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# SCHATTEN $p$–NORM AND EUCLIDEAN OPERATOR RADIUS INEQUALITIES

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*Abstract.* From the positivity of $2\times 2$ block matrices, we study the Schatten $p$-norm inequalities for the sum and product of $n\times n$ complex matrices via the singular values and the Moore-Penrose inverses. Using these inequalities, we deduce the $p$-numerical radius and the classical numerical radius bounds. We also study the Euclidean operator radius inequalities for a pair of bounded linear operators via the Moore-Penrose inverse, and we deduce new classical numerical radius bounds.

## 1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space and $\mathbb{B}(\mathscr{H})$ be the $C^*$-algebra of all bounded linear operators that act on $\mathscr{H}$ to itself. For $T\in\mathbb{B}(\mathscr{H})$, $T^*$ denotes the adjoint of $T$ and $|T|=(T^*T)^{\frac{1}{2}}$. An operator $T\in\mathbb{B}(\mathscr{H})$ is said to be positive if $\langle Tx,x\rangle\geqslant 0$, for all $x\in\mathscr{H}$, and it is denoted as $T\geqslant 0$. The numerical range of $T\in\mathbb{B}(\mathscr{H})$, denoted by $W(T)$, is defined as $W(T)=\{\langle Tx,x\rangle : x\in\mathscr{H}, \|x\|=1\}$. It is well known that $W(T)$ is a bounded convex subset of $\mathbb{C}$, though the exact shape of it is in general uncertain. The radius of the smallest circular disc with center at the origin containing $W(T)$ is defined to be the numerical radius $w(T)$, that is, $w(T)=\sup\{|\langle Tx,x\rangle| : x\in\mathscr{H}, \|x\|=1\}$. The numerical radius defines a norm on $\mathbb{B}(\mathscr{H})$ and satisfies

$$\frac{1}{2}\|T\|\leqslant w(T)\leqslant \|T\|, \tag{1.1}$$

where the operator norm $\|T\|=\sup\{\|Tx\| : x\in\mathscr{H}, \|x\|=1\}$. We refer to recent books [7, 24] for further reading on the numerical radius. For any $T\in\mathbb{B}(\mathscr{H})$ with closed range, the Moore-Penrose inverse of $T$ is the operator $P\in\mathbb{B}(\mathscr{H})$, which satisfies the following four Penrose equations (see [13]): $TPT=T$, $PTP=P$, $(TP)^*=TP$ and $(PT)^*=PT$. Note that the Moore-Penrose inverse of $T$ is unique and is denoted by $T^\dagger$. Throughout, we restrict the notation $\mathscr{CR}(\mathscr{H})\subseteq\mathbb{B}(\mathscr{H})$ for all operators having closed ranges. It is clear that if $\mathscr{H}$ is finite-dimensional, then all operators in $\mathbb{B}(\mathscr{H})$ have closed ranges. Hence, each matrix has the Moore-Penrose inverse. It is easy to

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verify that for $T \in \mathscr{CR}(\mathscr{H})$, $T = TT^\dagger T = (TT^\dagger)^* T = (T^\dagger)^* |T|^2$ and $T = TT^\dagger T = T(T^\dagger T)^* = |T^*|^2 (T^\dagger)^*$. We refer to [9, 12] for further discussion of the Moore-Penrose inverse of operators in $\mathbb{B}(\mathscr{H})$. The $p$-numerical radius of an $n \times n$ matrix $T \in \mathscr{M}_n(\mathbb{C})$, denoted by $w_p(T)$, is defined as

$$\begin{aligned} w_p(T) &= \max\left\{\|Re(\lambda T)\|_p : \lambda \in \mathbb{C}, |\lambda| = 1\right\} \\ &= \sup\left\{\|Re(e^{i\theta} T)\|_p : \theta \in \mathbb{R}\right\}, \quad p > 0, \end{aligned}$$

where $\|\cdot\|_p$ is the Schatten $p$-norm and $Re(T) = \frac{1}{2}(T + T^*)$. Recall that the Schatten $p$-norm of $T \in \mathscr{M}_n(\mathbb{C})$ is defined as

$$\|T\|_p = \left(\sum_{j=1}^{n} s_j^p(T)\right)^{1/p} = (tr(|T|^p))^{1/p},$$

where $s_1(T) \geqslant s_2(T) \geqslant \ldots \geqslant s_n(T)$ are the singular values of $T$, that is, the eigenvalues of $|T|$. For $p = \infty$, $\|T\|_p = \|T\| = s_1(T)$ and so $w_p(T) = w(T)$ is the classical numerical radius. For $p \geqslant 1$, the Schatten $p$-norm defines a norm on $\mathscr{M}_n(\mathbb{C})$. For $1 \leqslant p \leqslant q \leqslant \infty$, $\|T\| \leqslant \|T\|_q \leqslant \|T\|_p \leqslant \|T\|_1$. The Schatten $p$-norm is very useful for studying matrix theory as well as operator theory; see [1, 3, 4] and the references therein. Due to importance of the concept of numerical radius, various generalizations have been studied over the last few years. One such generalization is the Euclidean operator radius; see [22]. Recall that for $B, C \in \mathbb{B}(\mathscr{H})$, the Euclidean operator radius of $B$ and $C$, denoted by $w_e(B,C)$, is defined as

$$w_e(B,C) = \sup\left\{\sqrt{|\langle Bx,x\rangle|^2 + |\langle Cx,x\rangle|^2} : x \in \mathscr{H}, \|x\| = 1\right\}.$$

The Euclidean operator radius defines a norm on $\mathbb{B}^2(\mathscr{H}) = \mathbb{B}(\mathscr{H}) \times \mathbb{B}(\mathscr{H})$, which satisfies

$$\frac{1}{8}\|B^*B + C^*C\| \leqslant w_e^2(B,C) \leqslant \|B^*B + C^*C\|, \tag{1.2}$$

see [22]. Here, the constants $\frac{1}{8}$ and 1 are the best possible. In [10, Theorem 1], Dragomir proved that

$$\frac{1}{2} w\left(B^2 + C^2\right) \leqslant w_e^2(B,C), \tag{1.3}$$

and the constant $\frac{1}{2}$ is the best possible. We refer to [2, 14, 15, 16, 17, 21, 23] for more generalizations of the Euclidean operator radius and related results.

In Section 2, we develop the Schatten $p$-norm and the $p$-numerical radius inequalities for the sum and product of $n \times n$ complex matrices using singular values. From these we deduce several classical numerical radius bounds. In Section 3, we develop upper bounds for the Euclidean operator radius for a pair of bounded linear operators and deduce the classical numerical radius bounds for bounded linear operators with closed range.

## 2. Schatten $p$-norm and $p$-numerical radius inequalities

In this section, we develop the Schatten $p$-norm inequalities, from which we deduce the $p$-numerical radius bounds. First, we note the following well-known lemmas, which are needed to present our results.

LEMMA 2.1. [20] *Let $A,B,C \in \mathbb{B}(\mathscr{H})$, where $A$ and $B$ are positive, and $x,y \in \mathscr{H}$. Then $\begin{bmatrix} A & C^* \\ C & B \end{bmatrix} \in \mathbb{B}(\mathscr{H} \bigoplus \mathscr{H})$ is positive if and only if $|\langle Cx,y\rangle|^2 \leqslant \langle Ax,x\rangle\langle By,y\rangle$.*

LEMMA 2.2. [18] *Let $T \in \mathbb{B}(\mathscr{H})$. Let $f$ and $g$ be non-negative continuous functions on $[0,\infty)$, which satisfy $f(t)g(t) = t$ for all $t \in [0,\infty)$. Then*

$$|\langle Tx,y\rangle|^2 \leqslant \langle f^2(|T|)x,x\rangle\langle g^2(|T^*|)y,y\rangle, \quad \textit{for every } x,y \in \mathscr{H}.$$

LEMMA 2.3. [8, Corollary III.1.2] *Let $T \in \mathscr{M}_n(\mathbb{C})$. Then*

$$s_j(T) = \max_{dimM=j} \left\{ \min_{x\in M, \|x\|=1} \|Tx\| \right\}.$$

Using the above lemmas, we now obtain a Schatten $p$-norm inequality for the sum of the product matrices via the singular values.

THEOREM 2.4. *Let $T,S,X,Y \in \mathscr{M}_n(\mathbb{C})$. Let $e,f,g,h$ be non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$ and $e(t)h(t) = t$ for all $t \geqslant 0$. Then*

$$s_j(Y(T+S)X) \leqslant \min \left\{ \begin{array}{l} \|Y\left(g^2(|T^*|)+h^2(|S^*|)\right)Y^*\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(X^*\left(f^2(|T|)+e^2(|S|)\right)X), \\ s_j^{\frac{1}{2}}(Y\left(g^2(|T^*|)+h^2(|S^*|)\right)Y^*)\|(X^*\left(f^2(|T|)+e^2(|S|)\right)X)\|^{\frac{1}{2}} \end{array} \right\}$$

*for every $j = 1,2,\ldots,n$.*

*Furthermore, we have*

$$\|Y(T+S)X\|_p \leqslant \min \left\{ \begin{array}{l} \|Y\left(g^2(|T^*|)+h^2(|S^*|)\right)Y^*\|^{\frac{1}{2}} \|(X^*\left(f^2(|T|)+e^2(|S|)\right)X)\|_{\frac{p}{2}}^{\frac{1}{2}}, \\ \|(Y\left(g^2(|T^*|)+h^2(|S^*|)\right)Y^*)\|_{\frac{p}{2}}^{\frac{1}{2}} \|(X^*\left(f^2(|T|)+e^2(|S|)\right)X)\|^{\frac{1}{2}} \end{array} \right\}$$

*for every $p > 0$.*

*Proof.* Following Lemmas 2.1 and 2.2, we get $\begin{bmatrix} f^2(|T|) & T^* \\ T & g^2(|T^*|) \end{bmatrix} \geqslant 0$ and $\begin{bmatrix} e^2(|S|) & S^* \\ S & h^2(|S^*|) \end{bmatrix} \geqslant 0$. Thus, $\begin{bmatrix} f^2(|T|)+e^2(|S|) & T^*+S^* \\ T+S & g^2(|T^*|)+h^2(|S^*|) \end{bmatrix} \geqslant 0$. Again, following Lemma 2.1, we have

$$|\langle (T+S)x,y\rangle|^2 \leqslant \langle\left(f^2(|T|)+e^2(|S|)\right)x,x\rangle\langle\left(g^2(|T^*|)+h^2(|S^*|)\right)y,y\rangle, \qquad (2.1)$$

for every $x,y \in \mathscr{H}$. By replacing $x$ with $Xx$ and $y$ with $Y^*y$ in (2.1), we obtain

$$|\langle (Y(T+S)X)x,y\rangle|^2 \leqslant \langle X^*\left(f^2(|T|)+e^2(|S|)\right)Xx,x\rangle\langle Y\left(g^2(|T^*|)+h^2(|S^*|)\right)Y^*y,y\rangle.$$

Taking the supremum over $y \in \mathscr{H}$ with $\|y\|=1$, we get

$$\|\,(Y(T+S)X)x\|^2 \leqslant \langle X^*\left(f^2(|T|)+e^2(|S|)\right)Xx,x\rangle\left\|Y\left(g^2(|T^*|)+h^2(|S^*|)\right)Y^*\right\|.$$

This implies that

$$\begin{aligned}&\max_{dimM=j}\min_{x\in M,\|x\|=1}\|\,(Y(T+S)X)x\|^2\\ &\leqslant \max_{dimM=j}\min_{x\in M,\|x\|=1}\langle X^*\left(f^2(|T|)+e^2(|S|)\right)Xx,x\rangle\|Y\left(g^2(|T^*|)+h^2(|S^*|)\right)Y^*\|,\end{aligned}$$

for every $j=1,2,3,\ldots,n$. Therefore, by Lemma 2.3, we get

$$s_j\left(Y(T+S)X\right) \leqslant s_j^{\frac{1}{2}}\left(X^*\left(f^2(|T|+e^2(|S|)\right)X\right)\|Y\left(g^2(|T^*|+h^2(|S^*|)\right)Y^*\|^{\frac{1}{2}}. \tag{2.2}$$

Again, by replacing $x$ with $Xy$ and $y$ with $Y^*x$ in (2.1) and using similar arguments, we get

$$s_j\left(Y(T+S)X\right) \leqslant \|\left(X^*\left(f^2(|T|)+e^2(|S|)\right)X\right)\|^{\frac{1}{2}}s_j^{\frac{1}{2}}\left(Y\left(g^2(|T^*|)+h^2(|S^*|)\right)Y^*\right), \tag{2.3}$$

for each $j=1,2,\ldots,n$. Hence, combining (2.2) and (2.3), we get the first desired inequality.

Now, it follows from (2.2) that

$$\begin{aligned}&\left(\sum_{j=1}^n s_j^p\left(Y(T+S)X\right)\right)^{\frac{1}{p}}\\ &\leqslant \left(\sum_{j=1}^n s_j^{\frac{p}{2}}\left(X^*\left(f^2(|T|)+e^2(|S|)\right)X\right)\right)^{\frac{1}{p}}\|Y\left(g^2(|T^*|)+h^2(|S^*|)\right)Y^*\|^{\frac{1}{2}}.\end{aligned} \tag{2.4}$$

This implies that

$$\|\,(Y(T+S)X)\,\|_p \leqslant \|\left(X^*\left(f^2(|T|+e^2(|S|)\right)X\right)\|_{\frac{p}{2}}^{\frac{1}{2}}\|Y\left(g^2(|T^*|+h^2(|S^*|)\right)Y^*\|^{\frac{1}{2}}. \tag{2.5}$$

Similarly, from (2.3), we also get

$$\|\,(Y(T+S)X)\,\|_p \leqslant \|\left(X^*\left(f^2(|T|+e^2(|S|)\right)X\right)\|^{\frac{1}{2}}\|Y\left(g^2(|T^*|+h^2(|S^*|)\right)Y^*\|_{\frac{p}{2}}^{\frac{1}{2}}. \tag{2.6}$$

Therefore, the desired second inequality follows from combining (2.5) and (2.6). □

As a consequence of Theorem 2.4, we get the following result.

COROLLARY 2.5. *Let* $T,S,X,Y \in \mathscr{M}_n(\mathbb{C})$. *Then*

$$\|Y(T+S)X\|_p \leqslant \min \left\{ \begin{array}{l} \|Y\left(|T^*|^{2t}+|S^*|^{2t}\right)Y^*\|^{\frac{1}{2}} \|(X^*\left(|T|^{2(1-t)}+|S|^{2(1-t)}\right)X)\|^{\frac{1}{2}}_{\frac{p}{2}}, \\ \|Y\left(|T^*|^{2t}+|S^*|^{2t}\right)Y^*\|^{\frac{1}{2}}_{\frac{p}{2}} \|(X^*\left(|T|^{2(1-t)}+|S|^{2(1-t)}\right)X)\|^{\frac{1}{2}} \end{array} \right\}$$

*and*

$$\|Y(T+S)X\|_p \leqslant \min \left\{ \begin{array}{l} \|Y\left(|T^*|^{2t}+|S^*|^{2(1-t)}\right)Y^*\|^{\frac{1}{2}} \|(X^*\left(|T|^{2(1-t)}+|S|^{2t}\right)X)\|^{\frac{1}{2}}_{\frac{p}{2}}, \\ \|Y\left(|T^*|^{2t}+|S^*|^{2(1-t)}\right)Y^*\|^{\frac{1}{2}}_{\frac{p}{2}} \|(X^*\left(|T|^{2(1-t)}+|S|^{2t}\right)X)\|^{\frac{1}{2}} \end{array} \right\}$$

*for every* $p>0$ *and* $t \in [0,1]$.

Taking $p \to \infty$ in Corollary 2.5, we get the following result.

COROLLARY 2.6. *Let* $T,S,X,Y \in \mathscr{M}_n(\mathbb{C})$. *Then*

$$\|Y(T+S)X\| \leqslant \|Y\left(|T^*|^{2t}+|S^*|^{2t}\right)Y^*\|^{\frac{1}{2}} \|(X^*\left(|T|^{2(1-t)}+|S|^{2(1-t)}\right)X)\|^{\frac{1}{2}}$$

*and*

$$\|Y(T+S)X\| \leqslant \|Y\left(|T^*|^{2t}+|S^*|^{2(1-t)}\right)Y^*\|^{\frac{1}{2}} \|(X^*\left(|T|^{2(1-t)}+|S|^{2t}\right)X)\|^{\frac{1}{2}}$$

*for every* $p>0$ *and* $t \in [0,1]$.

Setting $t=\frac{1}{2}$ in Corollary 2.6, we get the following result.

COROLLARY 2.7. *Let* $T,S,X,Y \in \mathscr{M}_n(\mathbb{C})$. *Then*

$$\|Y(T+S)X\| \leqslant \|Y\left(|T^*|+|S^*|\right)Y^*\|^{\frac{1}{2}} \|X^*\left(|T|+|S|\right)X\|^{\frac{1}{2}}.$$

Now, setting $S=T$ and $t=\frac{1}{2}$ in Corollary 2.5, we get an upper bound for the Schatten $p$-norm of the product of three matrices.

COROLLARY 2.8. *Let* $T,X,Y \in \mathscr{M}_n(\mathbb{C})$. *Then*

$$\|YTX\|_p \leqslant \min\left\{ \|Y|T^*|Y^*\|^{\frac{1}{2}} \|X^*|T|X\|^{\frac{1}{2}}_{\frac{p}{2}}, \|Y|T^*|Y^*\|^{\frac{1}{2}}_{\frac{p}{2}} \|X^*|T|X\|^{\frac{1}{2}} \right\}$$

*for every* $p>0$.

Considering $Y=T$ and $X=I$ (the identity operator) in Corollary 2.8, we get the following corollary.

COROLLARY 2.9. *Let $T \in \mathscr{M}_n(\mathbb{C})$. Then*

$$\|T^2\|_p \leqslant \min\left\{ \sqrt{\|T|T^*|T^*\|\|T\|_{\frac{p}{2}}}, \sqrt{\|T|T^*|T^*\|_{\frac{p}{2}}\|T\|} \right\} \tag{2.7}$$

*for every $p > 0$. Taking $p \to \infty$, we get*

$$\left\|T^2\right\|^2 \leqslant \|T|T^*|T^*\|\,\|T\|.$$

We now deduce an upper bound for the $p$-numerical radius. Setting $X = Y = I$, $S = e^{-i\theta}T^*$ and $T = e^{i\theta}T$ in Theorem 2.4, and then taking the supremum over all $\theta \in \mathbb{R}$, we get the following result.

COROLLARY 2.10. *Let $T \in \mathscr{M}_n(\mathbb{C})$. Let $e, f, g, h$ be non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$ and $e(t)h(t) = t$ for all $t \geqslant 0$. Then*

$$w_p(T) \leqslant \frac{1}{2}\min\left\{ \begin{array}{l} \|g^2(|T^*|)+h^2(|T|)\|^{\frac{1}{2}}\|f^2(|T|)+e^2(|T^*|)\|_{\frac{p}{2}}^{\frac{1}{2}}, \\ \|g^2(|T^*|)+h^2(|T|)\|_{\frac{p}{2}}^{\frac{1}{2}}\|f^2(|T|)+e^2(|T^*|)\|^{\frac{1}{2}} \end{array} \right\}$$

*for every $p > 0$.*

REMARK 2.11. Let $T \in \mathscr{M}_n(\mathbb{C})$. Let $e, f, g, h$ be non-negative continuous functions on $[0,\infty)$ such that $f(t)g(t) = t$ and $e(t)h(t) = t$ for all $t \geqslant 0$. Taking $p \to \infty$ in Corollary 2.10, we get

$$w(T) \leqslant \frac{1}{2}\|g^2(|T^*|)+h^2(|T|)\|^{\frac{1}{2}}\|f^2(|T|)+e^2(|T^*|)\|^{\frac{1}{2}},$$

which is also recently given in [6]. From this inequality one can deduce many classical numerical radius bounds.

Next, we obtain the Schatten $p$-norm inequalities for $n \times n$ matrices by the Moore-Penrose inverse. For this purpose, we need the following known lemma.

LEMMA 2.12. [19] *Let $T \in \mathscr{CR}(\mathscr{H})$. Then*

$$|\langle Tx, y\rangle|^2 \leqslant \left\langle |T|^2 x, x\right\rangle \left\langle TT^\dagger y, y\right\rangle, \quad \textit{for any } x, y \in \mathscr{H}.$$

THEOREM 2.13. *Let $S, T, X, Y \in \mathscr{M}_n(\mathbb{C})$. Then*

$$s_j(Y(T+S)X) \leqslant \min\left\{ \begin{array}{l} \|Y\left(T^\dagger T+|S|^2\right)Y^*\|^{\frac{1}{2}} s_j^{\frac{1}{2}}(X^*\left(|T^*|^2+SS^\dagger\right)X), \\ s_j^{\frac{1}{2}}(Y\left(T^\dagger T+|S|^2\right)Y^*)\|(X^*\left(|T^*|^2+SS^\dagger\right)X)\|^{\frac{1}{2}} \end{array} \right\}$$

*for every $j = 1, 2, \ldots, n$.*

*Furthermore, we have*

$$\|Y(T+S)X\|_p \leqslant \min\left\{ \begin{array}{l} \|Y\left(T^{\dagger}T+|S|^2\right)Y^*\|^{\frac{1}{2}}\|(X^*\left(|T^*|^2+SS^{\dagger}\right)X)\|_{\frac{p}{2}}^{\frac{1}{2}}, \\ \|(Y\left(T^{\dagger}T+|S|^2\right)Y^*)\|_{\frac{p}{2}}^{\frac{1}{2}}\|(X^*\left(|T^*|^2+SS^{\dagger}\right)X)\|^{\frac{1}{2}} \end{array} \right\}$$

*for all* $p>0$.

*Proof.* From Lemma 2.12, it is easy to see that $\begin{bmatrix} T^{\dagger}T & T^* \\ T & |T^*|^2 \end{bmatrix} \geqslant 0$ and $\begin{bmatrix} |S|^2 & S^* \\ S & SS^{\dagger} \end{bmatrix} \geqslant 0$. Therefore, $\begin{bmatrix} T^{\dagger}T+|S|^2 & T^*+S^* \\ T+S & |T^*|^2+SS^{\dagger} \end{bmatrix} \geqslant 0$. By Lemma 2.1, we have

$$|\langle (T+S)x,y\rangle|^2 \leqslant \left\langle \left(T^{\dagger}T+|S|^2\right)x,x\right\rangle \left\langle \left(|T^*|^2+SS^{\dagger}\right)y,y\right\rangle \tag{2.8}$$

for every $x,y \in \mathscr{H}$. Now, by replacing $x$ with $Xx$ and $y$ with $Y^*y$ in (2.8), and proceeding similarly as Theorem 2.4, we get the desired result. □

Considering $p\to\infty$ in Theorem 2.13, we get the following result.

COROLLARY 2.14. *Let* $T,S,X,Y \in \mathscr{M}_n(\mathbb{C})$. *Then*

$$\|Y(T+S)X\| \leqslant \|Y\left(T^{\dagger}T+|S|^2\right)Y^*\|^{\frac{1}{2}}\|(X^*\left(|T^*|^2+SS^{\dagger}\right)X)\|^{\frac{1}{2}}.$$

Setting $X=Y=I$ in Theorem 2.13, we get the following corollary.

COROLLARY 2.15. *Let* $T,S \in \mathscr{M}_n(\mathbb{C})$. *Then*

$$\|(T+S)\|_p \leqslant \min\left\{ \|T^{\dagger}T+|S|^2\|^{\frac{1}{2}}\||T^*|^2+SS^{\dagger}\|_{\frac{p}{2}}^{\frac{1}{2}}, \|T^{\dagger}T+|S|^2\|_{\frac{p}{2}}^{\frac{1}{2}}\||T^*|^2+SS^{\dagger}\|^{\frac{1}{2}} \right\},$$

*for all* $p>0$.

Now, replacing $T$ by $e^{i\theta}T$ and $S$ by $e^{-i\theta}T^*$ in Corollary 2.15, and taking the supremum all over $\theta\in\mathbb{R}$, we get the following bound for the Schatten $p$-numerical radius.

COROLLARY 2.16. *Let* $T \in \mathscr{M}_n(\mathbb{C})$. *Then*

$$w_p(T) \leqslant \frac{1}{2}\left\|T^{\dagger}T+|T^*|^2\right\|^{\frac{1}{2}}\left\|T^{\dagger}T+|T^*|^2\right\|_{\frac{p}{2}}^{\frac{1}{2}}$$

*for all* $p>0$. *Considering* $p\to\infty$, *we get*

$$w(T) \leqslant \frac{1}{2}\left\|T^{\dagger}T+|T^*|^2\right\|. \tag{2.9}$$

The numerical radius bound (2.9) is also studied in [5].

Another numerical radius bound is as follows.

THEOREM 2.17. *Let $T \in \mathscr{M}_n(\mathbb{C})$. Then*

$$w(T) \leqslant \frac{1}{4}\left\|T^{\dagger}T + TT^{\dagger} + TT^{*} + T^{*}T\right\|.$$

*Proof.* Setting $S = T$ and $x = y$ in (2.8), we obtain

$$\begin{aligned}
|\langle Tx,x\rangle| &\leqslant \frac{1}{2}\sqrt{\langle (T^{\dagger}T + |T|^2)x,x\rangle \langle (|T^*|^2 + TT^{\dagger})x,x\rangle} \\
&\leqslant \frac{1}{4}\left(\left\langle \left(T^{\dagger}T + |T|^2\right)x,x\right\rangle + \left\langle \left(|T^*|^2 + TT^{\dagger}\right)x,x\right\rangle\right) \\
&= \frac{1}{4}\left\langle \left(T^{\dagger}T + |T|^2 + |T^*|^2 + TT^{\dagger}\right)x,x\right\rangle \\
&\leqslant \frac{1}{4}\|T^{\dagger}T + |T|^2 + |T^*|^2 + TT^{\dagger}\|.
\end{aligned}$$

Taking the supremum over $x$ with $\|x\| = 1$, we get the desired result. □

## 3. Euclidean operator radius inequalities

In this section, we obtain the Euclidean operator radius inequalities for a pair of bounded linear operators via the Moore-Penrose inverse, and from these inequalities we deduce the classical numerical radius bounds. For this purpose, we need the following inner product inequality.

LEMMA 3.1. *Let $T \in \mathscr{CR}(\mathscr{H})$. Then*

$$\left|\left\langle |T|^2 x,y\right\rangle\right|^2 \leqslant \left|\left\langle |T|^4 x,x\right\rangle\right| \left|\left\langle T^{\dagger}Ty,y\right\rangle\right|,$$

*for every $x,y \in \mathscr{H}$.*

*Proof.* Following Lemma 2.12, we have

$$\begin{aligned}
\left|\left\langle |T|^2 x,y\right\rangle\right|^2 &= |\langle T^*Tx,y\rangle|^2 \\
&\leqslant \left|\left\langle |T|^4 x,x\right\rangle\right| \left|\left\langle T^*T\left(T^*T\right)^{\dagger} y,y\right\rangle\right| \\
&= \left|\left\langle |T|^4 x,x\right\rangle\right| \left|\left\langle T^*TT^{\dagger}{T^*}^{\dagger} y,y\right\rangle\right| \\
&= \left|\left\langle |T|^4 x,x\right\rangle\right| \left|\left\langle T^*\left(TT^{\dagger}\right)^* {T^*}^{\dagger} y,y\right\rangle\right| \\
&= \left|\left\langle |T|^4 x,x\right\rangle\right| \left|\left\langle T^*{T^*}^{\dagger}T^*{T^*}^{\dagger} y,y\right\rangle\right| \\
&= \left|\left\langle |T|^4 x,x\right\rangle\right| \left|\left\langle T^*{T^*}^{\dagger} y,y\right\rangle\right| \\
&= \left|\left\langle |T|^4 x,x\right\rangle\right| \left|\left\langle \left(T^{\dagger}T\right)^* y,y\right\rangle\right| \\
&= \left|\left\langle |T|^4 x,x\right\rangle\right| \left|\left\langle T^{\dagger}Ty,y\right\rangle\right| \quad \text{(since } T^{\dagger}T \text{ is self-adjoint).}
\end{aligned}$$

□

We also need the following known lemmas.

LEMMA 3.2. [18] *Let* $T \in \mathscr{B}(\mathscr{H})$ *be positive and let* $x \in \mathscr{H}$ *with* $\|x\| = 1$. *Then* $\langle Tx, x\rangle^r \leqslant \langle T^r x, x\rangle$ *for all* $r \geqslant 1$.

LEMMA 3.3. [11] *Let* $a, b, c \in \mathscr{H}$. *Then*

$$|\langle a,b\rangle|^2 + |\langle a,c\rangle|^2 \leqslant \|a\|^2 \sqrt{|\langle b,b\rangle|^2 + 2|\langle b,c\rangle| + |\langle c,c\rangle|^2}.$$

Using the above lemmas, we now obtain the Euclidean operator radius inequalities.

THEOREM 3.4. *Let* $T, S \in \mathscr{C}\mathscr{R}(\mathscr{H})$. *Then*

$$w_e(T,S) \leqslant \frac{\sqrt{2}}{2} \left\| |T|^4 + |S|^4 + TT^\dagger + SS^\dagger \right\|^{\frac{1}{2}}.$$

*Proof.* Take $x \in \mathscr{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
&|\langle Tx,x\rangle|^2 + |\langle Sx,x\rangle|^2 \\
&\leqslant \left\langle |T|^2 x,x\right\rangle \langle TT^\dagger x,x\rangle + \left\langle |S|^2 x,x\right\rangle \langle SS^\dagger x,x\rangle \quad \text{(by Lemma 2.12)} \\
&\leqslant \frac{1}{2}\left(\left\langle |T|^2 x,x\right\rangle^2 + \langle TT^\dagger x,x\rangle^2\right) + \frac{1}{2}\left(\left\langle |S|^2 x,x\right\rangle^2 + \langle SS^\dagger x,x\rangle^2\right) \\
&\text{(by the arithmetic-geometric mean inequality)} \\
&\leqslant \frac{1}{2}\left(\left\langle |T|^4 x,x\right\rangle + \left\langle \left(TT^\dagger\right)^2 x,x\right\rangle\right) + \frac{1}{2}\left(\left\langle |S|^4 x,x\right\rangle + \left\langle \left(SS^\dagger\right)^2 x,x\right\rangle\right) \\
&\text{(by Lemma 3.2)} \\
&= \frac{1}{2}\left\langle \left(|T|^4 + |S|^4 + \left(TT^\dagger\right)^2 + \left(SS^\dagger\right)^2\right) x,x\right\rangle \\
&= \frac{1}{2}\left\langle \left(|T|^4 + |S|^4 + TT^\dagger + SS^\dagger\right) x,x\right\rangle \\
&\leqslant \frac{1}{2}\left\| |T|^4 + |S|^4 + TT^\dagger + SS^\dagger \right\|.
\end{aligned}$$

By taking the supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$, we get

$$w_e^2(T,S) \leqslant \frac{1}{2}\left\| |T|^4 + |S|^4 + TT^\dagger + SS^\dagger \right\|,$$

as desired. □

Next, we provide the following Euclidean operator radius inequalities.

THEOREM 3.5. *Let* $T, S \in \mathscr{C}\mathscr{R}(\mathscr{H})$. *Then*

$$w_e(T,S) \leqslant \left\| |T|^2 + |S^*|^2 \right\|^{\frac{1}{2}} \left\| TT^\dagger + S^\dagger S \right\|^{\frac{1}{2}}$$

*and*

$$w_e(T,S) \leqslant \left\| |T^*|^2 + |S|^2 \right\|^{\frac{1}{2}} \left\| T^\dagger T + SS^\dagger \right\|^{\frac{1}{2}}.$$

*Proof.* Take $x \in \mathscr{H}$ with $\|x\| = 1$. We have

$$\begin{aligned}
&\left(|\langle Tx,x\rangle|^2 + |\langle Sx,x\rangle|^2\right)^{\frac{1}{2}} \\
&\leqslant |\langle Tx,x\rangle| + |\langle Sx,x\rangle| \\
&= \left|\langle TT^{\dagger}Tx,x\rangle\right| + \left|\langle SS^{\dagger}Sx,x\rangle\right| \\
&= \left|\left\langle Tx,\left(TT^{\dagger}\right)^{*}x\right\rangle\right| + \left|\langle SS^{\dagger}Sx,x\rangle\right| \\
&= \left|\langle Tx,TT^{\dagger}x\rangle\right| + \left|\langle S^{\dagger}Sx,S^{*}x\rangle\right| \quad (\text{since } TT^{\dagger} \text{ is self-adjoint}) \\
&\leqslant \|Tx\| \left\|TT^{\dagger}x\right\| + \|S^{*}x\| \left\|S^{\dagger}Sx\right\| \quad (\text{by the Cauchy-Schwarz inequality}) \\
&\leqslant \left(\|Tx\|^2 + \|S^{*}x\|^2\right)^{\frac{1}{2}} \left(\left\|TT^{\dagger}x\right\|^2 + \left\|S^{\dagger}Sx\right\|^2\right)^{\frac{1}{2}} \\
&= \left\langle \left(|T|^2 + |S^{*}|^2\right)x,x\right\rangle^{\frac{1}{2}} \left\langle \left(TT^{\dagger} + S^{\dagger}S\right)x,x\right\rangle^{\frac{1}{2}} \\
&\leqslant \left\||T|^2 + |S^{*}|^2\right\|^{\frac{1}{2}} \left\|TT^{\dagger} + S^{\dagger}S\right\|^{\frac{1}{2}}.
\end{aligned}$$

Taking the supremum over all $x$ in $\mathscr{H}$ with $\|x\| = 1$, we get

$$w_e(T,S) \leqslant \left\||T|^2 + |S^{*}|^2\right\|^{\frac{1}{2}} \left\|TT^{\dagger} + S^{\dagger}S\right\|^{\frac{1}{2}}.$$

Interchanging $S$ and $T$ in the above inequality, we also get

$$w_e(T,S) \leqslant \left\||T^{*}|^2 + |S|^2\right\|^{\frac{1}{2}} \left\|T^{\dagger}T + SS^{\dagger}\right\|^{\frac{1}{2}},$$

as desired. □

Setting $T = S$ in Theorem 3.5, we get the following corollary.

COROLLARY 3.6. *Let $T \in \mathscr{CR}(\mathscr{H})$. Then*

$$w(T) \leqslant \frac{1}{\sqrt{2}} \left\||T|^2 + |T^{*}|^2\right\|^{\frac{1}{2}} \left\|TT^{\dagger} + T^{\dagger}T\right\|^{\frac{1}{2}}.$$

Again, using similar techniques as used in Theorem 3.5, we obtain another upper bound for the Euclidean operator radius of a pair of bounded linear operators with closed ranges.

THEOREM 3.7. *Let $T,S \in \mathscr{CR}(\mathscr{H})$. Then*

$$w_e(T,S) \leqslant \left\||T|^4 + |S^{*}|^4\right\|^{\frac{1}{4}} \left\|TT^{\dagger} + S^{\dagger}S\right\|^{\frac{1}{4}}$$

*and*

$$w_e(T,S) \leqslant \left\||T^{*}|^4 + |S|^4\right\|^{\frac{1}{4}} \left\|T^{\dagger}T + SS^{\dagger}\right\|^{\frac{1}{4}}.$$

As a simple consequence of Theorem 3.7, we get the following result.

COROLLARY 3.8. *Let $T \in \mathscr{CR}(\mathscr{H})$. Then*

$$w^4(T) \leqslant \frac{1}{4}\left\| |T|^4 + |T^*|^4 \right\| \left\| TT^\dagger + T^\dagger T \right\|.$$

Another bound for the Euclidean operator radius of a pair of operators is as follows.

THEOREM 3.9. *Let $T,S \in \mathscr{CR}(\mathscr{H})$. Then*

$$w_e^4(T,S) \leqslant \frac{1}{2}\left\| |T|^8 + |S|^8 + T^\dagger T + S^\dagger S \right\| + \left\| |T|^2 + |S|^2 \right\|.$$

*Proof.* Let $x \in \mathscr{H}$ with $\|x\| = 1$. Setting $a = x, b = Tx$ and $c = Sx$ in Lemma 3.3, we get

$$\begin{aligned}
&\left(|\langle Tx,x\rangle|^2 + |\langle Sx,x\rangle|^2\right)^2 \\
&\leqslant |\langle Tx,Tx\rangle|^2 + 2\,|\langle Tx,Sx\rangle| + |\langle Sx,Sx\rangle|^2 \\
&= |\langle T^*Tx,x\rangle|^2 + |\langle S^*Sx,x\rangle|^2 + 2\,|\langle Tx,Sx\rangle| \\
&\leqslant \left\langle |T|^4 x,x\right\rangle \left\langle T^\dagger Tx,x\right\rangle + \left\langle |S|^4 x,x\right\rangle \left\langle S^\dagger Sx,x\right\rangle + 2\,\|Tx\|\,\|Sx\| \quad \text{(by Lemma 3.1)} \\
&\leqslant \frac{1}{2}\left(\left\langle |T|^4 x,x\right\rangle^2 + \left\langle T^\dagger Tx,x\right\rangle^2\right) + \frac{1}{2}\left(\left\langle |S|^4 x,x\right\rangle^2 + \left\langle S^\dagger Sx,x\right\rangle^2\right) \\
&\quad + \|Tx\|^2 + \|Sx\|^2 \quad \text{(by the arithmetic-geometric mean inequality)} \\
&\leqslant \frac{1}{2}\left(\left\langle |T|^8 x,x\right\rangle + \left\langle \left(T^\dagger T\right)^2 x,x\right\rangle\right) + \frac{1}{2}\left(\left\langle |S|^8 x,x\right\rangle + \left\langle \left(S^\dagger S\right)^2 x,x\right\rangle\right) \\
&\quad + \left\langle \left(|T|^4 + |S|^4\right) x,x\right\rangle \quad \text{(by Lemma 3.2)} \\
&= \frac{1}{2}\left(\left\langle \left(|T|^8 + |S|^8 + T^\dagger T + S^\dagger S\right) x,x\right\rangle\right) + \left\langle \left(|T|^2 + |S|^2\right) x,x\right\rangle \\
&\leqslant \frac{1}{2}\left\| |T|^8 + |S|^8 + T^\dagger T + S^\dagger S \right\| + \left\| |T|^2 + |S|^2 \right\|.
\end{aligned}$$

By taking the supremum over all $x \in \mathscr{H}$ with $\|x\| = 1$, we get

$$w_e^4(T,S) \leqslant \frac{1}{2}\left\| |T|^8 + |S|^8 + T^\dagger T + S^\dagger S \right\| + \left\| |T|^2 + |S|^2 \right\|. \quad \square$$

The next inequality is as follows.

THEOREM 3.10. *Let $T,S \in \mathscr{CR}(\mathscr{H})$. Then*

$$w_e^2(T,S) \leqslant \frac{\sqrt{2}}{2} w\left(|T|^4 + |S|^4 + i\left(TT^\dagger + SS^\dagger\right)\right).$$

*Proof.* Let $x \in \mathscr{H}$ with $\|x\| = 1$. Then

$$
\begin{aligned}
& |\langle Tx,x\rangle|^2 + |\langle Sx,x\rangle|^2 \\
\leqslant\ & \left\langle |T|^2 x,x \right\rangle \langle TT^{\dagger}x,x\rangle + \left\langle |S|^2 x,x \right\rangle \langle SS^{\dagger}x,x\rangle \quad \text{(by Lemma 2.12)} \\
\leqslant\ & \frac{1}{2}\left(\left\langle |T|^2 x,x \right\rangle^2 + \langle TT^{\dagger}x,x\rangle^2\right) + \frac{1}{2}\left(\left\langle |S|^2 x,x \right\rangle^2 + \langle SS^{\dagger}x,x\rangle^2\right) \\
& \text{(by the arithmetic-geometric mean inequality)} \\
\leqslant\ & \frac{1}{2}\left(\left\langle |T|^4 x,x \right\rangle + \left\langle \left(TT^{\dagger}\right)^2 x,x \right\rangle\right) + \frac{1}{2}\left(\left\langle |S|^4 x,x \right\rangle + \left\langle \left(SS^{\dagger}\right)^2 x,x \right\rangle\right) \text{(by Lemma 3.2)} \\
=\ & \frac{1}{2}\left(\left\langle \left(|T|^4 + |S|^4\right) x,x \right\rangle + \left\langle \left(TT^{\dagger} + SS^{\dagger}\right) x,x \right\rangle\right) \\
\leqslant\ & \frac{\sqrt{2}}{2}\left\langle \left(|T|^4 + |S|^4 + i\left(TT^{\dagger} + SS^{\dagger}\right)\right) x,x \right\rangle \\
& \text{(as } |a+b| \leqslant \sqrt{2}\,|a+ib| \text{ for } a,b \in \mathbb{R}) \\
\leqslant\ & \frac{\sqrt{2}}{2} w\left(|T|^4 + |S|^4 + i\left(TT^{\dagger} + SS^{\dagger}\right)\right).
\end{aligned}
$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$, we get the desired inequality. □

Again, using similar arguments as in Theorem 3.10, we get the following results.

THEOREM 3.11. *Let $T,S \in \mathscr{CR}(\mathscr{H})$. Then*

$$
w_e(T,S) \leqslant \frac{\sqrt{2}}{2} w\left(|T|^2 + |S|^2 + i\left(TT^{\dagger} + SS^{\dagger}\right)\right).
$$

THEOREM 3.12. *Let $T,S \in \mathscr{CR}(\mathscr{H})$. Then*

$$
w_e^2(T,S) \leqslant \frac{\sqrt{2}}{2} w\left(|T|^4 + TT^{\dagger} + i\left(|S|^4 + SS^{\dagger}\right)\right)
$$

*and*

$$
w_e^2(T,S) \leqslant \frac{\sqrt{2}}{2} w\left(|T|^4 + SS^{\dagger} + i\left(|T|^4 + TT^{\dagger}\right)\right).
$$

Setting $T = S$ in Theorem 3.12, we get the following numerical radius bound, which was also recently observed in [5, Theorem 2.5].

COROLLARY 3.13. *Let $T \in \mathscr{CR}(\mathscr{H})$. Then*

$$
w^2(T) \leqslant \frac{1}{2}\left\| |T|^4 + TT^{\dagger} \right\|.
$$

Finally, we prove the following bound.

THEOREM 3.14. *Let $T,S \in \mathscr{CR}(\mathscr{H})$ and let $\alpha,\beta \geqslant 0$. Then*

$$w_e^2(T,S) \leqslant \frac{1}{2(\alpha+1)}\left\||T|^4+TT^{\dagger}\right\|+\frac{\alpha}{2(\alpha+1)}w(T)\left\||T|^2+TT^{\dagger}\right\| + \frac{1}{2(\beta+1)}\left\||S|^4+SS^{\dagger}\right\|+\frac{\beta}{2(\beta+1)}w(S)\left\||S|^2+SS^{\dagger}\right\|.$$

*Proof.* Take $x \in \mathscr{H}$ with $\|x\|=1$. We have

$$\begin{aligned}
&|\langle Tx,x\rangle|^2+|\langle Sx,x\rangle|^2\\
&= \frac{1}{\alpha+1}|\langle Tx,x\rangle|^2+\frac{\alpha}{\alpha+1}|\langle Tx,x\rangle|^2+\frac{1}{\beta+1}|\langle Sx,x\rangle|^2+\frac{\beta}{\beta+1}|\langle Sx,x\rangle|^2\\
&\leqslant \frac{1}{\alpha+1}\left\langle |T|^2x,x\right\rangle\langle TT^{\dagger}x,x\rangle+\frac{\alpha}{\alpha+1}|\langle Tx,x\rangle|\sqrt{\left\langle |T|^2x,x\right\rangle\langle TT^{\dagger}x,x\rangle}\\
&\quad+\frac{1}{\beta+1}\left\langle |S|^2x,x\right\rangle\langle SS^{\dagger}x,x\rangle+\frac{\beta}{\beta+1}|\langle Sx,x\rangle|\sqrt{\left\langle |S|^2x,x\right\rangle\langle SS^{\dagger}x,x\rangle}\\
&\leqslant \frac{1}{2(\alpha+1)}\left(\left\langle |T|^2x,x\right\rangle^2+\langle TT^{\dagger}x,x\rangle^2\right)+\frac{\alpha}{2(\alpha+1)}|\langle Tx,x\rangle|\left(\left\langle |T|^2x,x\right\rangle+\langle TT^{\dagger}x,x\rangle\right)\\
&\quad+\frac{1}{2(\beta+1)}\left(\left\langle |S|^2x,x\right\rangle^2+\langle SS^{\dagger}x,x\rangle^2\right)+\frac{\beta}{2(\beta+1)}|\langle Sx,x\rangle|\left(\left\langle |S|^2x,x\right\rangle+\langle SS^{\dagger}x,x\rangle\right)\\
&\leqslant \frac{1}{2(\alpha+1)}\left(\left\langle |T|^4x,x\right\rangle+\left\langle \left(TT^{\dagger}\right)^2x,x\right\rangle\right)+\frac{\alpha}{2(\alpha+1)}|\langle Tx,x\rangle|\left\langle\left(|T|^2+TT^{\dagger}\right)x,x\right\rangle\\
&\quad+\frac{1}{2(\beta+1)}\left(\left\langle |S|^4x,x\right\rangle+\left\langle \left(SS^{\dagger}\right)^2x,x\right\rangle\right)+\frac{\beta}{2(\beta+1)}|\langle Sx,x\rangle|\left\langle\left(|S|^2+SS^{\dagger}\right)x,x\right\rangle\\
&\leqslant \frac{1}{2(\alpha+1)}\left\||T|^4+TT^{\dagger}\right\|+\frac{\alpha}{2(\alpha+1)}w(T)\left\||T|^2+TT^{\dagger}\right\|\\
&\quad+\frac{1}{2(\beta+1)}\left\||S|^4+SS^{\dagger}\right\|+\frac{\beta}{2(\beta+1)}w(S)\left\||S|^2+SS^{\dagger}\right\|.
\end{aligned}$$

Taking the supremum over $x \in \mathscr{H}$ with $\|x\|=1$, we get the desired inequality. □

As a consequence of Theorem 3.14, we deduce

COROLLARY 3.15. *Let $T \in \mathscr{CR}(\mathscr{H})$ and let $\alpha \geqslant 0$. Then*

$$w^2(T) \leqslant \frac{1}{2(\alpha+1)}\left\||T|^4+TT^{\dagger}\right\|+\frac{\alpha}{2(\alpha+1)}w(T)\left\||T|^2+TT^{\dagger}\right\|.$$

*In particular, for $\alpha=1$,*

$$w^2(T) \leqslant \frac{1}{4}\left\||T|^4+TT^{\dagger}\right\|+\frac{1}{4}w(T)\left\||T|^2+TT^{\dagger}\right\|. \tag{3.1}$$

Note that (3.1) gives a stronger numerical radius bound than Corollary 3.13 (via (2.9)).

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