Operators and Matrices
Volume 20, Number 2 (2026), 129–145
doi:10.7153/oam-2026-20-08

CHARACTERIZATIONS OF CLOSED EP OPERATORS ON HILBERT SPACES

ARUP MAJUMDAR*, P. SAM JOHNSON AND RAM N. MOHAPATRA

(*Communicated by D. Han*)

Abstract. In this paper, we present intriguing findings that characterize both the closed (unbounded) and bounded EP operators on Hilbert spaces. We establish the necessary and sufficient conditions for the product of a closed EP operator and a bounded EP operator to also be EP. Additionally, we demonstrate the result $\gamma(T) \leqslant r(T)$, where T is a bounded EP operator, and $\gamma(T)$ and $r(T)$ represent the reduced minimum modulus and the spectral radius of T, respectively.

1. Introduction

The term EP or Equal Projection was first introduced by Schwerdtfeger in 1950 [14] to describe a square matrix T over a complex field $\mathbb{C}$ for which the null spaces of T and T^* are identical. In 1965, M. H. Pearl established an equivalent condition of the EP matrix T commutes with its Moore-Penrose inverse $T^\dagger$ [13]. Subsequently, Campbell and Meyer expanded the concept of EP matrices to bounded linear operators on complex Hilbert spaces, defining EP operators as those for which the closed ranges of T and T^* are equal [2]. Itoh later introduced the hypo-EP operator, characterized by the conditions $T^\dagger T \geqslant TT^\dagger$ and $R(T)$ is closed [8]. Research has continued into the characterization of EP operators on Hilbert spaces, with several authors investigating their properties within C^*-algebras. In 2007, Boasso studied EP operators in the context of Banach space operators and Banach algebra elements [1], while Johnson focused on unbounded closed EP and hypo-EP operators on Hilbert spaces in 2021 [9]. This paper delves into the exploration of the characterizations of closed (possibly unbounded) EP operators on Hilbert spaces, presenting intriguing examples in Section 2. Section 3 is dedicated to the examination of bounded EP operators on Hilbert spaces.

From now on, we shall mean H, K, H_i, K_i $(i = 1, 2, \ldots, n)$ as Hilbert spaces. The specification of a domain is an essential part of the definition of an unbounded operator, usually defined on a subspace. Consequently, for an operator T, the specification of the subspace D on which T is defined, called the domain of T, denoted by $D(T)$, is to be given. The null space and range space of T are denoted by $N(T)$ and $R(T)$, respectively. $W_1^\perp$ denotes the orthogonal complement of a set W_1 whereas $W_1 \oplus^\perp W_2$

Mathematics subject classification (2020): 47A05, 47B02.
Keywords and phrases: EP operator, closed operator, Moore-Penrose inverse.
* Corresponding author.

Paper OaM-20-08

denotes the orthogonal direct sum of the subspaces W_1 and W_2 of a Hilbert space. Moreover, $T_{|W}$ denotes the restriction of T to a subspace W of a specified Hilbert space. We call $D(T)\cap N(T)^{\perp}$, the carrier of T and it is denoted by $C(T)$. T^* denotes the adjoint of T, when $D(T)$ is densely defined in the specified Hilbert space. Here, P_V is the orthogonal projection on the closed subspace V in the specified Hilbert space and the set of bounded operators from H into K is denoted by $B(H,K)$. Similarly, the set of all bounded operators on H is denoted by $B(H)$. For the sake of completeness of exposition, we first begin with the definition of a closed operator.

DEFINITION 1.1. Let T be an operator from a Hilbert space H with domain $D(T)$ to a Hilbert space K. If the graph of T defined by

$$G(T) = \{(x,Tx) : x \in D(T)\}$$

is closed in $H\times K$, then T is called a closed operator. Equivalently, T is a closed operator if $\{x_n\}$ in $D(T)$ such that $x_n \to x$ and $Tx_n \to y$ for some $x\in H, y\in K$, then $x\in D(T)$ and $Tx = y$. That is, $G(T)$ is a closed subspace of $H\times K$ with respect to the graph norm $\|(x,y)\|_T = (\|x\|^2+\|y\|^2)^{1/2}$. It is easy to show that the graph norm $\|(x,y)\|_T$ is equivalent to the norm $\|x\|+\|y\|$. We note that, for any densely defined closed operator T, the closure of $C(T)$, that is, $\overline{C(T)}$ is $N(T)^{\perp}$. $C(H)$ denotes the set of all closed operators from H into H. Meanwhile, $C(H,K)$ denotes the set of all closed operators from H into K.

We say that S is an extension of T (denoted by $T\subset S$) if $D(T)\subset D(S)$ and $Sx = Tx$ for all $x\in D(T)$. Moreover, $T\in C(H)$ commutes $A\in B(H)$ if $AT\subset TA$.

DEFINITION 1.2. Let T be a closed operator from $D(T)\subset H$ to K. The Moore-Penrose inverse of T is the map $T^{\dagger} : R(T)\oplus^{\perp} R(T)^{\perp} \to H$ defined by

$$T^{\dagger}y = \begin{cases} (T|_{C(T)})^{-1}y & \text{if } y\in R(T) \\ 0 & \text{if } y\in R(T)^{\perp}. \end{cases} \tag{1}$$

It can be shown that $T^{\dagger}$ is bounded if and only if $R(T)$ is closed, when T is closed.

DEFINITION 1.3. [2] An operator $T\in B(H)$ is called an EP operator if $R(T)$ is closed and $R(T) = R(T^*)$.

DEFINITION 1.4. [8] An operator $T\in B(H)$ is called a hypo-EP operator if $R(T)$ is closed with $R(T)\subset R(T^*)$.

THEOREM 1.5. [8] *Let $T\in B(H,K)$ be a closed range operator and let $\{T_n\}$ be a sequence of closed range operators in $B(H,K)$. Let $T_n^{\dagger}$ be the Moore-Penrose inverse of T_n for every n. Suppose that $T_n \to T$ (with respect to the norm of $B(H,K)$). Then the following conditions are equivalent:*

1. *$T_n^{\dagger} \to T^{\dagger}$;*

2. $T_n^\dagger T_n \to T^\dagger T$;

3. $\sup\{\|T_n^\dagger\| : n \in \mathbb{N}\} < \infty$.

Definitions 1.3 and 1.4 can be extended to densely defined closed operators, as discussed in the paper [9]. Below, we present these definitions.

DEFINITION 1.6. [9] Let T be a densely defined closed operator on H. T is said to be an EP operator if T has a closed range and $R(T) = R(T^*)$.

Moreover, the densely defined closed operator $T \in C(H)$ is called a hypo-EP operator if $R(T)$ is closed with $R(T) \subset R(T^*)$.

THEOREM 1.7. [12] *Let T be a densely defined closed operator from $D(T) \subset H$ into K. Then the following statements hold:*

1. *$T^\dagger$ is a closed operator from K into H;*

2. *$D(T^\dagger) = R(T) \oplus^\perp N(T^*)$; $N(T^\dagger) = N(T^*)$;*

3. *$R(T^\dagger) = C(T)$;*

4. *$T^\dagger T x = P_{\overline{R(T^\dagger)}} x$, for all $x \in D(T)$;*

5. *$T T^\dagger y = P_{\overline{R(T)}} y$, for all $y \in D(T^\dagger)$;*

6. *$(T^\dagger)^\dagger = T$;*

7. *$(T^*)^\dagger = (T^\dagger)^*$;*

8. *$N((T^*)^\dagger) = N(T)$;*

9. *$(T^* T)^\dagger = T^\dagger (T^*)^\dagger$;*

10. *$(T T^*)^\dagger = (T^*)^\dagger T^\dagger$.*

THEOREM 1.8. [10] *Let $T_1 : D(T_1) \subset H_1 \to K_1$ and $T_2 : D(T_2) \subset H_2 \to K_2$ be two closed operators with closed ranges. Then $T = T_1 \bigoplus T_2 : D(T_1) \bigoplus D(T_2) \subset H_1 \bigoplus H_2 \to K_1 \bigoplus K_2$ has the Moore-Penrose inverse. Moreover,*

$$T^\dagger = (T_1 \bigoplus T_2)^\dagger = T_1^\dagger \bigoplus T_2^\dagger.$$

THEOREM 1.9. [10] *Let $T_i : D(T_i) \subset H_i \to K_i$ $(i = 1, 2)$ be two closed operators with closed ranges $R(T_i)$, $i = 1, 2$. Then $\gamma(T_1 \bigoplus T_2) = \min\{\gamma(T_1), \gamma(T_2)\} > 0$, where $\gamma(T)$ is the reduced minimum modulus of T.*

2. Characterizations of closed EP operators on the Hilbert space H

Throughout the section, we consider T as the densely defined closed operator on H. Theorem 2.1 has been mentioned in the paper [9], but we present a different approach to prove the theorem.

THEOREM 2.1. *Let $T \in C(H)$ be a closed-range operator. Then T is EP if and only if $T^\dagger T = TT^\dagger$ on $D(T)$.*

Proof. Since, $R(T) = R(T^*)$. Then

$$T^\dagger T = P_{N(T)^\perp|D(T)} = P_{R(T)|D(T)} \subset P_{R(T)} = TT^\dagger.$$

Conversely, Suppose $T^\dagger T = TT^\dagger$ on $D(T)$. So,

$$R(T) = R(TT^\dagger) \supset \overline{R(T^\dagger T)} = \overline{R(T^\dagger)} = N(T)^\perp = R(T^*). \tag{2}$$

Again, $N(T) \subset R(T)^\perp$ because of $P_{R(T)}x = 0$, for all $x \in N(T)$. Thus,

$$R(T) \subset N(T)^\perp = R(T^*). \tag{3}$$

By (2) and (3), we get $R(T) = R(T^*)$. $\square$

Now, we present some examples of EP operators.

EXAMPLE 2.2. [9] Let $\phi : [0,1] \to \mathbb{C}$ by

$$\phi(t) = \begin{cases} 1 \text{ if } t = 0 \\ \frac{1}{\sqrt{t}} \text{ if } 0 < t \leqslant 1. \end{cases}$$

Define an operator $Tf = \phi f$ on $D(T) = \{f \in L^2[0,1] : \phi f \in L^2[0,1]\}$. T is a self-adjoint operator. Moreover. $R(T) = L^2[0,1]$ because $|\phi(t)| \geqslant 1$. Therefore, T is an EP operator.

EXAMPLE 2.3. [11] Define T on ℓ^2 by

$$T(x_1, x_2, x_3, \ldots, x_n, \ldots) = (x_1, 2x_2, 3x_3, \ldots, nx_n, \ldots)$$

with domain $D(T) = \{(x_1, x_2, x_3, \ldots, x_n, \ldots) \in \ell^2 : \sum_{n=1}^{\infty} |nx_n|^2 < \infty\}$. Since $D(T)$ contains the space c_{00} of all finitely non-zero sequences, $D(T)$ is a proper dense subspace of ℓ^2. One can show that T is a self-adjoint operator and $R(T) = N(T^*)^\perp = \ell^2 = R(T^*)$. Therefore, T is EP.

EXAMPLE 2.4. [11] Define T on ℓ^2 by

$$T(x_1, x_2, \ldots, x_n, \ldots) = \left(x_1, 2x_2, \frac{x_3}{3}, 4x_4, \frac{x_5}{5}, \ldots\right)$$

with domain $D(T) = \{(x_1, x_2, x_3, \ldots, x_n, \ldots) : (x_1, 2x_2, \frac{x_3}{3}, 4x_4, \frac{x_5}{5}, \ldots) \in \ell^2\}$. One can show that T is closed and $R(T)$ is a proper dense subspace of ℓ^2. Therefore, $R(T)$ is not closed, which confirms that T is not EP.

THEOREM 2.5. *Let $T_1 \in C(H_1)$ and $T_2 \in C(H_2)$ be two densely defined operators. Then T_1 and T_2 both are EP if and only if $T_1 \bigoplus T_2$ is also EP.*

Proof. Since T_1 and T_2 are EP operators, their ranges $R(T_1)$ and $R(T_2)$ both are closed. Therefore, the range of their direct sum, $R(T_1 \bigoplus T_2) = R(T_1) \bigoplus R(T_2)$, is also closed. From Theorem 1.8, the Moore-Penrose inverse of the direct sum satisfies

$$\begin{aligned}(T_1 \bigoplus T_2)^\dagger (T_1 \bigoplus T_2) &= T_1^\dagger T_1 \bigoplus T_2^\dagger T_2 \\ &= T_1 T_1^\dagger \bigoplus T_2 T_2^\dagger \quad (\text{on } D(T_1) \bigoplus D(T_2)) \\ &= (T_1 \bigoplus T_2)(T_1 \bigoplus T_2)^\dagger \quad (\text{on } D(T_1 \bigoplus T_2)).\end{aligned}$$

This identity confirms that $(T_1 \bigoplus T_2)$ is an EP operator.

Conversely, assume that $T_1 \bigoplus T_2$ is EP. we claim that $R(T_1)$ is closed. To establish this, let $u \in \overline{R(T_1)}$. Then there exists a sequence $\{T_1 u_n\}$ in $R(T_1)$ such that $T_1 u_n \to u$ as $n \to \infty$. Since $R(T_1 \bigoplus T_2)$ is closed. So, there is an element (v_1, v_2) in $D(T_1 \bigoplus T_2)$ such that

$$(T_1 \bigoplus T_2)(u_n, 0) \to (u, 0) = (T_1 v_1, T_2 v_2) \text{ as } n \to \infty,$$

which implies $u = T_1 v_1$ in $R(T_1)$. Thus, $R(T_1)$ is closed. By a similar argument, $R(T_2)$ is closed. Now, for all $h_1 \in D(T_1)$, we get

$$\begin{aligned}(T_1^\dagger T_1 h_1, 0) &= (T_1 \bigoplus T_2)^\dagger (T_1 \bigoplus T_2)(h_1, 0) \\ &= (T_1 \bigoplus T_2)(T_1 \bigoplus T_2)^\dagger (h_1, 0) \\ &= (T_1 T_1^\dagger h_1, 0).\end{aligned}$$

This shows that $T_1^\dagger T_1 = T_1 T_1^\dagger$ on $D(T_1)$. Similarly, one can show that $T_2^\dagger T_2 = T_2 T_2^\dagger$ on $D(T_2)$. Therefore, T_1 and T_2 both are EP operators. □

THEOREM 2.6. *Let $T \in C(H)$ be a closed range operator and $T = U_T|T|$ be the polar decomposition of T. Then the following conditions are equivalent:*

1. *T is EP;*
2. *U_T is EP.*

Proof. 1) ⇒ 2) Since $R(T) = R(T^*)$. Then

$$R(|T|) = R(T^* T) = R(T^*) = R(T).$$

So, $R(U_T) = R(T)$ is closed and $N(U_T) = R(T)^\perp = N(T)$. Now, $U_T U_T^\dagger = P_{R(U_T)} = P_{R(T)}$. Moreover, $U_T^\dagger U_T = P_{N(U_T)^\perp} = P_{N(T)^\perp} = P_{R(T)}$. Thus, U_T is EP.

2) ⇒ 1) Since U_T is EP. Then

$$\begin{aligned}&U_T U_T^\dagger = U_T^\dagger U_T \\ &P_{R(U_T)} = P_{R((U_T)^*)} \\ &P_{R(T)} = P_{R(T^*)} \quad (\text{because } R(U_T) = R(T) \text{ and } R((U_T)^*) = R(U_{T^*}) = R(T^*)).\end{aligned}$$

Hence, $R(T) = R(T^*)$. Therefore, T is EP. □

THEOREM 2.7. *Let $T \in C(H)$ be of the form*

$$\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} C(T) \\ N(T) \end{bmatrix} \to \begin{bmatrix} R(T) \\ R(T)^{\perp} \end{bmatrix}.$$

Then T is EP if and only if T_1 is EP.

Proof. We claim that the restriction $T_1 = T_{|C(T)}$ is EP. It is easy to show that T_1 is closed. Since T is EP. So, $R(T) = R(T^*)$. Again, $R(T_1) = R(T)$ is closed. Now, observe that:

$$T_1 T_1^{\dagger} = I_{R(T)} = I_{N(T)^{\perp}} \text{ and } T_1^{\dagger} T_1 = I_{C(T)}.$$

Hence, $T_1^{\dagger} T_1 = T_1 T_1^{\dagger}$ on $D(T_1) = C(T)$, which shows that T_1 is EP.

Conversely, assume that T_1 is EP. Then $R(T) = R(T_1)$ is closed. By Lemma 3.3 [10], it follows that

$$R(T^*) = R(T_1^*).$$

SInce T_1 is EP, we also have

$$R(T^*) = R(T_1^*) = R(T_1) = R(T).$$

Therefore, T is EP. □

THEOREM 2.8. *Let $T \in C(H)$ be an EP operator and S be a bounded operator on H. Then T commutes with S ($ST \subset TS$) if and only if $ST^{\dagger} = T^{\dagger} S$.*

Proof. Since T is EP, its range is closed and satisfies $R(T) = R(T^*)$. Accordingly, the operator T admits the following block representation:

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} C(T) \\ N(T) \end{bmatrix} \to \begin{bmatrix} R(T) \\ N(T) \end{bmatrix},$$

and

$$T^{\dagger} = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} \text{ is bounded.}$$

Now, the bounded operator S represents in block form as:

$$S = \begin{bmatrix} S_1 & S_2 \\ S_3 & S_4 \end{bmatrix} : \begin{bmatrix} R(T) \\ N(T) \end{bmatrix} \to \begin{bmatrix} R(T) \\ N(T) \end{bmatrix}.$$

Let $u = (u_1, u_2) \in D(ST)$, so that $STu = TSu$. Applying the block forms gives:

$$S_1 T_1 u_1 = T_1(S_1 u_1 + S_2 u_2) \tag{4}$$

$$S_3 T_1 u_1 = 0. \tag{5}$$

Now, take $w_1 \in R(T)$. Since T_1 is invertible, there exists an element $v_1 \in C(T)$ such that $T_1^{-1}w_1 = v_1$. From (5), we get $S_3w_1 = S_3T_1v_1 = 0$. Thus, $S_3 = 0$.

Now, set $u_1 = 0$ in (4), we get $T_1S_2u_2 = 0$ which implies $S_2 = 0$ (Since T_1 is invertible).

We claim that $S_1T_1^{-1} = T_1^{-1}S_1$. When $u_2 = 0$, then for all $u_1 \in C(T)$, (4) says that $S_1T_1u_1 = T_1S_1u_1$.

Let $p_1 \in R(T_1)$. Then there is $z_1 \in C(T)$ such that $z_1 = T_1^{-1}p_1$. Again, $S_1p_1 = T_1S_1T_1^{-1}p_1$ implies that $T_1^{-1}S_1p_1 = S_1T_1^{-1}p_1$. Hence, $T_1^{-1}S_1 = S_1T_1^{-1}$. It is evident that

$$ST^\dagger = \begin{bmatrix} S_1T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$T^\dagger S = \begin{bmatrix} T_1^{-1}S_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, $ST^\dagger = T^\dagger S$.

Conversely, assume that $ST^\dagger = T^\dagger S$. It is again evident that $T_1T_1^{-1} = I_{R(T)}$ and $T_1^{-1}T_1 = I_{C(T)}$. Then for all $x_1 \in R(T)$ and $x_2 \in N(T)$, we have

$$S_1T_1^{-1}x_1 = T_1^{-1}(S_1x_1 + S_2x_2), \tag{6}$$

and

$$S_3T_1^{-1}x_1 = 0. \tag{7}$$

From the equality (7), $S_3 = 0$. Now consider $x_1 = 0$, we get $S_2x_2 = 0$, for all $x_2 \in N(T)$. Thus, $S_2 = 0$. Taking $x_2 = 0$, we obtain $S_1T_1^{-1}x_1 = T_1^{-1}S_1x_1$ for all $x_1 \in R(T)$ which implies $T_1S_1y = S_1T_1y$ for all $y \in C(T)$. So, for all $w_1 \in C(T), w_2 \in N(T)$, we have

$$ST\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} S_1T_1w_1 \\ 0 \end{pmatrix} = \begin{pmatrix} T_1S_1w_1 \\ 0 \end{pmatrix} = TS\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Therefore, $ST \subset TS$. $\square$

THEOREM 2.9. *Let $T \in C(H)$ be a closed-range operator. Then T is EP if and only if T^n is EP for every $n \in \mathbb{N}$, assuming that the domain $D(T^n)$ is densely defined for all $n \in \mathbb{N}$.*

Proof. Suppose T is EP. Then $R(T) = R(T^*)$ is closed. So,

$$R((T^*)^2) = T^*R(T^*) = T^*R(T) = R(T^*) = R(T). \tag{8}$$

Again

$$R(T^2) = TR(T^*) = R(TT^*) = R(T), \tag{9}$$

and

$$R((T^*)^2) \subset R((T^2)^*) \text{ (because } (T^*)^2 \subset (T^2)^*). \tag{10}$$

We know that $N(T) \subset N(T^2)$. Thus,

$$R((T^2)^*) \subset N(T^2)^\perp \subset N(T)^\perp = R(T^*) = R(T) = R((T^*)^2). \tag{11}$$

The identities (8)–(11) verify that $R((T^2)^*) = R((T^*)^2) = R(T) = R(T^2)$. Since $R(T)$ is closed which implies that $R(T^2)$ is closed. Now we will show that T^2 is closed. Let $(x,y) \in \overline{G(T^2)}$, where $G(T^2)$ is the graph of T^2. Then there exists a sequence $\{x_n\}$ in $D(T^2)$ such that $x_n \to x$ and $T^2x_n \to y$ as $n \to \infty$. The closed range of T^2 guarantees that there exists an element $z \in D(T^2)$ with $y = T^2z$.

Furthermore, $\gamma(T) > 0$, where $\gamma(T)$ is the reduced minimal modulus of T. So, from the relation $\|T^2x_n - T^2z\| \geqslant \gamma(T)\|Tx_n - Tz\|$, we get $Tx_n \to Tz$ as $n \to \infty$. Since T is closed. So, $Tx = Tz$ which implies $x \in D(T^2)$ and $y = T^2z = T^2x$.

Hence, T^2 is EP. By induction hypothesis, it follows that T^n is EP for all $n \in \mathbb{N}$. The converse part is obviously true. □

Theorem 7.1 in [15] states that for a bounded normal operator T of finite descent on a Hilbert space H, the relation $(T^n)^\dagger = (T^\dagger)^n$, holds for all $n \in \mathbb{N}$. Using Theorem 2.9, we can derive the same relation $(T^n)^\dagger = (T^\dagger)^n$ holds for all $n \in \mathbb{N}$, when $T \in C(H)$ is an EP operator.

COROLLARY 2.10. *Let $T \in C(H)$ be an EP operator. Then, for every $n \in \mathbb{N}$, the Moore-Penrose inverse satisfies $(T^n)^\dagger = (T^\dagger)^n$, provided that the domain $D(T^n)$ is densely defined.*

Proof. Theorem 2.9 says that T^n is EP. So, $R((T^n)^*) = R(T^n) = R(T) = R(T^*)$. Now, to verify $(T^n)^\dagger = (T^\dagger)^n$, observe the following identities:

$$(T^\dagger)^n(T^n)(T^\dagger)^n = (T^\dagger)^n \text{ because } TT^\dagger = T^\dagger T \text{ on } D(T). \tag{12}$$

$$(T^n)(T^\dagger)^n(T^n) = T^n \text{ because } TT^\dagger = T^\dagger T \text{ on } D(T). \tag{13}$$

$$(T^n)(T^n)^\dagger = TT^\dagger = P_{R(T)} = P_{R(T^n)} \tag{14}$$

is symmetric.

$$(T^n)^\dagger(T^n) = T^\dagger T|_{D(T^n)} = P_{N(T)^\perp}|_{D(T^n)} = P_{N(T^n)^\perp}|_{D(T^n)} \tag{15}$$

is symmetric because $D(T^n)$ is densely defined. Therefore, Theorem 5.7 in [15] and the four identities (12)–(15) verify that $(T^n)^\dagger = (T^\dagger)^n$. □

THEOREM 2.11. *Let $T \in C(H)$ be an EP operator. Then for all non zero $\lambda \in \mathbb{C}$, $\lambda \in \rho(T)$ if and only if $\lambda \in \rho(T_{|C(T)})$, where $T_{|C(T)} : C(T) \subset N(T)^\perp \to N(T)^\perp = R(T)$.*

Proof. Let $\lambda \in \rho(T)$, the resolvent set of T. Then $(T-\lambda)^{-1}$ exists and is bounded. We first observe that that $N(T_{|C(T)}-\lambda)=\{0\}$.

Now take any $y \in N(T)^{\perp}$. There exists $x=x_1+x_2 \in D(T)$, where $x_1 \in C(T), x_2 \in N(T)$, such that

$$y=(T-\lambda)x=Tx_1-\lambda x_1-\lambda x_2.$$

Rewriting gives that

$$y-(Tx_1-\lambda x_1)=-\lambda x_2 \in N(T)^{\perp} \cap N(T)=\{0\}.$$

So, $y=(T_{|C(T)}-\lambda)x_1$, which shows that $(T_{|C(T)}-\lambda)$ is onto. To show that $T_{|C(T)}-\lambda)^{-1}$ is bounded, we estimate

$$\|(T_{|C(T)}-\lambda)^{-1}y\|=\|x_1\| \leqslant \|x\|=\|(T-\lambda)^{-1}y\| \leqslant \|(T-\lambda)^{-1}\|\|y\|.$$

Hence, $(T_{|C(T)}-\lambda)^{-1}$ is bounded, and so $\lambda \in \rho(T_{|C(T)})$.

Conversely, Let $0 \neq \mu \in \rho(T_{|C(T)})$. Then $(T_{|C(T)}-\mu)^{-1}$ is bounded. We claim that $(T-\mu)^{-1}$ is bounded on H. Let $z=z_1+z_2 \in N(T-\mu)$, where $z_1 \in C(T)$, $z_2 \in N(T)$. Then $(T_{|C(T)}-\mu)z_1=\mu z_2 \in N(T)^{\perp} \cap N(T)=\{0\}$. So, $z_2=0$.

Since $N(T_{|C(T)}-\mu)=\{0\}$ and the operator is injective, we also have $z_1=0$. Hence, $N(T-\mu)=\{0\}$.

To prove surjectivity, let $w=w_1+w_2 \in H$, with $w_1 \in R(T)=N(T)^{\perp}$, $w_2 \in N(T)$. There is an element $u_1 \in C(T)$ with $(T_{|C(T)}-\mu)u_1=w_1$. Then consider the element $(u_1-\frac{w_2}{\mu}) \in D(T)$, we obtain

$$(T-\mu)\Big(u_1-\frac{w_2}{\mu}\Big)=(T_{|C(T)}-\mu)u_1+w_2=w_1+w_2=w.$$

This confirms that $T-\mu$ is onto. Finally, to show that $(T-\mu)^{-1}$ is bounded, we observe the following:

$$\begin{aligned}\|(T-\mu)^{-1}w\|=\Big\|u_1-\frac{w_2}{\mu}\Big\| \leqslant \|(T_{|C(T)}-\mu)^{-1}w_1\|+\frac{\|w_2\|}{|\mu|} &\leqslant M(\|w_1\|+\|w_2\|)\\ &\leqslant M\sqrt{2}\|w_1+w_2\|\\ &=M\sqrt{2}\|w\|,\end{aligned}$$

where $M=\max\{\|(T_{|C(T)}-\mu)^{-1}\|,\frac{1}{|\mu|}\}$. Therefore, $\mu \in \rho(T)$. □

COROLLARY 2.12. *Let $T \in C(H)$ be an EP operator. Then 0 is not a limit point of the spectrum of T.*

Proof. Since T is EP. Then $T_{|C(T)}$ is closed, one-one and onto, where $T_{|C(T)}: C(T) \to R(T)$. For all $y \in R(T)$, we have $\|(T_{|C(T)})^{-1}y\|=\|T^{\dagger}y\| \leqslant \|T^{\dagger}\|\|y\|$. So, $0 \in \rho(T_{|C(T)})$. We know that $\rho(T_{|C(T)})$ is open. Then there exists $\varepsilon>0$ such that

$$\{\lambda \in \rho(T_{|C(T)}): 0<|\lambda|<\varepsilon\} \subset \rho(T)$$

by using Theorem 2.11. Therefore, 0 is not a limit point of the spectrum of T. □

COROLLARY 2.13. *Let $T \in C(H)$ be an EP operator with finite-dimensional kernel, that is,* $\dim N(T) < \infty$. *Then $0 \in \Delta_k(T)$ for all $k \in \{1,2,3,4,5\}$, where $\Delta_k(T) = \mathbb{C} \setminus \sigma_{ek}(T)$ and $\sigma_{ek}(T)$ denotes the k-th essential spectrum as defined in* [5].

Proof. Since $R(T)$ is closed and $R(T)^{\perp} = N(T)$. Then $0 \in \Delta_k(T)$ for all $k = 1,2,3,4$. Moreover, $\Delta_5(T)$ is the union of all the components of $\Delta_1(T)$ which intersect the resolvent set $\rho(T)$ of T. Consequently, by Corollary 2.12, it follows that $0 \in \Delta_5(T)$. □

We consider two operators $S \in B(H)$ and $T \in C(H)$, but ST is not closed in general. We illustrate an example to show that ST is not closed.

EXAMPLE 2.14. Let S is defined on ℓ^2 by:

$$S(x_1,x_2,\ldots,x_n,\ldots) = \left(x_1, \frac{1}{2}x_2,\ldots,\frac{1}{n}x_n,\ldots\right)$$

Then, S is bounded. Consider T on ℓ^2 by:

$$T(x_1,x_2,\ldots,x_n,\ldots) = (x_1,2x_2,\ldots,nx_n,\ldots)$$

T is self-adjoint. So, T is closed. But $D(ST) = D(T) = \{x \in \ell^2 : Tx \in \ell^2\}$ is densely defined in ℓ^2. $ST = I_{D(T)}$ is bounded. If ST is closed then $D(T) = \ell^2$, which is a contradiction. Therefore, ST is not closed.

The next Lemma 2.15 gives a sufficient condition to have the closed operator ST when $S \in B(H)$ and $T \in C(H)$.

LEMMA 2.15. *Let $T \in C(H)$ and $S \in B(H)$ be an EP operator with $R(ST) = R(T)$. Then ST is closed.*

Proof. The operators T and S can be expressed as follows:

$$T = \begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix} : \begin{bmatrix} C(T) \\ N(T) \end{bmatrix} \to \begin{bmatrix} R(S) \\ N(S) \end{bmatrix},$$

and

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} R(S) \\ N(S) \end{bmatrix} \to \begin{bmatrix} R(S) \\ N(S) \end{bmatrix}.$$

It is straightforward to verify that S_1^{-1} is bounded. From the given condition $R(ST) = R(T)$, it follows that $T_2 = 0$. Hence, T_1 is closed, and consequently, the composition S_1T_1 is also closed.

Now, let us consider an arbitrary element $(p,q) \in \overline{G(ST)}$. Then there is a sequence $\{p_n\} = \{p'_n + p''_n\}$ in $D(T)$ with $p'_n \in C(T)$, $p''_n \in N(T)$, for all $n \in \mathbb{N}$, such that

$$p'_n \to p' \quad \text{and} \quad p''_n \to p'', \quad \text{as} \quad n \to \infty,$$

where $p = p' + p''$, $p' \in R(T)$, and $p'' \in N(T)$. Furthermore, we have

$$STp_n \to q = q' + q'' \quad \text{as} \quad n \to \infty,$$

where $q' \in R(S)$, $q'' \in N(S)$. So, $q'' = 0$ and $S_1T_1p'_n \to q'$ as $n \to \infty$. By the closedness of S_1T_1, we conclude that $S_1T_1p' = q'$. Thus,

$$q = \begin{pmatrix} S_1T_1p' \\ 0 \end{pmatrix} = ST \begin{pmatrix} p' \\ p'' \end{pmatrix} = STp.$$

Therefore, ST is closed. □

If A and B are two EP operators, it remains an open question under what conditions the product AB is also EP. In [6], Hartwig and Katz provided the necessary and sufficient conditions for the product of two square EP matrices to be EP. Subsequently, Dragan S. Djordjević established the necessary and sufficient conditions for the product of two bounded EP operators on $B(H)$ to also be EP [4]. In Theorem 2.16, we will present the necessary and sufficient conditions for the product of a closed EP operator and a bounded EP operator to also be EP.

THEOREM 2.16. *Let $T \in C(H)$ and $S \in B(H)$ both be EP with having T^* has a matrix representation. Then, ST is EP if and only if*

1. $R(ST) = R(T)$*;*
2. $N(ST) = N(T)$.

Proof. Since ST is EP. Then it must be closed and satisfy

$$R(ST) = R(T^*S^*) \subset R(T^*) = R(T).$$

Now consider an arbitrary element

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \begin{bmatrix} C(T) \\ N(T) \end{bmatrix}.$$

Then there exists

$$\begin{pmatrix} w \\ t \end{pmatrix} \in \begin{bmatrix} R(S) \\ N(S) \end{bmatrix}$$

such that

$$\begin{bmatrix} T_1 & 0 \\ T_2 & 0 \end{bmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} T_1^* & T_2^* \\ 0 & 0 \end{bmatrix} \begin{pmatrix} w \\ t \end{pmatrix}.$$

This yields the relations:

$$T_1u = T_1^*w + T_2^*t \quad \text{and} \quad T_2u = 0 \tag{16}$$

So, $T_2 = 0$ and $R(T) = R(T_1) \subset R(S)$. Furthermore, $N(T_1) = \{0\}$. Now consider the element $w_1 + w_2 \in N(ST)$, where $w_1 \in C(T), w_2 \in N(T)$. Then $S_1T_1w_1 = 0$ which implies $w_1 = 0$. Thus, $N(ST) \subset N(T) \subset N(ST)$. It follows that $N(ST) = N(T)$ establishing condition (2).

Next, let $y \in R(T) \cap R(ST)^{\perp}$. Then, $y \in R(ST)^{\perp} = N(ST) = N(T) = N(T^*) = R(T)^{\perp}$ which implies $y = 0$. Hence, $R(T) \subset R(ST) \subset R(T)$. Therefore, the condition (1), $R(ST) = R(T)$ is also verified.

Conversely, By Lemma 2.15, it follows that ST is closed. $R(ST) = R(T)$ is closed. Now

$$\begin{aligned} R((ST)^*) &= N(ST)^{\perp} \\ &= N(T)^{\perp} \text{ (by condition (2))} \\ &= R(T) \\ &= R(ST) \text{ (by condition (1))}. \end{aligned}$$

It shows that ST is EP. □

THEOREM 2.17. *Let $T \in C(H,K)$ be a closed range operator. Then $R(|T|) = R(|T|^{\alpha})$ for all $\alpha \in (0,\infty)$.*

Proof. Since $R(T)$ is closed. Then $R(T^*) = R(T^*T) = R(|T|)$ is closed. Now,

$$R(|T|) = R(|T|^{\frac{1}{2}}) = \ldots = R(|T|^{\frac{1}{2^n}}), \text{ for all } n \in \mathbb{N}. \tag{17}$$

When $1 < m \leqslant 2^n$ and $m \in \mathbb{N}$, we have

$$\begin{aligned} R(|T|^{\frac{m}{2^n}}) &= |T|^{\frac{m-1}{2^n}} R(|T|^{\frac{1}{2^n}}) \\ &= |T|^{\frac{m-1}{2^n}} R(|T|^{\frac{1}{2^{n-1}}}) \\ &= R(|T|^{\frac{m+1}{2^n}}). \end{aligned} \tag{18}$$

By induction hypothesis, we can say that

$$R(|T|^{\frac{m+k}{2^n}}) = R(|T|^{\frac{m}{2^n}}), \text{ for all } k \in \mathbb{N}. \tag{19}$$

Let us choose $k = 2^n - m$, then $R(|T|^{\frac{m}{2^n}}) = R(|T|)$, for all $m, n \in \mathbb{N}$ and $1 \leqslant m \leqslant 2^n$.

For all $\alpha \in (0,1)$, we have $|T| = |T|^{\alpha}|T|^{1-\alpha}$. So,

$$R(|T|) \subset R(|T|^{\alpha}). \tag{20}$$

Again, we also have $m', n' \in \mathbb{N}$ such that $1 \leqslant m' \leqslant 2^{n'}$ and $\frac{m'}{2^{n'}} \leqslant \alpha$. Thus,

$$R(|T|^{\alpha}) \subset R(|T|^{\frac{m'}{2^{n'}}}) = R(|T|). \tag{21}$$

By (20) and (21), $R(|T|^{\alpha}) = R(|T|)$, for all $\alpha \in (0,1]$. We know that $R(|T|^2) = R(|T|)$. Let us assume that $R(|T|^q) = R(|T|)$, where $q \geqslant 2$.

$$R(|T|^{q+1}) = |T|^{q-1}R(|T|^2) = |T|^{q-1}R(|T|) = R(|T|^q) = R(|T|).$$

Then $R(|T|^n) = R(|T|)$, for all $n \in \mathbb{N}$.

Let us consider an arbitrary element $1 < \beta < \infty$. Then there exists $\mu \in \mathbb{N}$ such that $\mu < \beta \leqslant \mu + 1$. So, $0 < \beta - \mu \leqslant 1$. Hence,

$$R(|T|^{\beta}) = |T|^{\mu} R(|T|^{\beta-\mu}) = R(|T|^{\mu+1}) = R(|T|).$$

Therefore, $R(|T|) = R(|T|^{\alpha})$, for all $\alpha \in (0,\infty)$. $\square$

COROLLARY 2.18. *Let $T \in C(H)$ be an EP operator. Then, $R(T) = R(|T|) = R(|T|^{\alpha})$, for all $\alpha \in (0,\infty)$.*

Proof. Since T is EP. By Theorem 2.17, we obtain

$$R(|T|^{\alpha}) = R(|T|) = R(T^*T) = R(T^*) = R(T). \quad \square$$

The following Theorem 2.19 says that the converse of Corollary 2.18 is true when $T \in B(H)$ and for some $\alpha \in (0,1)$.

THEOREM 2.19. *Let $T \in B(H)$ satisfy the condition $R(T) = R(|T|) = R(|T|^{\alpha})$, for some $\alpha \in (0,1)$. Then T is EP.*

Proof. We begin by claiming that the range of $|T|$, $R(|T|)$, is closed. By McCarthy inequality, it holds that

$$N(|T|^{\beta}) = N(|T|), \text{ for all } \beta \in (0,1].$$

Since T is bounded, so $H = D(|T|^{\alpha}) = D(|T|^{\alpha}) + R(|T|)$. Now, let $x \in N(|T|)^{\perp}$. Then there are $x_1 \in D(|T|^{\alpha})$, $x_2 \in R(|T|)$ such that $x = x_1 + x_2$. Then $x_1 \in N(|T|)^{\perp}$.

As $x_1 \in D(|T|^{\alpha})$, there exists y such that $|T|^{\alpha}x_1 = |T|y$ which implies

$$(x_1 - |T|^{1-\alpha}y) \in N(|T|^{\alpha}) = N(|T|).$$

Again, $|T|^{1-\alpha}y \in R(|T|^{1-\alpha}) \subset N(|T|^{1-\alpha})^{\perp} = N(|T|)^{\perp}$. We obtain that

$$(x_1 - |T|^{1-\alpha}y) \in N(|T|) \cap N(|T|)^{\perp} = \{0\},$$

which shows that $x_1 \in R(|T|^{1-\alpha})$. Moreover, $x = x_1 + x_2 \in R(|T|^{1-\alpha})$. Hence,

$$\overline{R(|T|^{1-\alpha})} = \overline{R(|T|)} = N(|T|)^{\perp} \subset R(|T|^{1-\alpha}) \subset \overline{R(|T|^{1-\alpha})}. \tag{22}$$

So, $R(|T|^{1-\alpha})$ is closed which implies $R(|T|) \subset R(|T|^{1-\alpha}) = R(|T|^{2^k(1-\alpha)}) \subset R(|T|)$, for some $k \in \mathbb{N}$. This concludes that $R(|T|) = R(T)$ is closed. It follows that $R(T^*) = R(|T|^2) = R(|T|) = R(T)$. Therefore, T is EP. $\square$

THEOREM 2.20. *Let $T \in C(H)$ be an EP operator and $S \in B(H)$ be also an EP operator with $\|Sx\| \leqslant a\|Tx\|$, for all $x \in D(T)$ and $0 < a < 1$. Then $T + S$ is hypo-EP.*

Proof. It is easy to show that $T+S$ is closed. The given condition says that

$$(1-a)\|Tx\| \leqslant \|(T+S)x\| \leqslant (1+a)\|Tx\|, \text{ for all } x \in D(T). \tag{23}$$

The above inequality (23) guarantees that $R(T+S)$ is closed. So, $R((T+S)^*)$ is also closed. $N(T) \subset N(S)$ says that $N(T) \subset N(T+S)$. For all $u \in D(T^*T) \subset D(T)$, we get

$$\|Su\|^2 \leqslant a^2\|Tu\|^2$$
$$\langle S^*Su, u\rangle \leqslant a^2\langle T^*Tu, u\rangle$$
$$S^*S \leqslant a^2T^*T.$$

By Douglas Theorem [3], there exists a contraction C such that $S^* \subset (aT)^*C$. So, $R(S^*) \subset R(T^*)$. When an arbitrary $p \in N(T+S)$, then $\|Tp\| = \|Sp\| \leqslant a\|Tp\| < \|Tp\|$ confirms that $p \in N(T)$. Thus, $N(T+S) = N(T)$. Again,

$$R(T+S)^* = R(T^*) = R(T^*) + R(S^*) = R(T) + R(S) \supset R(T+S).$$

Therefore, $T+S$ is hypo-EP. □

COROLLARY 2.21. *Let $T \in C(H)$ and $S \in B(H)$ be EP with the following conditions:*

1. $\|Sx\| \leqslant a\|Tx\|$, *where $a < 1$ and for all $x \in D(T)$;*
2. $\|S^*z\| \leqslant b\|T^*z\|$, *where $b < 1$ and for all $z \in D(T^*)$.*

Then, $T+S$ is EP.

Proof. From Theorem 2.20, we have $R((T+S)^*) = R(T)$ is closed and $T+S$ is closed. Again, the mentioned condition (2) guarantees that $R(T+S) = R(T)$. Thus, $R(T+S) = R((T+S)^*) = R(T)$. Therefore, $T+S$ is EP. □

REMARK 2.22. Let $T \in C(H)$ and $S \in B(H)$ be two normal and EP operators with $\|Sx\| < a\|Tx\|$, where $a < 1$ and for all $x \in D(T)$. Then, $T+S$ is EP.

THEOREM 2.23. *Let $T \in C(H)$ be a closed-range operator. Then T is EP if and only if*

1. $T(I-TT^\dagger)x = 0$, *for all $x \in H$;*
2. $T^*(I-T^*(T^*)^\dagger)x = 0$, *for all $x \in H$.*

Proof. Suppose T is EP. Then $N(T) = N(T^*)$. For all $x \in H$, we have

$$T(I-TT^\dagger)x = TP_{R(T)^\perp}x = TP_{N(T)}x = 0,$$

and

$$T^*(I-T^*(T^*)^\dagger)x = T^*P_{N(T)}x = T^*P_{N(T^*)}x = 0.$$

Conversely, From the condition (1), we get $TP_{R(T)^\perp}x = 0$, for all $x \in H$. So, $R(T)^\perp = N(T^*) \subset N(T)$. Condition (2) says that $R(T^*)^\perp = N(T) \subset N(T^*)$. Therefore, T is EP. □

3. Characterizations of bounded EP operators on Hilbert spaces

Let us consider the set E of all EP operators in $B(H)$. The following example shows that E is not closed in $B(H)$.

EXAMPLE 3.1. Let us define $T_n : \ell^2 \to \ell^2$ by

$$T_n(x_1, x_2, \ldots, x_k, \ldots) = \left(x_1, \frac{1}{2}x_2, \ldots, \frac{1}{n}x_n, 0, \ldots, 0, \ldots\right), \text{ for all } n \in \mathbb{N}.$$

Here, each $T_n \in B(\ell^2)$ is self-adjoint closed range operator. So, T_n is EP for all $n \in \mathbb{N}$. Moreover, $T_n \to T$, as $n \to \infty$, where $T : \ell^2 \to \ell^2$ defined by:

$$T(x_1, x_2, \ldots, x_k, x_{k+1} \ldots) = \left(x_1, \frac{1}{2}x_2, \ldots, \frac{1}{k}x_k, \frac{1}{k+1}x_{k+1}, \ldots\right).$$

It is easy to show that $\gamma(T) = 0$, where $\gamma(T)$ is the reduced minimum modulus of T. Thus, T is not EP because $R(T) = R(T^*)$ but $R(T)$ is not closed.

The following Theorem 3.2 says that there is a closed set in $B(H)$ whose all elements are EP. Before stating Theorem 3.2, a new set E_δ is defined by:

$$E_\delta = \{T \in E : \gamma(T) \geqslant \delta > 0\}. \tag{24}$$

THEOREM 3.2. *E_δ is closed set in $B(H)$.*

Proof. Let us consider $T \in \overline{E_\delta}$, the closure of E_δ. Then there is a sequence $\{T_n\}$ in E_δ such that $T_n \to T$ as $n \to \infty$. So, $\gamma(T_n) \geqslant \delta$ implies $\gamma(T) \geqslant \delta$ [7]. Thus, $R(T)$ is closed. We know that $\gamma(T_n) = \frac{1}{\|{T_n}^\dagger\|}$, for all $n \in \mathbb{N}$. Thus, $\sup\{\|{T_n}^\dagger\| : n \in \mathbb{N}\} \leqslant \frac{1}{\delta}$. By Theorem 1.5, we get ${T_n}^\dagger \to T^\dagger$ as $n \to \infty$. Hence,

$$\begin{aligned}
&\|T^\dagger T - TT^\dagger\| \\
&= \|T^\dagger(T - T_n) + T^\dagger T_n - {T_n}^\dagger T_n + T_n {T_n}^\dagger - T_n T^\dagger + T_n T^\dagger - TT^\dagger\| \\
&\leqslant \|T^\dagger\|\|T - T_n\| + \|T^\dagger - {T_n}^\dagger\|\|T_n\| + \|T_n\|\|{T_n}^\dagger - T^\dagger\| + \|T_n - T\|\|T^\dagger\|.
\end{aligned}$$

The right-hand side of the above inequality goes to 0 as $n \to \infty$. Hence, T is EP. Therefore, E_δ is closed in $B(H)$. □

COROLLARY 3.3. $E = (\bigcup_{\delta>0} E_\delta) \bigcup \{0\}$.

Proof. Let $S \in E$ be a non-zero EP operator. Then $\gamma(S) \geqslant \delta_1$, for some $\delta_1 > 0$. So, $S \in E_{\delta_1}$. Thus, $E \subset (\bigcup_{\delta>0} E_\delta) \bigcup \{0\}$. The opposite inclusion is obvious. Therefore, $E = (\bigcup_{\delta>0} E_\delta) \bigcup \{0\}$. □

THEOREM 3.4. *Let $T \in B(H)$ be an EP operator. Then*

$$\gamma(T) \leqslant r(T).$$

Proof. Theorem 2.9 says that T^n is EP and $R(T^n) = R(T)$, for all $n \in \mathbb{N}$. Moreover, $(T^n)^\dagger = (T^\dagger)^n$, for all $n \in \mathbb{N}$. We know that $r(T^\dagger) = \lim_{n\to\infty} \|(T^\dagger)^n\|^{\frac{1}{n}}$. Again,

$$\frac{1}{\|(T^\dagger)^n\|} = \frac{1}{\|(T^n)^\dagger\|} = \gamma(T^n) = \inf_{x\in R(T^n)} \frac{\|T^n x\|}{\|x\|} \leqslant \left(\inf_{x\in R(T)} \frac{\|Tx\|}{\|x\|} \right) \|T\|^{n-1} = \|T\|^{n-1}\gamma(T). \tag{25}$$

Thus, $\frac{1}{\|T\|} = \lim_{n\to\infty} \frac{1}{\|T\|^{1-\frac{1}{n}}} \leqslant r(T^\dagger)$. Therefore, $\gamma(T) = \frac{1}{\|T^\dagger\|} \leqslant r(T)$. $\square$

Acknowledgements. We would like to thank the anonymous referees for their thorough and critical reading of the paper and their numerous useful comments and suggestions, which helped to improve it substantially.

The present work of the second author was partially supported by the Anusandhan National Research Foundation (ANRF), Department of Science and Technology, Government of India (Reference Number: MTR/2023/000471) under the scheme “Mathematical Research Impact Centric Support (MATRICS)”.

Data availability. Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflict of interest. The authors declare no conflicts of interest.

ORCID ID. https://orcid.org/0009-0004-4812-0535

REFERENCES

[1] ENRICO BOASSO, *On the Moore–Penrose inverse, EP Banach space operators, and EP Banach algebra elements*, J. Math. Anal. Appl., **339** (2): 1003–1014, 2008.
[2] STEPHEN L. CAMPBELL AND CARL D. MEYER, *EP operators and generalized inverses*, Canad. Math. Bull., **18** (3): 327–333, 1975.
[3] R. G. DOUGLAS, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proc. Amer. Math. Soc., **17**: 413–415, 1966.
[4] DRAGAN S. DJORDJEVIĆ, *Products of EP operators on Hilbert spaces*, Proc. Amer. Math. Soc., **129** (6): 1727–1731, 2001.
[5] DAVID ERIC EDMUNDS, W. DESMOND EVANS, *Spectral Theory and Differential Operators*, Oxford University Press, 1987.
[6] ROBERT E. HARTWIG, IRVING J. KATZ, *On products of EP matrices*, Linear Algebra Appl., **252**: 339–345, 1997.
[7] IN SUNG HWANG, WOO YOUNG LEE, *The reduced minimum modulus of operators*, J. Math. Anal. Appl., **267** (2): 679–694, 2002.
[8] MASUO ITOH, *On some EP operators*, Nihonkai Math. J., **16** (1): 49–56, 2005.
[9] P. SAM JOHNSON, *Closed EP and hypo-EP operators on Hilbert spaces*, The Journal of Analysis., **30** (4): 1377–1390, 2022.

[10] ARUP MAJUMDAR, P. SAM JOHNSON, *The Moore-Penrose inverses of unbounded closable operators and the direct sum of closed operators in Hilbert spaces*, Linear and Multilinear Algebra, **73** (8): 1668–1684, 2025.

[11] ARUP MAJUMDAR, P. SAM JOHNSON, AND RAM N MOHAPATRA, *Hyers-Ulam Stability of Unbounded Closable Operators in Hilbert Spaces*, Math. Nachr., 1–17, 2024.

[12] ARUP MAJUMDAR, P. SAM JOHNSON, RAM N. MOHAPATRA, *On the generalized Cauchy dual of closed operators in Hilbert spaces*, Acta Scientiarum Mathematicarum, 2025, (to appear).

[13] M. H. PEARL, *On generalized inverses of matrices*, Proc. Cambridge Philos. Soc., **62**: 673–677, 1966.

[14] HANS SCHWERDTFEGER, *Introduction to Linear Algebra and the Theory of Matrices*, P. Noordhoff, Groningen, 1950.

[15] M. ZUHAIR NASHED, *Generalized inverses and applications*, Proceedings of an Advanced Seminar sponsored by the Mathematics Research Center at the University of Wisconsin-Madison, October 8–10, 1973, Publication of Mathematics Research Center, the University of Wisconsin-Madison, Elsevier Science, 1976.

(Received April 10, 2025)

Arup Majumdar
Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka (NITK)
Surathkal, Mangaluru 575025, India
e-mail: arupmajumdar93@gmail.com

P. Sam Johnson
Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka (NITK)
Surathkal, Mangaluru 575025, India

Ram N. Mohapatra
Department of Mathematics
University of Central Florida
Orlando, FL. 32816, USA