Operators
and
Matrices
Volume 20, Number 2 (2026), 211–227
doi:10.7153/oam-2026-20-12

NUMERICAL RADIUS INEQUALITIES FOR OPERATOR MATRICES THROUGH MIXED SCHWARZ TYPE INEQUALITIES

YONGHUI REN, PINTU BHUNIA AND MOHAMED AMINE IGHACHANE*

(*Communicated by F. Kittaneh*)

Abstract. We establish new numerical radius inequalities for $n \times n$ operator matrices based on mixed Schwarz type inequalities. Using polar decomposition $\mathscr{A}_{ij} = \mathscr{U}_{ij}|\mathscr{A}_{ij}|$ and continuous nonnegative functions $f, g : [0,\infty) \to [0,\infty)$, we develop a Cauchy Schwarz type numerical radius inequality for composite operator blocks of the form

$$\widetilde{\mathscr{T}_{ij}} = g\left(|\mathscr{B}_{ij}|\right) f\left(|\mathscr{A}_{ij}^*|\right) \mathscr{U}_{ij}, \qquad 1 \leqslant i, j \leqslant n.$$

This unified approach extends several classical inequalities for operator matrices and yields the estimate

$$w(\widetilde{\mathscr{T}}) \leqslant w\left(\widehat{\mathscr{T}}^{(2)}\right),$$

where $\widetilde{\mathscr{T}} = [\widetilde{\mathscr{T}_{ij}}]$ is an $n \times n$ operator matrix and $\widehat{\mathscr{T}}^{(2)}$ is an upper triangular scalar comparison matrix, whose entries depend only on the functions f and g.

As an application, choosing suitable functions f and g together with a specific choice of the block operator $\mathscr{B}_{ij}$ leads to a new inequality involving the Moore-Penrose inverse. The resulting estimates recover several known inequalities and also provide new sharper bounds for the numerical radius of operator matrices.

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ equipped with inner product $\langle \cdot, \cdot \rangle$. An operator $\mathscr{A} \in \mathbb{B}(\mathscr{H})$ is called positive, denoted $\mathscr{A} \geqslant 0$, if it satisfies $\langle \mathscr{A}x, x \rangle \geqslant 0$ for all $x \in \mathscr{H}$. For an operator $\mathscr{A} \in \mathbb{B}(\mathscr{H})$, we denote $|\mathscr{A}| := (\mathscr{A}^*\mathscr{A})^{\frac{1}{2}}$. A fundamental concept in the theory of bounded linear operators on a Hilbert space is the numerical radius. For an operator $\mathscr{A} \in \mathbb{B}(\mathscr{H})$, the numerical radius is defined by

$$w(\mathscr{A}) = \sup_{\|x\|=1} |\langle \mathscr{A}x, x \rangle|,$$

Mathematics subject classification (2020): Primary 47A12; Secondary 47A30, 47B15.

Keywords and phrases: Numerical radius, block operator matrices, mixed Schwarz inequality, polar decomposition, Moore-Penrose inverse, operator inequalities.

[*] Corresponding author.

© ELEMENT, Zagreb

which represents the supremum of the absolute values of the inner products $\langle \mathscr{A}x, x\rangle$ over all unit vectors $x \in \mathscr{H}$. It is well known that $w(\cdot)$ defines a norm on $\mathbb{B}(\mathscr{H})$, that is equivalent to the usual operator norm $\|\cdot\|$, satisfying the inequalities

$$\frac{1}{2}\|\mathscr{A}\| \leqslant w(\mathscr{A}) \leqslant \|\mathscr{A}\|, \qquad \mathscr{A} \in \mathbb{B}(\mathscr{H}). \tag{1}$$

The numerical radius plays a crucial role in spectral theory, perturbation analysis, and in the study of various operator inequalities. In particular, several classical and modern results, including those of Kato, Kittaneh, and their extensions, have been formulated in terms of the numerical radius, as it often provides sharper bounds than the operator norm.

The classical mixed Schwarz inequality, which introduced the operator powers $|\mathscr{A}|^{\alpha}$ and $|\mathscr{A}^*|^{1-\alpha}$ for $0 \leqslant \alpha \leqslant 1$, states that

$$|\langle \mathscr{A}x, y\rangle|^2 \leqslant \langle |\mathscr{A}|^{2\alpha}x, x\rangle \, \langle |\mathscr{A}^*|^{2(1-\alpha)}y, y\rangle. \tag{2}$$

An important extension of this fundamental inequality has inspired a wide range of refined estimates and structural generalizations in operator theory. In particular, several recent works have developed mixed Schwarz-type inequalities adapted to block operator matrices, operator means, and generalized numerical radius settings. These advancements not only deepen our understanding of operator factorizations but also serve as essential tools for deriving sharper upper bounds for quantities such as the numerical radius, spectral radius, and various operator norms.

Later, Kittaneh [13] refined and generalized this inequality by introducing two continuous nonnegative functions $f, g : [0,\infty) \to [0,\infty)$ satisfying $f(t)g(t) = t$, obtaining

$$|\langle \mathscr{A}x, y\rangle|^2 \leqslant \langle f^2(|\mathscr{A}|)x, x\rangle \, \langle g^2(|\mathscr{A}^*|)y, y\rangle, \qquad \mathscr{A} \in \mathbb{B}(\mathscr{H}),\ x, y \in \mathscr{H}. \tag{3}$$

Inequality (3) has become a cornerstone in the study of operator inequalities and has found numerous applications in numerical radius and norm estimates, spectral bounds, and perturbation analysis.

A fundamental concept in the theory of operators is the Moore-Penrose inverse [8], denoted by $\mathscr{A}^{\dagger}$, which extends the notion of an inverse to bounded operators that may fail to be invertible. For every operator $\mathscr{A}$ with closed range, the Moore-Penrose inverse exists uniquely and is characterized by the four Penrose equations: $\mathscr{A}\mathscr{A}^{\dagger}\mathscr{A} = \mathscr{A}$, $\mathscr{A}^{\dagger}\mathscr{A}\mathscr{A}^{\dagger} = \mathscr{A}^{\dagger}$, $(\mathscr{A}\mathscr{A}^{\dagger})^* = \mathscr{A}\mathscr{A}^{\dagger}$, and $(\mathscr{A}^{\dagger}\mathscr{A})^* = \mathscr{A}^{\dagger}\mathscr{A}$. These identities further imply that $\mathscr{A}\mathscr{A}^{\dagger}$ and $\mathscr{A}^{\dagger}\mathscr{A}$ are orthogonal projections onto $\overline{\mathscr{R}(\mathscr{A})}$ and $\overline{\mathscr{R}(\mathscr{A}^*)}$, respectively. We refer the interested readers to [5, 7] for additional insights into the Moore-Penrose inverses of operators in $\mathbb{B}(\mathscr{H})$.

The interplay between $\mathscr{A}^{\dagger}$ and mixed Schwarz-type inequalities has recently become an effective framework for establishing refined numerical radius estimates. In particular, Sababheh et al. [18] re-established a strengthened form of the mixed Schwarz inequality: for any $\mathscr{A} \in \mathbb{B}(\mathscr{H})$ with closed range and all $x, y \in \mathscr{H}$,

$$|\langle \mathscr{A}x, y\rangle| \leqslant \sqrt{\langle |\mathscr{A}|^2 x, x\rangle \, \langle \mathscr{A}\mathscr{A}^{\dagger}y, y\rangle}. \tag{4}$$

Such inequalities, when combined with projection structures encoded in $\mathscr{A}^\dagger$, lead to several new operator bounds and sharper numerical radius estimates.

Very recently, Ren *et al.* [17] established a new mixed Schwarz-type inequality which unifies and extends a large number of existing inequalities of this kind, including those of Kato, Kittaneh, Furuta, and other classical results. Their approach is based on the positivity of certain block operator matrices and allows a systematic derivation of various mixed inequalities as particular cases obtained by appropriate choices of the functions f and g. The following theorem, which serves as the foundation of the present work, provides this unified formulation of the mixed Schwarz-type inequality.

In particular, if $\mathscr{A},\mathscr{B}\in\mathbb{B}(\mathscr{H})$ have polar decompositions $\mathscr{A}=\mathscr{U}|\mathscr{A}|$ and $\mathscr{B}=\mathscr{V}|\mathscr{B}|$, then for all $x,y\in\mathscr{H}$ the following inequalities hold:

$$\left|\langle g(|\mathscr{B}|)\,f(|\mathscr{A}^*|)\,\mathscr{U}\,x,\ y\rangle\right|^2 \leqslant \langle f^2(|\mathscr{A}|)x,x\rangle\,\langle g^2(|\mathscr{B}|)y,y\rangle, \tag{5}$$

$$\left|\langle g(|\mathscr{A}|)\,f(|\mathscr{B}^*|)\,\mathscr{V}\,x,\ y\rangle\right|^2 \leqslant \langle f^2(|\mathscr{B}|)x,x\rangle\,\langle g^2(|\mathscr{A}|)y,y\rangle. \tag{6}$$

where $f,g:[0,\infty)\to[0,\infty)$ are continuous functions.

Kittaneh [4] sharpened the classical upper bound for the numerical radius in (1) by proving that, for every $\mathscr{A}\in\mathbb{B}(\mathscr{H})$,

$$w(\mathscr{A})\leqslant\frac{1}{2}\left\|\,|\mathscr{A}|+|\mathscr{A}^*|\,\right\|\leqslant\frac{1}{2}\left(\|\mathscr{A}\|+\|\mathscr{A}^2\|^{1/2}\right).$$

In a subsequent paper [14], he established a two sided estimate for the square of the numerical radius:

$$\frac{1}{4}\left\|\mathscr{A}\mathscr{A}^*+\mathscr{A}^*\mathscr{A}\right\|\ \leqslant\ w^2(\mathscr{A})\ \leqslant\ \frac{1}{2}\left\|\mathscr{A}\mathscr{A}^*+\mathscr{A}^*\mathscr{A}\right\|. \tag{7}$$

As a direct consequence of inequality (4), one obtains the following numerical radius estimate (see [18], Theorem 2.2):

$$w(\mathscr{A})\leqslant\frac{1}{2}\left\|\,|\mathscr{A}|^2+\mathscr{A}\mathscr{A}^\dagger\,\right\|. \tag{8}$$

This inequality illustrates that $w(\mathscr{A})$ is governed simultaneously by the modulus $|\mathscr{A}|$ and the range projection encoded in $\mathscr{A}\mathscr{A}^\dagger$.

Building upon (8), Bhunia *et al.* [4] derived several significant inequalities concerning the operator norm and the numerical radius of the sum of two operators via their Moore-Penrose inverses.

THEOREM 1. ([4]) *Let $\mathscr{A},\mathscr{B}\in\mathbb{B}(\mathscr{H})$ be two operators with closed range. Then*

$$w(\mathscr{A}+\mathscr{B})\leqslant\frac{1}{2}\left\|\,|\mathscr{A}^*|^2+|\mathscr{B}|^2+\mathscr{A}^\dagger\mathscr{A}+\mathscr{B}\mathscr{B}^\dagger\,\right\|,$$

$$w(\mathscr{A}+\mathscr{B})\leqslant\frac{1}{2}\left\|\,|\mathscr{A}|^2+|\mathscr{B}^*|^2+\mathscr{B}^\dagger\mathscr{B}+\mathscr{A}\mathscr{A}^\dagger\,\right\|.$$

These inequalities play a crucial role in the study of numerical radius bounds for operator matrices.

Further developments of inequalities involving the numerical radius through the use of the Moore-Penrose inverse are discussed in the following references [4, 6, 9, 12, 16, 17].

The numerical radius inequalities for $n \times n$ operator matrices have attracted considerable attention in recent years, due to their fundamental role in operator theory, matrix analysis, and applications to perturbation estimates. Given an operator matrix $[\mathscr{A}_{ij}]_{i,j=1}^{n}$ with entries in $\mathbb{B}(\mathscr{H})$, researchers have developed various sharp upper bounds for $w([\mathscr{A}_{ij}])$, aiming to refine the classical estimates and to obtain more effective tools for analyzing the structure of block operators. In this direction, the foundational contributions of Hou and Du, and later Abu-Omar and Kittaneh, provide the starting point for many recent developments. In particular, they established the following two fundamental numerical radius inequalities:

For instance, Hou and Du [11] proved the bound

$$w\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} & \cdots & \mathscr{A}_{1n} \\ \mathscr{A}_{21} & \mathscr{A}_{22} & \cdots & \mathscr{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{A}_{n1} & \mathscr{A}_{n2} & \cdots & \mathscr{A}_{nn} \end{bmatrix}\right) \leqslant w\left(\begin{bmatrix} \|\mathscr{A}_{11}\| & \|\mathscr{A}_{12}\| & \cdots & \|\mathscr{A}_{1n}\| \\ \|\mathscr{A}_{21}\| & \|\mathscr{A}_{22}\| & \cdots & \|\mathscr{A}_{2n}\| \\ \vdots & \vdots & \ddots & \vdots \\ \|\mathscr{A}_{n1}\| & \|\mathscr{A}_{n2}\| & \cdots & \|\mathscr{A}_{nn}\| \end{bmatrix}\right), \tag{9}$$

which was considerably refined by Abu-Omar and Kittaneh [1], who showed

$$w\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} & \cdots & \mathscr{A}_{1n} \\ \mathscr{A}_{21} & \mathscr{A}_{22} & \cdots & \mathscr{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{A}_{n1} & \mathscr{A}_{n2} & \cdots & \mathscr{A}_{nn} \end{bmatrix}\right) \leqslant w\left(\begin{bmatrix} w(\mathscr{A}_{11}) & \|\mathscr{A}_{12}\| & \cdots & \|\mathscr{A}_{1n}\| \\ \|\mathscr{A}_{21}\| & w(\mathscr{A}_{22}) & \cdots & \|\mathscr{A}_{2n}\| \\ \vdots & \vdots & \ddots & \vdots \\ \|\mathscr{A}_{n1}\| & \|\mathscr{A}_{n2}\| & \cdots & w(\mathscr{A}_{nn}) \end{bmatrix}\right). \tag{10}$$

These inequalities constitute the basis for several recent improvements in the theory of numerical radius inequalities for operator matrices.

Very recently, Bhunia [3] established a interesting and powerful numerical radius inequality for $n \times n$ operator matrices. More precisely, for a block operator matrix $[\mathscr{A}_{ij}]$ with entries in $\mathbb{B}(\mathscr{H})$, he obtained a refined upper bound for $w([\mathscr{A}_{ij}])$ which significantly improves several classical numerical radius inequalities. The theorem is stated as follows:

THEOREM 2. ([3]) *Let $\mathscr{A}_{ij} \in \mathbb{B}(\mathscr{H})$ for all $i, j = 1, \ldots, n$. Let f_{ij} and g_{ij} be non-negative continuous functions on $[0, \infty)$ such that*

$$f_{ij}(t)\, g_{ij}(t) = t, \qquad \forall t \geqslant 0.$$

Then

$$w\left(\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} & \cdots & \mathscr{A}_{1n} \\ \mathscr{A}_{21} & \mathscr{A}_{22} & \cdots & \mathscr{A}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{A}_{n1} & \mathscr{A}_{n2} & \cdots & \mathscr{A}_{nn} \end{bmatrix}\right) \leqslant w\left(\begin{bmatrix} w(\mathscr{A}_{11}) & a_{12} & \cdots & a_{1n} \\ 0 & w(\mathscr{A}_{22}) & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w(\mathscr{A}_{nn}) \end{bmatrix}\right),$$

where for every $i<j$,

$$a_{ij}=\left\|f_{ij}^2(|\mathscr{A}_{ij}|)+g_{ji}^2(|\mathscr{A}_{ji}^*|)\right\|^{1/2}\left\|g_{ij}^2(|\mathscr{A}_{ij}^*|)+f_{ji}^2(|\mathscr{A}_{ji}|)\right\|^{1/2}.$$

For some numerical radius inequalities for $n\times n$ matrices and applications to the estimation of zeros of polynomials, the reader is encouraged to consult the following paper [15].

Building on this line of research, the present paper develops a unified framework for numerical radius inequalities of block operator matrices by combining Kittaneh's functional approach with the recently established mixed Schwarz-type inequalities (5) and (6). Given an $n\times n$ operator matrix $\mathscr{A}=[\mathscr{A}_{ij}]_{i,j=1}^n$ with polar decompositions $\mathscr{A}_{ij}=\mathscr{U}_{ij}|\mathscr{A}_{ij}|$, we introduce the composite blocks

$$\widetilde{\mathscr{A}_{ij}}:=g_{ij}\big(|\mathscr{B}_{ij}|\big)\,f_{ij}\big(|\mathscr{A}_{ij}^*|\big)\,\mathscr{U}_{ij},\qquad 1\leqslant i,j\leqslant n,$$

which simultaneously extend Kittaneh's inequality to the multi-operator setting and capture the interactions between different entries of the block matrix. We show that the associated block operator matrix $\widetilde{\mathscr{A}}=[\widetilde{\mathscr{A}_{ij}}]$ admits a numerical radius bound of the form

$$w(\widetilde{\mathscr{A}})\leqslant w\big(\widehat{\mathscr{A}^{(2)}}\big),$$

where $\widehat{\mathscr{A}^{(2)}}$ is an explicitly constructed upper triangular comparison matrix depending on the functions f_{ij},g_{ij}.

A further contribution of this work is the incorporation of the Moore-Penrose inverse into the mixed Schwarz framework. By choosing $f_{ij}(t)=t$ and $g_{ij}(t)=\sqrt{t}$ and taking $\mathscr{B}_{ij}=\mathscr{A}_{ij}\mathscr{A}_{ij}^\dagger$, we obtain

$$\widetilde{\mathscr{A}_{ij}}:=g_{ij}\big(|\mathscr{A}_{ij}\mathscr{A}_{ij}^\dagger|\big)\,f_{ij}\big(|\mathscr{A}_{ij}^*|\big)\,\mathscr{U}_{ij},$$

which introduces the canonical projection $\mathscr{A}_{ij}\mathscr{A}_{ij}^\dagger$ onto the range of $\mathscr{A}_{ij}$. This choice enables us to obtain new numerical radius inequalities for block operator matrices through the use of the Moore-Penrose inverse. The results established here unify and extend several classical inequalities, including those of Kato, Kittaneh, Hou-Du, Abu-Omar-Kittaneh, and Bhunia, and further provide sharper bounds for operator matrices involving the Moore-Penrose inverse.

The paper is organized as follows. In Section 2, we develop a unified framework for generalized mixed Schwarz-type inequalities for block operator matrices, combining Kittaneh's functional approach with the recent mixed inequalities of Ren-Ighachane-Bhunia. This section forms the theoretical core of the paper, presenting numerical radius bounds that simultaneously extend, refine, and connect several classical results, including those of Hou-Du, Abu-Omar-Kittaneh, and Bhunia. Section 3 focuses on applications of the general framework, particularly to operator matrices involving the Moore-Penrose inverse. New upper bounds for the numerical radius are obtained in terms of range projections and canonical decompositions, showing how the developed machinery naturally incorporates non-invertible operators and block matrices with projection components. Several illustrative cases including 2×2 off-diagonal forms and symmetric configurations are also discussed.

2. Generalized numerical radius inequalities for operator matrices

In this section, we develop a unified framework for deriving numerical radius inequalities for $n \times n$ operator matrices by combining the functional approach of Kittaneh with the recent mixed Schwarz-type inequalities introduced in [17]. These inequalities allow us to control the off-diagonal interactions between the entries of an operator matrix through appropriate choices of continuous functions f_{ij} and g_{ij}, together with the polar decompositions of the component operators.

Our first main result, which generalizes and strengthens several known numerical radius estimates for operator matrices including those of Hou-Du, Abu Omar-Kittaneh, and Bhunia establishes a bound for block matrices whose entries are of the form

$$g_{ij}\big(|\mathscr{B}_{ij}|\big)\, f_{ij}\big(|\mathscr{A}_{ij}^*|\big)\, \mathscr{U}_{ij}.$$

This theorem serves as the cornerstone of our approach and will later be used to derive a variety of consequences and applications, including inequalities involving the Moore-Penrose inverse.

Our first main result is the following theorem.

THEOREM 3. *Let $\mathscr{A}_{ij}, \mathscr{B}_{ij} \in \mathbb{B}(\mathscr{H})$ for all $i,j = 1,\ldots,n$, and the polar decomposition of $\mathscr{A}_{ij}$ be $\mathscr{A}_{ij} = \mathscr{U}_{ij}|\mathscr{A}_{ij}|$. Let $f_{ij}, g_{ij} : [0,\infty) \to [0,\infty)$ be continuous functions and let the composite operator blocks $\widetilde{\mathscr{T}_{ij}} := g_{ij}\big(|\mathscr{B}_{ij}|\big)\, f_{ij}\big(|\mathscr{A}_{ij}^*|\big)\, \mathscr{U}_{ij}$, $1 \leqslant i,j \leqslant n$. Then the operator matrix $\widetilde{\mathscr{T}} = [\widetilde{\mathscr{T}_{ij}}]_{i,j=1}^n$ satisfies $w(\widetilde{\mathscr{T}}) \leqslant w\big(\widehat{\mathscr{T}}^{(2)}\big)$, where*

$$\widehat{\mathscr{T}}^{(2)} = \big[\widehat{t}_{ij}^{(2)}\big]_{i,j=1}^n,$$

$$\widehat{t}_{ij}^{(2)} = \begin{cases} w\left(\widetilde{\mathscr{T}_{ii}}\right) & i = j, \\ \left\| f_{ij}^2(|\mathscr{A}_{ij}|) + g_{ji}^2(|\mathscr{B}_{ji}|) \right\|^{1/2} \left\| f_{ji}^2(|\mathscr{A}_{ji}|) + g_{ij}^2(|\mathscr{B}_{ij}|) \right\|^{1/2} & i < j, \\ 0 & i > j. \end{cases}$$

Proof. Let $x = (x_1,\ldots,x_n)^T \in \bigoplus_{k=1}^n \mathscr{H}$ be a unit vector, i.e. $\sum_{k=1}^n \|x_k\|^2 = 1$. The numerical radius of $\widetilde{\mathscr{T}}$ is given by

$$w(\widetilde{\mathscr{T}}) = \sup_{\|x\|=1} \big|\langle \widetilde{\mathscr{T}}x, x\rangle\big|, \qquad \langle \widetilde{\mathscr{T}}x, x\rangle = \sum_{i,j=1}^n \langle \widetilde{\mathscr{T}_{ij}}x_j, x_i\rangle.$$

Applying the triangle inequality gives

$$\big|\langle \widetilde{\mathscr{T}}x, x\rangle\big| \leqslant \sum_{i=1}^n \big|\langle \widetilde{\mathscr{T}_{ii}}x_i, x_i\rangle\big| + \sum_{i<j} \Big(\big|\langle \widetilde{\mathscr{T}_{ij}}x_j, x_i\rangle\big| + \big|\langle \widetilde{\mathscr{T}_{ji}}x_i, x_j\rangle\big|\Big).$$

For the diagonal blocks,

$$\big|\langle \widetilde{\mathscr{T}_{ii}}x_i, x_i\rangle\big| \leqslant w(\widetilde{\mathscr{T}_{ii}})\,\|x_i\|^2.$$

For off-diagonal terms ($i<j$), by inequality (5), for all $x_i, x_j \in \mathscr{H}$,

$$\big|\langle \widetilde{\mathscr{T}_{ij}} x_j, x_i\rangle\big| \leqslant \|f_{ij}(|\mathscr{A}_{ij}|)x_j\|\,\|g_{ij}(|\mathscr{B}_{ij}|)x_i\|,$$

$$\big|\langle \widetilde{\mathscr{T}_{ji}} x_i, x_j\rangle\big| \leqslant \|f_{ji}(|\mathscr{A}_{ji}|)x_i\|\,\|g_{ji}(|\mathscr{B}_{ji}|)x_j\|.$$

Set

$$u := \|f_{ij}(|\mathscr{A}_{ij}|)x_j\|,\ \ v := \|g_{ij}(|\mathscr{B}_{ij}|)x_i\|,\quad s := \|f_{ji}(|\mathscr{A}_{ji}|)x_i\|,\ \ t := \|g_{ji}(|\mathscr{B}_{ji}|)x_j\|.$$

Using the Cauchy-Schwarz inequality,

$$uv+st \leqslant (u^2+t^2)^{1/2}(v^2+s^2)^{1/2}.$$

Thus,

$$\big|\langle \widetilde{\mathscr{T}_{ij}} x_j, x_i\rangle\big| + \big|\langle \widetilde{\mathscr{T}_{ji}} x_i, x_j\rangle\big| \leqslant \Big(\|f_{ij}(|\mathscr{A}_{ij}|)x_j\|^2 + \|g_{ji}(|\mathscr{B}_{ji}|)x_j\|^2\Big)^{1/2} \times \Big(\|g_{ij}(|\mathscr{B}_{ij}|)x_i\|^2 + \|f_{ji}(|\mathscr{A}_{ji}|)x_i\|^2\Big)^{1/2}.$$

Using $\|Tz\|^2 = \langle |T|^2 z, z\rangle$,

$$\Big(\|f_{ij}(|\mathscr{A}_{ij}|)x_j\|^2 + \|g_{ji}(|\mathscr{B}_{ji}|)x_j\|^2\Big)^{1/2} \leqslant \|\,f_{ij}^2(|\mathscr{A}_{ij}|) + g_{ji}^2(|\mathscr{B}_{ji}|)\|^{1/2}\|x_j\|,$$

and likewise,

$$\Big(\|g_{ij}(|\mathscr{B}_{ij}|)x_i\|^2 + \|f_{ji}(|\mathscr{A}_{ji}|)x_i\|^2\Big)^{1/2} \leqslant \|\,f_{ji}^2(|\mathscr{A}_{ji}|) + g_{ij}^2(|\mathscr{B}_{ij}|)\|^{1/2}\|x_i\|.$$

Thus,

$$\big|\langle \widetilde{\mathscr{T}_{ij}} x_j, x_i\rangle\big| + \big|\langle \widetilde{\mathscr{T}_{ji}} x_i, x_j\rangle\big| \leqslant \widehat{t}_{ij}^{(2)}\,\|x_i\|\,\|x_j\|,$$

where

$$\widehat{t}_{ij}^{(2)} = \Big\|f_{ij}^2(|\mathscr{A}_{ij}|) + g_{ji}^2(|\mathscr{B}_{ji}|)\Big\|^{1/2}\Big\|f_{ji}^2(|\mathscr{A}_{ji}|) + g_{ij}^2(|\mathscr{B}_{ij}|)\Big\|^{1/2}.$$

Summing over i, j,

$$\big|\langle \widetilde{\mathscr{T}} x, x\rangle\big| \leqslant \sum_{i=1}^{n} w(\widetilde{\mathscr{T}_{ii}})\|x_i\|^2 + \sum_{i<j} \widehat{t}_{ij}^{(2)}\,\|x_i\|\,\|x_j\|.$$

Let $|x| = (\|x_1\|,\ldots,\|x_n\|)^T$ so that $\||x|\| = 1$. Then the right-hand side equals $\langle \widehat{\mathscr{T}}^{(2)}|x|, |x|\rangle$, hence

$$\big|\langle \widetilde{\mathscr{T}} x, x\rangle\big| \leqslant \langle \widehat{\mathscr{T}}^{(2)}|x|, |x|\rangle \leqslant w(\widehat{\mathscr{T}}^{(2)}).$$

Taking the supremum over all unit vectors x, we obtain $w(\widetilde{\mathscr{T}}) \leqslant w(\widehat{\mathscr{T}}^{(2)})$, as desired. □

The next theorem is similar to Theorem 3, derived from the second mixed Schwarz inequality (6), which involves polar decomposition $\mathscr{B} = \mathscr{V}|\mathscr{B}|$. In this setting, the roles of the operators $\mathscr{A}$ and $\mathscr{B}$ are interchanged, and the resulting block operator matrix involves the partial isometry V_{ij} instead of U_{ij}.

THEOREM 4. *Let $\mathscr{A}_{ij}, \mathscr{B}_{ij} \in \mathbb{B}(\mathscr{H})$ for all $i, j = 1, \dots, n$, and the polar decomposition of $\mathscr{B}_{ij}$ be $\mathscr{B}_{ij} = \mathscr{V}_{ij}|\mathscr{B}_{ij}|$. Let $f_{ij}, g_{ij} : [0,\infty) \to [0,\infty)$ be continuous functions and let the composite operator blocks $\widetilde{\mathscr{S}_{ij}} := g_{ij}(|\mathscr{A}_{ij}|)\, f_{ij}\Big(|\mathscr{B}_{ij}^*|\Big)\, \mathscr{V}_{ij}$, $1 \leqslant i, j \leqslant n$. Then the block operator matrix $\widetilde{\mathscr{S}} = \big[\widetilde{\mathscr{S}_{ij}}\big]_{i,j=1}^n$ satisfies $w(\widetilde{\mathscr{S}}) \leqslant w\big(\widehat{\mathscr{S}^{(2)}}\big)$, where*

$$\widehat{\mathscr{S}^{(2)}} = \big[\widehat{s}_{ij}^{(2)}\big]_{i,j=1}^n,$$

$$\widehat{s}_{ij}^{(2)} = \begin{cases} w(\widetilde{\mathscr{S}_{ii}}) & i = j, \\ \Big\| f_{ij}^2(|\mathscr{B}_{ij}|) + g_{ji}^2(|\mathscr{A}_{ji}|) \Big\|^{1/2} \Big\| f_{ji}^2(|\mathscr{B}_{ji}|) + g_{ij}^2(|\mathscr{A}_{ij}|) \Big\|^{1/2} & i < j, \\ 0 & i > j. \end{cases}$$

As a direct consequence of Theorem 3, we obtain the following corollary, which was recently developed by Bhunia [3].

COROLLARY 1. *Let $\mathscr{A}_{ij} \in \mathbb{B}(\mathscr{H})$ for all $i, j = 1, \dots, n$. Let $f_{ij}, g_{ij} : [0,\infty) \to [0,\infty)$ be continuous functions satisfying $f_{ij}(t)\, g_{ij}(t) = t$ for all $t \geqslant 0$. Then, the operator matrix $\widetilde{\mathscr{T}} = [\mathscr{A}_{ij}]_{i,j=1}^n$ satisfies $w(\widetilde{\mathscr{T}}) \leqslant w(\widehat{\mathscr{T}})$, where*

$$\widehat{\mathscr{T}} = [\widehat{t}_{ij}]_{i,j=1}^n,$$

$$\widehat{t}_{ij} = \begin{cases} w(\mathscr{A}_{ii}) & i = j, \\ \Big\| f_{ij}^2(|\mathscr{A}_{ij}|) + g_{ji}^2(|\mathscr{A}_{ji}^*|) \Big\|^{1/2} \Big\| f_{ji}^2(|\mathscr{A}_{ji}|) + g_{ij}^2(|\mathscr{A}_{ij}^*|) \Big\|^{1/2} & i < j, \\ 0 & i > j. \end{cases}$$

Proof. By functional calculus, the relation $f_{ij}(t)\, g_{ij}(t) = t$ implies that

$$g_{ij}(|\mathscr{A}_{ij}^*|)\, f_{ij}(|\mathscr{A}_{ij}^*|) = |\mathscr{A}_{ij}^*|.$$

Hence $\widetilde{\mathscr{T}_{ij}} = g_{ij}(|\mathscr{A}_{ij}^*|)\, f_{ij}(|\mathscr{A}_{ij}^*|)\, \mathscr{U}_{ij} = |\mathscr{A}_{ij}^*|\, \mathscr{U}_{ij}$. Now recall that if $\mathscr{A}_{ij} = \mathscr{U}_{ij}|\mathscr{A}_{ij}|$ is the polar decomposition, then

$$|\mathscr{A}_{ij}^*| = (\mathscr{A}_{ij}\mathscr{A}_{ij}^*)^{1/2} = (\mathscr{U}_{ij}|\mathscr{A}_{ij}|^2\mathscr{U}_{ij}^*)^{1/2} = \mathscr{U}_{ij}\, |\mathscr{A}_{ij}|\, \mathscr{U}_{ij}^*,$$

where the last equality follows because conjugation by the partial isometry $\mathscr{U}_{ij}$ preserves the functional calculus on the supporting subspace. Multiplying on the right by $\mathscr{U}_{ij}$ gives

$$|\mathscr{A}_{ij}^*|\, \mathscr{U}_{ij} = \mathscr{U}_{ij}\, |\mathscr{A}_{ij}|\, \mathscr{U}_{ij}^*\, \mathscr{U}_{ij} = \mathscr{U}_{ij}\, |\mathscr{A}_{ij}|\, (\mathscr{U}_{ij}^*\mathscr{U}_{ij}) = \mathscr{U}_{ij}\, |\mathscr{A}_{ij}| = \mathscr{A}_{ij}.$$

Thus, $\widetilde{\mathscr{T}} = [\mathscr{A}_{ij}]$, and so applying Theorem 3 to this matrix yields

$$w([\mathscr{A}_{ij}]) = w(\widetilde{\mathscr{T}}) \leqslant w(\widehat{\mathscr{T}}). \quad \square$$

We now present a Furuta type numerical radius inequality for the block operator matrix introduced in Theorem 3. This result arises from a power type factorization of each entry $\mathscr{A}_{ij}$ involving the parameters θ_{ij} and γ_{ij}, and provides a direct Furuta type extension of the estimate obtained in Theorem 3. It also serves as a foundation for the subsequent, more flexible mixed Furuta type inequality presented in the next corollary.

COROLLARY 2. *Let* $\mathscr{A}_{ij} \in \mathbb{B}(\mathscr{H})$ *for all* $i,j = 1,\dots,n$, *and* $\widetilde{\mathscr{T}_{ij}} = \mathscr{A}_{ij}|\mathscr{A}_{ij}|^{\theta_{ij}+\gamma_{ij}-1}$, *where* $\theta_{ij}+\gamma_{ij} \geqslant 1$ *for all* i,j. *Then the operator matrix* $\widetilde{\mathscr{T}} = [\widetilde{\mathscr{T}_{ij}}]_{i,j=1}^{n}$ *satisfies*

$$w\big(\widetilde{\mathscr{T}}\big) \leqslant w\big(\widehat{\mathscr{T}}^{(2)}\big), \tag{11}$$

where

$$\widehat{\mathscr{T}}^{(2)} = \big[\widehat{t}_{ij}^{(2)}\big]_{i,j=1}^{n},$$

$$\widehat{t}_{ij}^{(2)} = \begin{cases} w\left(\mathscr{A}_{ii}\,|\mathscr{A}_{ii}|^{\theta_{ii}+\gamma_{ii}-1}\right) & i=j,\\[2mm] \left\|\,|\mathscr{A}_{ij}|^{2\theta_{ij}} + |\mathscr{A}_{ji}^{*}|^{2\gamma_{ji}}\,\right\|^{1/2} \left\|\,|\mathscr{A}_{ji}|^{2\theta_{ji}} + |\mathscr{A}_{ij}^{*}|^{2\gamma_{ij}}\,\right\|^{1/2} & i<j,\\[2mm] 0 & i>j. \end{cases}$$

Proof. We apply Theorem 3 with the specified choices $f_{ij}(t) = t^{\theta_{ij}}$, $g_{ij}(t) = t^{\gamma_{ij}}$ and $\mathscr{B}_{ij} = \mathscr{A}_{ij}^{*}$. The composite block is $\widetilde{\mathscr{T}_{ij}} = g_{ij}\big(|\mathscr{B}_{ij}|\big)\, f_{ij}\big(|\mathscr{A}_{ij}^{*}|\big)\, \mathscr{U}_{ij}$. Substituting the power functions and using $|\mathscr{B}_{ij}| = |\mathscr{A}_{ij}^{*}|$ yields

$$\widetilde{\mathscr{T}_{ij}} = |\mathscr{A}_{ij}^{*}|^{\gamma_{ij}}\, |\mathscr{A}_{ij}^{*}|^{\theta_{ij}}\, \mathscr{U}_{ij} = |\mathscr{A}_{ij}^{*}|^{\gamma_{ij}+\theta_{ij}}\, \mathscr{U}_{ij}.$$

When $\theta_{ij}+\gamma_{ij} \geqslant 1$, the operator $|\mathscr{A}_{ij}|^{\theta_{ij}+\gamma_{ij}-1}$ is well defined and we may use the intertwining property of the partial isometry $\mathscr{U}_{ij}$ (coming from the polar decomposition)

$$|\mathscr{A}_{ij}^{*}|^{\alpha}\, \mathscr{U}_{ij} = \mathscr{U}_{ij}\, |\mathscr{A}_{ij}|^{\alpha}, \qquad (\alpha \geqslant 0),$$

with $\alpha = \theta_{ij}+\gamma_{ij}$. Thus $|\mathscr{A}_{ij}^{*}|^{\theta_{ij}+\gamma_{ij}}\, \mathscr{U}_{ij} = \mathscr{U}_{ij}\, |\mathscr{A}_{ij}|^{\theta_{ij}+\gamma_{ij}}$. Recalling $\mathscr{A}_{ij} = \mathscr{U}_{ij}|\mathscr{A}_{ij}|$, we get

$$\mathscr{A}_{ij}\, |\mathscr{A}_{ij}|^{\theta_{ij}+\gamma_{ij}-1} = \mathscr{U}_{ij} |\mathscr{A}_{ij}|^{\theta_{ij}+\gamma_{ij}} = |\mathscr{A}_{ij}^{*}|^{\theta_{ij}+\gamma_{ij}}\, \mathscr{U}_{ij} = \widetilde{\mathscr{T}_{ij}}.$$

To obtain the numerical radius estimate (11) and the explicit formulas for the entries $\widehat{t}_{ij}^{(2)}$, one simply substitutes the power-function identities

$$f_{ij}^{2}(|\mathscr{A}_{ij}|) = |\mathscr{A}_{ij}|^{2\theta_{ij}} \qquad \text{and} \qquad g_{ji}^{2}(|\mathscr{A}_{ji}^{*}|) = |\mathscr{A}_{ji}^{*}|^{2\gamma_{ji}}$$

(and the analogous expressions for f_{ji}^{2} and g_{ij}^{2}) into the corresponding terms in Theorem 3. This yields the asserted formulas for $\widehat{t}_{ii}^{(2)}$ and $\widehat{t}_{ij}^{(2)}$ (for $i<j$), and completes the proof. $\square$

Another corollary is given below, in which we present a new numerical radius inequality for block operator matrices associated with the Moore-Penrose inverse.

The main idea is to extend the framework of mixed Schwarz type inequalities by incorporating the projection operator $\mathscr{A}\mathscr{A}^{\dagger}$, which naturally arises when $\mathscr{A}$ has a closed range. Using suitable choices of functions in the general mixed inequality, namely

$$f(t)=t \quad \text{and} \quad g(t)=\sqrt{t},$$

we obtain a new family of operator blocks involving both $\mathscr{A}$ and its Moore-Penrose inverse. This leads to a refined bound for the numerical radius of block operator matrices, expressed in terms of the orthogonal projections onto the ranges of the component operators. The following theorem establishes this result precisely.

THEOREM 5. *Let $\mathscr{A}_{ij} \in \mathbb{B}(\mathscr{H})$ have closed ranges, and let $\mathscr{A}_{ij} = \mathscr{U}_{ij}\,|\mathscr{A}_{ij}|$ be the polar decomposition. Then the operator matrix $\widetilde{\mathscr{A}} = [\mathscr{A}_{ij}]_{i,j=1}^{n}$ satisfies $w(\widetilde{\mathscr{A}}) \leqslant w\big(\widehat{\mathscr{A}^{(2)}}\big)$, where $\widehat{\mathscr{A}^{(2)}} = [\widehat{t}_{ij}^{(2)}]_{i,j=1}^{n}$ and*

$$\widehat{t}_{ij}^{(2)} = \begin{cases} w(\mathscr{A}_{ii}) & i=j, \\ \left\| \, |\mathscr{A}_{ij}|^2 + \mathscr{A}_{ji}\mathscr{A}_{ji}^{\dagger} \right\|^{1/2} \left\| \, |\mathscr{A}_{ji}|^2 + \mathscr{A}_{ij}\mathscr{A}_{ij}^{\dagger} \right\|^{1/2} & i<j, \\ 0 & i>j. \end{cases}$$

Proof. Take $\mathscr{B}_{ij} = \mathscr{A}_{ij}\mathscr{A}_{ij}^{\dagger} = \mathscr{P}_{ij}$ is the orthogonal projection onto $\mathscr{R}(\mathscr{A}_{ij})$. Consider the functions $f_{ij}(t)=t,\ g_{ij}(t)=\sqrt{t},\ t \geqslant 0$. It follows that $f_{ij}(|\mathscr{A}_{ij}|) = |\mathscr{A}_{ij}|$, $f_{ij}^2(|\mathscr{A}_{ij}|) = |\mathscr{A}_{ij}|^2$. Moreover, since $\mathscr{P}_{ij}$ is an orthogonal projection, we obtain

$$g_{ij}(|\mathscr{B}_{ij}|) = (\mathscr{A}_{ij}\mathscr{A}_{ij}^{\dagger})^{1/2} = P_{ij}, \qquad g_{ij}^2(|\mathscr{B}_{ij}|) = P_{ij}.$$

Therefore, the operator

$$\widetilde{\mathscr{T}}_{ij} = g_{ij}(|\mathscr{B}_{ij}|)\, f_{ij}(|\mathscr{A}_{ij}^{*}|)\, \mathscr{U}_{ij} = P_{ij}\,|\mathscr{A}_{ij}^{*}|\,\mathscr{U}_{ij}.$$

Moreover, since

$$P_{ij}\,|\mathscr{A}_{ij}^{*}|\,\mathscr{U}_{ij} = \mathscr{A}_{ij}\mathscr{A}_{ij}^{\dagger}\mathscr{U}_{ij}|\mathscr{A}_{ij}| = \mathscr{A}_{ij}\mathscr{A}_{ij}^{\dagger}\mathscr{A}_{ij} = \mathscr{A}_{ij},$$

it follows that

$$\widetilde{\mathscr{T}}_{ij} = \mathscr{A}_{ij}.$$

Applying Theorem 3, we finally obtain $w([\mathscr{A}_{ij}]) = w(\widetilde{\mathscr{T}}) \leqslant w(\widehat{\mathscr{T}^{(2)}})$, which completes the proof using the fact that if $\mathscr{P}$ is an orthogonal projection, then $|\mathscr{P}|^{\alpha} = \mathscr{P}$ for every $\alpha > 0$. $\square$

We now observe that Theorem 3 admits a natural interpretation in terms of an Aluthge-type transformation. More precisely, for two continuous functions $f,g:[0,\infty)\to[0,\infty)$, we consider the generalized transformation introduced in [19],

$$\Phi_{\mathscr{A}}^{(f,g)}:=f(|\mathscr{A}|)\,\mathscr{U}\,g(|\mathscr{A}|),$$

which provides a broad extension of the classical Aluthge map. Indeed, the symmetric choice $f(t)=g(t)=t^{1/2}$ yields precisely the standard Aluthge transformation, whereas arbitrary pairs (f,g) give rise to a richer class of nonlinear operator transforms. This substitution leads to an alternative formulation of our mixed Schwarz-type inequalities, presented below.

In the next theorem, we introduce a new numerical radius inequality for $n\times n$ operator matrices via the generalized Aluthge transformation associated with the polar decomposition $\mathscr{A}=\mathscr{U}|\mathscr{A}|$. This nonlinear operator map significantly extends the classical Aluthge transform and enables us to derive a mixed inequality involving the functions f and g through the operator $\Phi_{\mathscr{A}}^{(g,f)}$. The resulting estimate unifies and extends both Aluthge's inequality within a single framework, providing a more general setting that accommodates a broad class of operator means and functional choices.

The precise formulation is stated in the following theorem.

THEOREM 6. *Let $\mathscr{A}_{ij}\in\mathbb{B}(\mathscr{H})$ for all $i,j=1,\ldots,n$, and let $\mathscr{A}_{ij}=\mathscr{U}_{ij}|\mathscr{A}_{ij}|$ be the polar decomposition of each block $\mathscr{A}_{ij}$. For continuous functions $f_{ij},g_{ij}:[0,\infty)\to[0,\infty)$, set*

$$\Phi_{\mathscr{A}_{ij}}^{(g_{ij},f_{ij})}:=g_{ij}\big(|\mathscr{A}_{ij}|\big)\mathscr{U}_{ij}f_{ij}\big(|\mathscr{A}_{ij}|\big),\qquad 1\leqslant i,j\leqslant n.$$

Then the operator matrix $\widetilde{\mathscr{T}}=\big[\Phi_{\mathscr{A}_{ij}}^{(g_{ij},f_{ij})}\big]_{i,j=1}^{n}$ satisfies $w(\widetilde{\mathscr{T}})\leqslant w\big(\widehat{\mathscr{T}}^{(2)}\big)$, where $\widehat{\mathscr{T}}^{(2)}=[\widehat{t}_{ij}^{(2)}]_{i,j=1}^{n}$ and

$$\widehat{t}_{ij}^{(2)}=\begin{cases} w\Big(\Phi_{\mathscr{A}_{ii}}^{(g_{ii},f_{ii})}\Big) & i=j,\\ \Big\|f_{ij}^{2}\big(|\mathscr{A}_{ij}|\big)+g_{ji}^{2}\big(|\mathscr{A}_{ji}|\big)\Big\|^{1/2}\Big\|f_{ji}^{2}\big(|\mathscr{A}_{ji}|\big)+g_{ij}^{2}\big(|\mathscr{A}_{ij}|\big)\Big\|^{1/2} & i<j,\\ 0 & i>j.\end{cases}$$

Proof. Apply Theorem 3 with $\mathscr{B}_{ij}=\mathscr{A}_{ij}$ and the same pair of functions (f_{ij},g_{ij}) for each block. Using the fact that for any operator $\mathscr{A}\in\mathbb{B}(\mathscr{H})$ with polar decomposition $\mathscr{A}=\mathscr{U}|\mathscr{A}|$, we have

$$|\mathscr{A}^{*}|=(\mathscr{A}\mathscr{A}^{*})^{\frac{1}{2}}=(\mathscr{U}|\mathscr{A}|^{2}\mathscr{U}^{*})^{\frac{1}{2}}=\mathscr{U}|\mathscr{A}|\mathscr{U}^{*},$$

which implies that $|\mathscr{A}^{*}|\mathscr{U}=\mathscr{U}|\mathscr{A}|=\mathscr{A}$. Consequently, for every continuous function f, we have

$$f(|\mathscr{A}^{*}|)\mathscr{U}=\mathscr{U}f(|\mathscr{A}|).$$

Then each composite block is given by

$$\widetilde{\mathscr{T}_{ij}} := g_{ij}\big(|\mathscr{B}_{ij}|\big)\, f_{ij}\big(|\mathscr{A}_{ij}^*|\big)\, \mathscr{U}_{ij}.$$

reduces to

$$\widetilde{\mathscr{T}_{ij}} = g_{ij}\big(|\mathscr{A}_{ij}|\big)\, f_{ij}\big(|\mathscr{A}_{ij}^*|\big)\, \mathscr{U}_{ij} = g_{ij}\big(|\mathscr{A}_{ij}|\big)\mathscr{U}_{ij} f_{ij}\big(|\mathscr{A}_{ij}|\big) = \Phi_{\mathscr{A}_{ij}}^{(g_{ij},f_{ij})}.$$

Consequently, the diagonal entries of $\widehat{\mathscr{T}}^{(2)}$ become $w(\Phi_{\mathscr{A}_{ii}}^{(g_{ii},f_{ii})})$, and the off-diagonal upper-triangular entries take the stated form with $\mathscr{B}_{ij}$ replaced by $\mathscr{A}_{ij}$. Therefore, the inequality $w(\widetilde{\mathscr{T}}) \leqslant w(\widehat{\mathscr{T}}^{(2)})$ follows directly from Theorem 3. □

We next present an important special case of Theorem 6 obtained by choosing the power functions $g(x) = x^t$ and $f(x) = x^{1-t}$ for $0 < t < 1$. This choice corresponds to the classical t-Aluthge type transform associated with the polar decomposition of each block $\mathscr{A}_{ij}$. The resulting operator matrix inequality is described in the following corollary.

COROLLARY 3. *Let $\mathscr{A}_{ij} \in \mathbb{B}(\mathscr{H})$ for all $i,j = 1,\ldots,n$ and let $\mathscr{A}_{ij} = \mathscr{U}_{ij}\,|\mathscr{A}_{ij}|$ be the polar decomposition of each block $\mathscr{A}_{ij}$. For a fixed exponent $t \in (0,1)$, and define the associated t-Aluthge type block transforms by*

$$\Phi_{\mathscr{A}_{ij}}^{(t)} := |\mathscr{A}_{ij}|^t\, \mathscr{U}_{ij}\, |\mathscr{A}_{ij}|^{1-t}.$$

Then the operator matrix $\widetilde{\mathscr{T}}^{(t)} := \big[\Phi_{\mathscr{A}_{ij}}^{(t)}\big]_{i,j=1}^n$ satisfies $w\big(\widetilde{\mathscr{T}}^{(t)}\big) \leqslant w\big(\widehat{\mathscr{T}}^{(2,t)}\big)$, where $\widehat{\mathscr{T}}^{(2,t)} = [\widehat{t}_{ij}^{(2,t)}]_{i,j=1}^n$ and

$$\widehat{t}_{ij}^{(2,t)} = \begin{cases} w\big(\Phi_{\mathscr{A}_{ii}}^{(t)}\big) & i = j, \\ \Big\||\mathscr{A}_{ij}|^{2(1-t)} + |\mathscr{A}_{ji}|^{2t}\Big\|^{1/2} \Big\||\mathscr{A}_{ji}|^{2(1-t)} + |\mathscr{A}_{ij}|^{2t}\Big\|^{1/2} & i < j, \\ 0 & i > j. \end{cases}$$

In particular, the case $t = \frac{1}{2}$ recovers the symmetric classical Aluthge transform $\Phi_{\mathscr{A}_{ij}}^{(1/2)} = |\mathscr{A}_{ij}|^{1/2}\,\mathscr{U}_{ij}\,|\mathscr{A}_{ij}|^{1/2}$.

3. Applications to special classes of operator matrices

In this final section, we apply the general numerical radius inequalities established in Section 2 to several special classes of block operator matrices. Our aim is to demonstrate how the unified framework developed earlier yields concrete and often sharper bounds for structured matrices, including 2×2 symmetric forms, off-diagonal configurations, and operator blocks involving the Moore-Penrose inverse. These applications show that the comparison matrices obtained in the previous section capture the essential interactions between the entries of a block operator matrix, and they allow us to derive refined estimates for expressions of the form $\mathscr{A} \pm \mathscr{B}$. The results presented below illustrate the flexibility and strength of the developed theory and recover, extend, or improve several known numerical radius inequalities in the literature.

We start with the following interesting lemma from [10].

LEMMA 1. *Let* $\mathscr{A}, \mathscr{B} \in \mathbb{B}(\mathscr{H})$. *Then*

$$w\left(\begin{bmatrix} \mathscr{A} & 0 \\ 0 & \mathscr{B} \end{bmatrix}\right) = \max\big(w(\mathscr{A}), w(\mathscr{B})\big),$$

and

$$w\left(\begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{B} & \mathscr{A} \end{bmatrix}\right) = \max\big(w(\mathscr{A}+\mathscr{B}), w(\mathscr{A}-\mathscr{B})\big).$$

In particular,

$$w\left(\begin{bmatrix} 0 & \mathscr{B} \\ \mathscr{B} & 0 \end{bmatrix}\right) = w(\mathscr{B}).$$

and

$$w\left(\begin{bmatrix} 0 & X \\ e^{i\theta}Y & 0 \end{bmatrix}\right) = w\left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix}\right), \qquad \textit{for all } \theta \in \mathbb{R}.$$

A direct consequence of Theorem 5 is the following numerical radius estimate for 2×2 block operator matrices involving the Moore-Penrose inverse.

COROLLARY 4. *Let* $\mathscr{A}, \mathscr{B} \in \mathbb{B}(\mathscr{H})$ *be operators with closed ranges. Consider the block operator matrix* $\mathscr{M} = \begin{bmatrix} 0 & \mathscr{A} \\ \mathscr{B} & 0 \end{bmatrix}$. *Then*

$$w(\mathscr{M}) \leqslant \frac{1}{2}\left\| |\mathscr{A}|^2 + \mathscr{B}\mathscr{B}^\dagger \right\|^{1/2} \left\| |\mathscr{B}|^2 + \mathscr{A}\mathscr{A}^\dagger \right\|^{1/2}.$$

In particular, if $\mathscr{A} = \mathscr{B}$, *then*

$$w(\mathscr{A}) = w\left(\begin{bmatrix} 0 & \mathscr{A} \\ \mathscr{A} & 0 \end{bmatrix}\right) \leqslant \frac{1}{2}\left\| |\mathscr{A}|^2 + \mathscr{A}\mathscr{A}^\dagger \right\|.$$

This is exactly the inequality (8).

As another application of Theorem 5, we obtain the following upper bound for the numerical radius of a symmetric 2×2 operator matrix, expressed in terms of the canonical projection $\mathscr{B}\mathscr{B}^\dagger$.

COROLLARY 5. *Let $\mathscr{A},\mathscr{B} \in \mathbb{B}(\mathscr{H})$ have closed ranges. Then*

$$\max w(\mathscr{A} \pm \mathscr{B}) = w\left(\begin{bmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{B} & \mathscr{A}\end{bmatrix}\right) \leqslant w(\mathscr{A}) + \frac{1}{2}\left\| |\mathscr{B}|^2 + \mathscr{B}\mathscr{B}^\dagger \right\|.$$

As an immediate consequence of Corollary 3, we obtain the following specialization for the symmetric 2×2 block matrix.

COROLLARY 6. *Let $\mathscr{A},\mathscr{B} \in \mathbb{B}(\mathscr{H})$ and the polar decompositions $\mathscr{A} = \mathscr{U}_A|\mathscr{A}|$, $\mathscr{B} = \mathscr{U}_B|\mathscr{B}|$. For fixed $t \in (0,1)$, set*

$$\Phi_{\mathscr{A}}^{(t)} := |\mathscr{A}|^t\, \mathscr{U}_A |\mathscr{A}|^{1-t}, \qquad \Phi_{\mathscr{B}}^{(t)} := |\mathscr{B}|^t\, \mathscr{U}_B |\mathscr{B}|^{1-t},$$

and form the block Aluthge transform $\widetilde{\mathscr{T}}^{(t)} := \begin{bmatrix}\Phi_{\mathscr{A}}^{(t)} & \Phi_{\mathscr{B}}^{(t)}\\ \Phi_{\mathscr{B}}^{(t)} & \Phi_{\mathscr{A}}^{(t)}\end{bmatrix}$. Then

$$w\big(\widetilde{\mathscr{T}}^{(t)}\big) \leqslant w\big(\Phi_{\mathscr{A}}^{(t)}\big) + \frac{1}{2}\left\| |\mathscr{B}|^{2(1-t)} + |\mathscr{B}|^{2t} \right\|.$$

Proof. Apply Corollary 3 with the block data $\mathscr{A}_{11} = \mathscr{A}_{22} = \mathscr{A}$, $\mathscr{A}_{12} = \mathscr{A}_{21} = \mathscr{B}$. Then the diagonal entries of $\widehat{\mathscr{T}}^{(2,t)}$ are $\widehat{t}_{11}^{(2,t)} = \widehat{t}_{22}^{(2,t)} = w(\Phi_{\mathscr{A}}^{(t)}) = \alpha$, while the single nonzero off-diagonal entry (for $i < j$) is

$$\widehat{t}_{12}^{(2,t)} = \left\| |\mathscr{B}|^{2(1-t)} + |\mathscr{B}|^{2t} \right\|^{1/2} \left\| |\mathscr{B}|^{2(1-t)} + |\mathscr{B}|^{2t} \right\|^{1/2} = S_t.$$

Hence $\widehat{\mathscr{T}}^{(2,t)} = \begin{pmatrix}\alpha & S_t\\ 0 & \alpha\end{pmatrix}$, and the claimed inequality follow from the triangle bound $w(\alpha I + N) \leqslant |\alpha| + w(N)$ together with $w\begin{pmatrix}0 & S_t\\ 0 & 0\end{pmatrix} = \frac{1}{2}S_t$. $\square$

Next result is for 3×3 special block matrices.

THEOREM 7. *Let $\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D} \in \mathbb{B}(\mathscr{H})$ have closed ranges. Consider the 3×3 block matrix $\mathscr{S} = \begin{bmatrix}0 & \mathscr{A} & \mathscr{C}\\ \mathscr{B} & 0 & 0\\ \mathscr{D} & 0 & 0\end{bmatrix}$. Assume the projections $\mathscr{P}_{BA} = \mathscr{B}\mathscr{B}^\dagger$, $\mathscr{P}_{AB} = \mathscr{A}\mathscr{A}^\dagger$, $\mathscr{P}_{DC} = \mathscr{D}\mathscr{D}^\dagger$, $\mathscr{P}_{CD} = \mathscr{C}\mathscr{C}^\dagger$. Then*

$$w(\mathscr{S}) \leqslant \frac{1}{2}\sqrt{(\widehat{t}_{12}^{(2)})^2 + (\widehat{t}_{13}^{(2)})^2},$$

where

$$\widehat{t}_{12}^{(2)} := \left\| |\mathscr{A}|^2 + \mathscr{P}_{BA} \right\|^{1/2} \left\| |\mathscr{B}|^2 + \mathscr{P}_{AB} \right\|^{1/2},$$

$$\widehat{t}_{13}^{(2)} := \left\| |\mathscr{C}|^2 + \mathscr{P}_{DC} \right\|^{1/2} \left\| |\mathscr{D}|^2 + \mathscr{P}_{CD} \right\|^{1/2}.$$

Moreover,

$$w(\mathscr{A}\mathscr{B}\pm\mathscr{C}\mathscr{D}) \leqslant w^2(\mathscr{S}) \leqslant \frac{1}{4}\big((\widehat{t}_{12}^{(2)})^2+(\widehat{t}_{13}^{(2)})^2\big).$$

Proof. Apply Theorem 5 to the 3×3 operator matrix $\mathscr{S}$ with the choice of entries

$$A_{11}=A_{22}=A_{33}=A_{23}=A_{32}=0, \quad A_{12}=\mathscr{A},\, A_{13}=\mathscr{C},\, A_{21}=\mathscr{B},\, A_{31}=\mathscr{D}.$$

By that theorem, all nonzero off-diagonal comparison entries appear only in the first row, and they coincide with the quantities given in the statement. Hence, the scalar upper-triangular comparison matrix provided by the theorem is exactly

$$\widehat{\mathscr{S}^{(2)}} = \begin{bmatrix} 0 & \widehat{t}_{12}^{(2)} & \widehat{t}_{13}^{(2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and Theorem 5 yields the estimate

$$w(\mathscr{S}) \leqslant w\big(\widehat{\mathscr{S}^{(2)}}\big).$$

It remains to compute the numerical radius of the 3×3 scalar matrix $\widehat{\mathscr{S}^{(2)}}$. Put $a := \widehat{t}_{12}^{(2)}$ and $c := \widehat{t}_{13}^{(2)}$, and set

$$M := \begin{bmatrix} 0 & a & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the characteristic polynomial of $M+M^*$ is $-\lambda(\lambda^2-(a^2+c^2))$, so the eigenvalues are $\sqrt{a^2+c^2}$, $-\sqrt{a^2+c^2}$, 0. Therefore, $\rho(M+M^*)=\sqrt{a^2+c^2}$, and since $M+M^*$ is Hermitian, we have $w(M)=\frac{1}{2}\rho(M+M^*)=\frac{1}{2}\sqrt{a^2+c^2}$. Combining this with the comparison estimate above gives

$$w(\mathscr{S}) \leqslant w\big(\widehat{\mathscr{S}^{(2)}}\big) = \frac{1}{2}\sqrt{\big(\widehat{t}_{12}^{(2)}\big)^2+\big(\widehat{t}_{13}^{(2)}\big)^2}.$$

Again, from the proof of [2, Theorem 2.8], we obtain $w(\mathscr{A}\mathscr{B}\pm\mathscr{C}\mathscr{D}) \leqslant w^2(\mathscr{S})$. Combining this with the estimate for $w(\mathscr{S})$ yields

$$w(\mathscr{A}\mathscr{B}\pm\mathscr{C}\mathscr{D}) \leqslant w^2(\mathscr{S}) \leqslant \frac{1}{4}\big((\widehat{t}_{12}^{(2)})^2+(\widehat{t}_{13}^{(2)})^2\big). \quad \square$$

We conclude this paper with the following general Moore-Penrose type numerical radius inequality. It extends our previous results by incorporating arbitrary continuous functions f and g together with the canonical range projections, and provides a unified bound for the mixed operator expression $g(|\mathscr{B}|)f(|\mathscr{A}^*|)\mathscr{U}$ arising from the polar decomposition of $\mathscr{A}$.

4. Conclusions

In this work, we developed a unified framework for deriving numerical radius inequalities for block operator matrices by employing mixed Schwarz type inequalities. The approach provides a versatile comparison technique that captures and extends several classical results, while also yielding new bounds involving the Moore-Penrose inverse. The obtained inequalities not only generalize existing estimates, but also introduce sharper and structurally transparent bounds applicable to a wide class of operator matrices. We believe that this framework opens up further possibilities for refining numerical radius estimates and for exploring new applications within operator theory and matrix analysis.

Acknowledgement. The authors would like to express their sincere gratitude to the referee for the careful reading of the manuscript and for the constructive comments and valuable suggestions, which have significantly improved the quality, clarity, and presentation of the paper.

Declarations

Ethical approval. This statement is not applicable here.

Competing interest. The authors declare no competing interests.

Authors' contributions. All authors have contributed equally to this work.

Availability of data and materials. This statement does not apply.

REFERENCES

[1] A. ABU-OMAR AND F. KITTANEH, *Numerical radius inequalities for $n \times n$ operator matrices*, Linear Algebra Appl. **468** (2015), 18–26.

[2] P. BHUNIA, *Sharper bounds for the numerical radius of $n \times n$ operator matrices*, Arch. Math. (Basel) **123** (2024), no. 2, 173–183.

[3] P. BHUNIA, *Numerical radius inequalities of operator matrices*, Indian J. Pure Appl. Math. (2025), `https://doi.org/10.1007/s13226-025-00792-8`.

[4] P. BHUNIA, F. KITTANEH AND S. SAHOO, *Improved numerical radius bounds using the Moore-Penrose inverse*, Linear Algebra Appl. **711** (2025), 1–16.

[5] G. CORACH, A. MAESTRIPIERI, *Weighted generalized inverses, oblique projections, and least-squares problems*, Numer. Funct. Anal. Optim. **26** (6) (2005), 659–673.

[6] X. DONG, Y. GUO, AND D. WU, NEW BOUNDS FOR THE DAVIS-WIELANDT RADIUS VIA THE MOORE-PENROSE INVERSE OF BOUNDED LINEAR OPERATORS, *Axioms* **14** (2025) no. 6, 439.

[7] G. FONGI AND M. GONZALEZ, *Moore-Penrose inverse and partial orders on Hilbert space operators*, Linear Algebra Appl. **674** (2023), 1–20.

[8] C. W. GROETSCH, *Generalized inverses of linear operators*, Representation and Approximation, Marcel Dekker, New York, 1977.

[9] M. GUESBA, S. BARIK AND K. PAUL, *Further Berezin number and Berezin norm inequalities for sums and products of operators*, Complex Anal. Oper. Theory **19** (2025), no. 2, Paper No. 32, 21 pp.

[10] O. HIRZALLAH, F. KITTANEH AND K. SHEBRAWI, *Numerical radius inequalities for certain 2×2 operator matrices*, Integr. Equ. Oper. Theory **71** (2011), 129–147.

[11] J. C. HOU AND H. K. DU, *Norm inequalities of positive operator matrices*, Integral Equations Operator Theory **22** (1995), 281–294.

[12] M. A. IGHACHANE, F. KITTANEH, Y. REN, *New improvements on numerical radius bounds via the Moore-Penrose inverse*, Georgian Math. J. (2025), `https://doi.org/10.1515/gmj-2025-2076`.

[13] F. KITTANEH, *Notes on some inequalities for Hilbert Space operators*, Publ. Res. Inst. Math. Sci. **24** (2), (1988), 283–293.

[14] F. KITTANEH, *Numerical radius inequalities for Hilbert space operators*, Studia Math. **168** (2005), 73–80.

[15] F. KITTANEH AND S. SAHOO, *On numerical radius inequalities for $n\times n$ operator matrices and applications to bounding the zeros of polynomials*, Linear Multilinear Algebra **74** (2026), no. 1, 1–19.

[16] S. MAHAPATRA, A. SEN, R. BIRBONSHI AND K. PAUL, *Berezin number and Berezin norm inequalities via Moore-Penrose inverse*, J. Pseudo-Differ. Oper. Appl. **16** (2025), no. 3, Paper No. 64, 19 pp.

[17] Y. REN, M. A. IGHACHANE AND P. BHUNIA, *A generalized mixed Schwarz inequality and its application to the numerical radius*, J. Pseudo-Differ. Oper. Appl. **16** (2025), no. 4, Paper No. 90, 20 pp.

[18] M. SABABHEH, D. S. DJORDJEVĆ AND H. R. MORADI, *Numerical radius and norm bounds via the Moore-Penrose inverse*, Complex Anal. Oper. Theory **18** (2024), no. 5, Paper No. 117, 11 pp.

[19] K. SHEBRAWI AND M. BAKHERAD, *Generalizations of the Aluthge transform of operators*, Filomat **32** (18) (2018), 6465–6474.

(Received December 26, 2025)

Yonghui Ren
School of Mathematics and Statistics
Zhoukou Normal University
Zhoukou 466001, China
e-mail: `yonghuiren1992@163.com`

Pintu Bhunia
Department of Mathematics
SRM University AP
Amaravati 522240, Andhra Pradesh, India
e-mail: `pintubhunia5206@gmail.com`
`pintu.b@srmap.edu.in`

Mohamed Amine Ighachane
Sciences and Technologies Team (ESTE)
Higher School of Education and Training of El Jadida
Chouaib Doukkali University
El Jadida, Morocco
e-mail: `mohamedamineighachane@gmail.com`