then

$$\begin{aligned}
\omega_\rho(XSY^*) &= \omega_\rho\left(\begin{bmatrix} XSY^* & O \\ O & O \end{bmatrix}\right) \\
&= \omega_\rho(ZTZ^*) \\
&\leqslant \frac{1}{2}\ \omega_\rho(|Z|^2 T + T|Z|^2) \quad \text{(by Proposition 3.1)} \\
&= \frac{1}{2}\ \omega_\rho\left(\begin{bmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y \end{bmatrix}\begin{bmatrix} O & S \\ O & O \end{bmatrix} + \begin{bmatrix} O & S \\ O & O \end{bmatrix}\begin{bmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y \end{bmatrix}\right) \\
&= \frac{1}{2}\omega_\rho\left(\begin{bmatrix} O & |X|^2 S + S|X|^2 \\ O & O \end{bmatrix}\right) \quad \text{(since } X^*Y = O). \qquad (3.11)
\end{aligned}$$

Finally, by invoking Lemma 3.2 in conjunction with inequality (3.11), we deduce that

$$\omega_\rho(XSY^*) \leqslant \frac{1}{2}\omega_\rho\left(\begin{bmatrix} O & |X|^2 S + S|X|^2 \\ O & O \end{bmatrix}\right) \leqslant \frac{1}{2\rho}\||X|^2 S + S|Y|^2\| \leqslant \frac{1}{2}\omega_\rho(|X|^2 S + S|Y|^2).$$

This completes the proof. □

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2. All the authors Ramkishan, Preeti Dharmarha and Amit Kumar have equally contributed in all aspects of manuscript. All authors read and approved the final manuscript.
3. The manuscript has no associated data.

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