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ULAM–HYERS–RASSIAS STABILITY OF A NONLINEAR STOCHASTIC INTEGRAL EQUATION OF VOLTERRA TYPE

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Abstract. The aim of this paper is to give some Ulam-Hyers-Rassias stability results for Volterra-type stochastic integral equations. The argument makes use of Gronwall lemma and Banach's fixed point theorem.

1. Introduction

The study of stability problems for various functional equations originated from a famous talk given by Ulam in 1940. In the talk, Ulam discussed a problem concerning the stability of homomorphisms (see [21] and [22]). More precisely, he proposed the following problem:

Given a group G_1, a metric group (G_2,d) and a positive number ε, does there exist a $\delta > 0$ such that if a function $f : G_1 \longrightarrow G_2$ satisfies the following inequality

$$d(f(xy), f(x)f(y)) < \delta,$$

for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \longrightarrow G_2$ such that:

$$d(f(x), T(x)) < \varepsilon,$$

for all $x \in G_1$?

When this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable, or that the equation defining group homomorphisms are stable (in the sense of Ulam).

In 1941, D. H. Hyers (see [8]) gave a partial solution of Ulam's problem under the assumption that G_1 and G_2 are Banach spaces. In 1950, T. Aoki (see [2]) studied the stability problem for additive mappings by using unbounded Cauchy differences (see also [14]). In 1978, Th. M. Rassias (see [18]) studied a similar problem. The stability considered in [18] is often called the Ulam-Hyers-Rassias stability.

In [17], V. Radu introduced a simple and nice proof for the Hyers-Ulam stability of the Cauchy additive functional equation. Using the idea of V. Radu, S.M. Jung proved

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in [10] the Hyers-Ulam-Rassias stability of some Volterra integral equations defined on a finite interval. After that, in [5], L. P. Castro and D. A. Ramos investigated the stability of Volterra integral equation of second kind for not only the finite case but also the infinite case. A simple proof of Jung's problem was later given in [19] by using some Gronwall lemmas.

In the references, at the end of this paper, we have listed other papers dealing with the stability of functional equations.

For a large amount of information on the stability of functional equations, the reader is invited to consult the books [6], [9] and [11] (see also the papers [1], [4], and others). Especially, in [4], the authors presented some recent developments in Ulam's type stability.

In this paper, we first introduce the notion of Hyers-Ulam-Rassias stability for a Volterra-type stochastic integral equation and then prove that kind of equation has the Hyers-Ulam-Rassias stability.

2. Definitions and Preliminaries

Fix a probability space $(\Omega,\mathscr{F},\mathbf{P})$. Let $\|\cdot\|_p=(E|\cdot|^p)^{\frac{1}{p}}$ be a norm of the space $L_p(\Omega,\mathbf{P})$, where $p>0$. Let W_t be a Brownian motion defined in $(\Omega,\mathscr{F},\mathbf{P})$ and let $\{\mathscr{F}_t, a\leqslant t\leqslant b\}$ be the natural filtration associated to W_t.

Denote by $L^p_{ad}([a,b],\Omega)$ the space of stochastic processes $f(t,\omega)$ such that each $f(t,\omega)$ is adapted to the filtration $\{\mathscr{F}_t\}$ and $E\left(\int_a^b |f(t)|^p dt\right)<\infty$.

Let $A(t,x)$ and $B(t,x)$ be measurable functions of $t\in[a,b]$ and $x\in\mathbb{R}$. Consider the stochastic integral equation of Volterra type:

$$X_t=\xi+\int_a^t A(s,X_s)ds+\int_a^t B(s,X_s)dWs,\ \ a\leqslant t\leqslant b, \tag{1}$$

where ξ is a $\mathscr{F}_a$ measurable random variable.

One has the following result for the existence and uniqueness of solution of Equation (1).

THEOREM 1. ([13]) *Let $A(t,x)$ and $B(t,x)$ be measurable functions on $[a,b]\times\mathbb{R}$ satisfying the Lipschitz and linear growth conditions in x. Suppose ξ is an $\mathscr{F}_a$ measurable random variable with $E(\xi^2)<\infty$. Then stochastic integral equation in Equation (1) has a unique continuous solution X_t.*

In the following definitions, we introduce the Ulam-Hyers-Rassias stability of a stochastic integral equation.

DEFINITION 1. Equation (1) is said to have the Ulam-Hyers stability with respect to ε if there exists a constant $c>0$ such that for each solution $X_t\in L^p_{ad}([a,b],\Omega)$ of the inequality

$$\|X_t-\xi-\int_a^t A(s,X_s)ds-\int_a^t B(s,X_s)dW_s\|_p\leqslant\varepsilon,\ \ a\leqslant t\leqslant b, \tag{2}$$

there exists a solution $U_t \in L^p_{ad}([a,b],\Omega)$ of Equation (1) such that:

$$\|X_t - U_t\|_p \leqslant c\varepsilon, \ \ t \in [a,b].$$

DEFINITION 2. Equation (1) is said to have the Ulam-Hyers-Rassias stability with respect to $\phi(t)$ if there exists a constant $M_\phi > 0$ such that for each solution $X_t \in L^p_{ad}([a,b],\Omega)$ of the inequation

$$\|X_t - \xi - \int_a^t A(s,X_s)ds - \int_a^t B(s,X_s)dW_s\|_p \leqslant \phi(t), \ \ a \leqslant t \leqslant b, \tag{3}$$

there exists a solution $U_t \in L^p_{ad}([a,b],\Omega)$ of Equation (1) such that:

$$\|X_t - U_t\|_p \leqslant M_\phi \phi(t), \ \ t \in [a,b],$$

where M_ϕ is a constant that does not depend on X_t.

In order to show that Equation (1) is stable in the sense of Ulam-Hyers-Rassias, we will need Gronwall lemma (see [7], [19], [20]), the Banach fixed point theorem and an inequality for the moment of Ito integral (see [24]).

LEMMA 1. *Let $\phi(t), \psi(t) \in C([a,b],\mathbb{R}_+)$ be two functions. We suppose that $\phi(t)$ is nondecreasing. If $x(t) \in C([a,b],\mathbb{R}_+)$ is a solution of the following inequation*

$$x(t) \leqslant \phi(t) + \int_a^t \psi(s)x(s)ds, t \in [a,b],$$

then

$$x(t) \leqslant \phi(t)\exp\left(\int_a^t \psi(s)ds\right).$$

THEOREM 2. ([3]) *(Banach's fixed point theorem) Suppose that (X,d) is a complete metric space and $T : X \to X$ is a contraction (for some $\lambda \in [0,1)$), $d(T(x),T(y) \leqslant \lambda d(x,y)$ for all $x,y \in X$. Suppose that there exist an element $u \in X$ and a number $\delta > 0$ such that*

$$d(u,T(u)) \leqslant \delta.$$

Then there exists a unique $p \in X$ such that $p = T(p)$. Moreover, $d(u,p) \leqslant \dfrac{\delta}{1-\lambda}$.

THEOREM 3. ([24]) *Let $p \geqslant 2$ and let $g \in L^2_{ad}([a,b],\Omega)$ be such that*

$$E\left[\int_a^b |g(t)|^2 dt\right] < \infty,$$

then

$$E\left|\int_a^b g(t)dW_t\right|^p \leqslant C_1.E\left[\int_a^b |g(t)|^p dt\right], \tag{4}$$

where $C_1 = \left(\dfrac{p(p-1)}{2}\right)^{\frac{p}{2}} (b-a)^{\frac{p-2}{2}}$.

In the next two sections, we will investigate Equation (1) under the following assumptions with respect to the random functions $A(t,x), B(t,y)$ and the random variable ξ defined for $a \leqslant t \leqslant b$ and $-\infty < x, y < \infty$:

(A1) $A(t,x)$ and $B(t,x)$ are measurable functions on $[a,b] \times \mathbb{R}$;

(A2) There exists a constant $K > 0$ such that

$$|A(t,x) - A(t,y)| \leqslant K|x-y|, \ \ \forall x,y \in \mathbb{R},$$

$$|B(t,x) - B(t,y)| \leqslant K|x-y|, \ \ \forall x,y \in \mathbb{R};$$

(A3) There exists a constant $L > 0$ such that

$$|A(t,x)| \leqslant L(1+|x|), \ \ \forall x \in \mathbb{R},$$

$$|B(t,x)| \leqslant L(1+|x|), \ \ \forall x \in \mathbb{R};$$

(A4) The random variable ξ is $\mathscr{F}_a$ measurable with $E(\xi^p) < \infty$, where $p \geqslant 2$.

3. Gronwall lemma approach

In the following theorem, we will use the Gronwall lemma approach to the Ulam-Hyers-Rassias stability of Equation (1).

THEOREM 4. ***(Ulam-Hyers-Rassias stability)***

Suppose that the assumptions (A1), (A2), (A3), (A4) together with the following assumption is satisfied:

(A5) The function $\phi(t)$ is nonnegative and the function $\phi^p(t)$ is nondecreasing;

Then:

a) Equation (1) has a unique continuous solution which belongs to the space $L^p_{ad}([a,b],\Omega)$.

b) Equation (1) has the Ulam-Hyers-Rassias stability with respect to $\phi(t)$ in the space $L^p_{ad}([a,b],\Omega)$.

Proof. a) According to Lyapunov's inequality, we have $\|\cdot\|_2 \leqslant \|\cdot\|_p, \forall p \geqslant 2$. Therefore, $\|\xi\|_2 \leqslant \|\xi\|_p$. Consequently, Equation (1) has a unique continuous solution. If U_t is the continuous solution of Equation (1) then we need to prove that U_t belongs to the space $L^p_{ad}([a,b],\Omega)$.

We have

$$U_t = \xi + \int_a^t A(s,U_s)dt + \int_a^t B(s,U_s)dW_s.$$

Using the inequalities $(a+b)^p \leqslant 2^{p-1}(a^p+b^p)$, $(a+b+c)^p \leqslant 3^{p-1}(a^p+b^p+c^p)$ and the linear growth conditions of $A(t,x), B(t,x)$, one gets

$$|U_t|^p \leqslant 3^{p-1}\left\{|\xi|^p + \left|\int_a^t A(s,U_s)ds\right|^p + \left|\int_a^t B(t,U_s)dW_s\right|^p\right\}$$

and

$$\left|\int_a^t A(s,U_s)ds\right|^p \leqslant \left(\int_a^t L(1+|U_s|)ds\right)^p \leqslant L^p 2^{p-1}\left(\left(\int_a^t ds\right)^p + \left(\int_a^t |U_s|ds\right)^p\right) \leqslant$$
$$\leqslant L^p 2^{p-1}\left((b-a)^p + \left(\int_a^t |U_s|ds\right)^p\right).$$

Applying the Hölder inequality, we obtain

$$\int_a^t |U_s|ds \leqslant \left(\int_a^t ds\right)^{\frac{p-1}{p}} \left(\int_a^t |U_s|^p ds\right)^{\frac{1}{p}} \leqslant (b-a)^{\frac{p-1}{p}} \left(\int_a^t |U_s|^p ds\right)^{\frac{1}{p}}.$$

Thus,

$$\left|\int_a^t A(s,U_s)ds\right|^p \leqslant L^p 2^{p-1}(b-a)^{p-1}\left(b-a+\int_a^t |U_s|^p ds\right).$$

Using the inequality (4) in Theorem 3, one obtains

$$E\left|\int_a^t B(s,U_s)dW_s\right|^p \leqslant C_1 E\int_a^t |B(s,U_s)|^p ds \leqslant C_1 L^p E\int_a^t (1+|U_s|)^p ds \leqslant$$
$$\leqslant C_1 L^p E\int_a^t 2^{p-1}(1+|U_s|^p)ds \leqslant$$
$$\leqslant C_1 L^p 2^{p-1}\left(b-a+\int_a^t E|U_s|^p ds\right),$$

where C_1 is the constant in Theorem 3.
Therefore,

$$E|U_t|^p \leqslant C_2 + C_3\int_a^t E|U_s|^p ds,$$

where $\begin{cases} C_2 = 3^{p-1}\left(E|\xi|^p + L^p 2^{p-1}(b-a)^p + C_1 L^p 2^{p-1}(b-a)\right), \\ C_3 = 3^{p-1}\left(L^p 2^{p-1}(b-a)^{p-1} + C_1 L^p 2^{p-1}\right). \end{cases}$

According to Lemma 1 , we have the following estimate

$$E|U_t|^p \leqslant C_2 \exp\left(\int_a^t C_3 ds\right) \leqslant C_2 \exp((b-a)C_3) < \infty.$$

Hence, $U_t \in L^p_{ad}([a,b],\Omega)$.

b) Let X_t be a solution of Inequation (3) and let U_t be the solution of Equation (1). Using again the inequality $(a+b+c)^p \leqslant 3^{p-1}(a^p+b^p+c^p)$, we obtain

$$|X_t - U_t|^p \leqslant 3^{p-1}\{\left|X_t - \xi - \int_a^t A(s,X_s)ds - \int_a^t B(s,X_s)dW_s\right|^p$$
$$+ \left|\int_a^t (A(s,X_s)-A(s,U_s))ds\right|^p + \left|\int_a^t (B(s,X_s)-B(s,U_s))dW_s\right|^p\}.$$

We have

$$E\left|X_t-\xi-\int_a^t A(s,X_s)ds-\int_a^t B(s,X_s)dW_s\right|^p \leqslant \phi^p(t).$$

Using the Lipschitz conditions and the Hölder inequality, we get

$$\left|\int_a^t (A(s,X_s)-A(s,U_s))ds\right|^p \leqslant \left(\int_a^t K|X_s-U_s|ds\right)^p \leqslant$$
$$\leqslant K^p\left(\int_a^t ds\right)^{p-1}\int_a^t |X_s-U_s|^p ds \leqslant K^p(b-a)^{p-1}\int_a^t |X_s-U_s|^p ds.$$

Using the inequality (4) in Theorem 3, we obtain

$$E\left|\int_a^t (B(s,X_s)-B(s,U_s))dW_s\right|^p \leqslant C_1.E\int_a^t |B(s,X_s)-B(s,U_s)|^p ds \leqslant$$
$$\leqslant C_1K^p\int_a^t E|X_s-U_s|^p ds.$$

where C_1 is the constant given in Theorem 3.
Therefore,

$$E|X_t-U_t|^p \leqslant C_4\phi^p(t)+C_5\int_a^t E|X_s-U_s|^p ds,$$

where $C_4=3^{p-1}, C_5=3^{p-1}K^p\left((b-a)^{p-1}+C_1\right)$.

According to Lemma 1, we obtain

$$E|X_t-U_t|^p \leqslant C_4\phi^p(t)\exp(\int_a^t C_5 ds) \leqslant C_4\phi^p(t)\exp(C_5(b-a)).$$

Hence,

$$\|X_t-U_t\|_p \leqslant M_\phi\phi(t),$$

where $M_\phi = C_4^{\frac{1}{p}}\exp\left(\frac{C_5(b-a)}{p}\right)$, which implies that Equation (1) has the Ulam-Hyers-Rassias stability.

REMARK 1. The constant M_ϕ in Theorem 4 does not depend on $\phi(t)$.

COROLLARY 1. ***(Ulam-Hyers stability)***
We suppose that the assumptions (A1), (A2), (A3) and (A4) are satisfied. Then:
a) Equation (1) *has a unique continuous solution belonging to the space* $L^p_{ad}([a,b],\Omega)$.
b) Equation (1) *has the Ulam-Hyers stability in the space* $L^p_{ad}([a,b],\Omega)$.

4. Fixed point approach

In the following theorems, we will use fixed point approach to the Ulam-Hyers-Rassias stability of Equation (1).

THEOREM 5. ***(Ulam-Hyers stability)***
Suppose that the assumptions (A1), (A2), (A3), (A4) together with the following assumption is satisfied:
(A5) $2^{\frac{p-1}{p}} K\{(b-a)^p + C_1(b-a)\}^{\frac{1}{p}} < 1$, *where* C_1 *is the constant in Theorem 3.*
Then:
a) Equation (1) *has a unique solution which belongs to the space* $L^p_{ad}([a,b],\Omega)$.
b) Equation (1) *has the Ulam-Hyers stability in the space* $L^p_{ad}([a,b],\Omega)$.

Proof.
Note that $L^p_{ad}([a,b],\Omega)$ is a Banach space when equipped with the norm

$$\|X_t\|_{p,\infty} = (\sup_{t\in[a,b]} E(|X_t|^p))^{\frac{1}{p}}.$$

Let us now introduce the operator T which is defined by:

$$T(X_t) = \xi + \int_a^t A(s,X_s)ds + \int_a^t B(s,X_s)dWs,$$

for all $X_t \in L^p_{ad}([a,b],\Omega)$ and $t \in [a,b]$.
As in Theorem 4, we have the following estimate

$$E|T(X_t)|^p \leqslant C_2 + C_3 \int_a^t E|X_s|^p ds$$

which implies that $\|T(X_t)\|_{p,\infty} < \infty$. Hence, $T(L^p_{ad}([a,b],\Omega)) \subset L^p_{ad}([a,b],\Omega)$.
For all $X_t, Y_t \in L^p_{ad}([a,b],\Omega)$, we have:

$$\begin{aligned} |T(X_t) - T(Y_t)|^p \leqslant\ & 2^{p-1}\{\left|\int_a^t (A(s,X_s) - A(s,Y_s))ds\right|^p \\ & + \left|\int_a^t (B(s,X_s) - B(s,Y_s))dW_s\right|^p\}, \end{aligned}$$

From the proof of Theorem 4, we obtain

$$E|T(X_t) - T(Y_t)|^p \leqslant 2^{p-1}K^p\left\{(b-a)^{p-1} + C_1\right\}\int_a^t E|X_s - Y_s|^p ds.$$

Therefore,

$$\sup_{t\in[a,b]} E|T(X_t) - T(Y_t)|^p \leqslant 2^{p-1}K^p\left\{(b-a)^{p-1} + C_1\right\}(b-a) \sup_{t\in[a,b]} E|X_t - Y_t|^p.$$

Hence,

$$\|T(X_t)-T(Y_t)\|_{p,\infty} \leqslant C_6\|X_t-Y_t\|_{p,\infty},$$

where $C_6 = 2^{\frac{p-1}{p}} K\{(b-a)^p + C_1(b-a)\}^{\frac{1}{p}}$.

Thus, by assumption $(A5)$, T is a contraction so that the fixed point theorem for contractions on Banach spaces ensures that there exists a unique $U_t \in L^p_{ad}([a,b],\Omega)$ such that $U_t = T(U_t)$.

We assume that X_t is a solution of Inequation (2). We have $\|X_t - T(X_t)\|_p \leqslant \varepsilon,\ \forall t \in [a,b]$ which implies that $\|X_t - T(X_t)\|_{p,\infty} \leqslant \varepsilon$. By the estimate in Theorem 2, we obtain

$$\|X_t - U_t\|_{p,\infty} \leqslant \frac{\varepsilon}{1-C_6}.$$

On the other hand, we have

$$\|X_t - U_t\|_p \leqslant \|X_t - U_t\|_{p,\infty},\ \ \forall t \in [a,b].$$

Thus, $\|X_t - U_t\|_p \leqslant \dfrac{\varepsilon}{1-C_6}$, which implies that Equation (1) has the Ulam-Hyers stability.

THEOREM 6. *(**Ulam-Hyers-Rassias stability**) Suppose that the assumptions (A1), (A2), (A3), (A4) together with the following assumptions is satisfied:*

(A5) The function $\phi(t)$ is positive and there exists a constant $N_\phi > 0$ such that

$$\int_a^t \phi^p(s)ds \leqslant N_\phi \phi^p(t),\ \ \forall t \in [a,b];$$

(A6) $2^{\frac{p-1}{p}} K\left((b-a)^{p-1}+C_1\right)^{\frac{1}{p}} N_\phi^{\frac{1}{p}} < 1$, where C_1 is the constant in the Theorem 3.

Then:

a) Equation (1) *has a unique solution which belongs to the space $L^p_{ad}([a,b],\Omega)$.*

b) Equation (1) *has the Ulam-Hyers-Rassias stability with respect to $\phi(t)$ in $L^p_{ad}([a,b],\Omega)$.*

Proof. We choose a continuous function $\psi : [a,b] \to (0,\infty)$ such that:

$$\int_a^t \psi^p(s)ds \leqslant N_\phi \psi^p(t).$$

Let α_ϕ and β_ϕ be two positive numbers such that:

$$\alpha_\phi \psi(t) \leqslant \phi(t) \leqslant \beta_\phi \psi(t),\ \ \forall t \in [a,b].$$

For all $X_t, Y_t \in L^p_{ad}([a,b],\Omega)$, we set

$$d_\psi(X_t,Y_t) = \sup_{t\in[a,b]} \frac{\|X_t - Y_t\|_p}{\psi(t)} < \infty.$$

It is known that $(L^p_{ad}([a,b],\Omega),d)$ is a complete metric space.

According to Theorem 4, we have $T(L^p_{ad}([a,b],\Omega)) \subset L^p_{ad}([a,b],\Omega)$, where $T(X_t) = \xi + \int_a^t A(s,X_s)ds + \int_a^t B(s,X_s)dWs$.

We assert that T is strictly contractive on $L^p_{ad}([a,b],\Omega)$. Given any $X_t, Y_t \in L^p_{ad}([a,b],\Omega)$, let $C_{X_t,Y_t} \in [0,\infty)$ be an arbitrary constant with $d_\psi(X_t,Y_t) \leqslant C_{X_t,Y_t}$, that is

$$\|X_t - Y_t\|_p \leqslant C_{X_t,Y_t}\psi(t), \ \ \forall t \in [a,b].$$

As in Theorem 5, we have the following estimate:

$$E|T(X_t) - T(Y_t)|^p \leqslant 2^{p-1}K^p\left\{(b-a)^{p-1} + C_1\right\}\int_a^t E|X_s - Y_s|^p ds.$$

Therefore,

$$\begin{aligned} E|T(X_t) - T(Y_t)|^p &\leqslant 2^{p-1}K^p\left\{(b-a)^{p-1} + C_1\right\}\int_a^t C^p_{X_t,Y_t}\psi(s)^p ds \\ &\leqslant 2^{p-1}K^p\left\{(b-a)^{p-1} + C_1\right\}N_\phi C^p_{X_t,Y_t}\psi(t)^p. \end{aligned}$$

Hence,

$$\|T(X_t) - T(Y_t)\|_p \leqslant C_7 C_{X_t,Y_t}\psi(t),$$

where $C_7 = 2^{\frac{p-1}{p}}K\left\{(b-a)^{p-1} + C_1\right\}^{\frac{1}{p}}N_\phi^{\frac{1}{p}}$. It implies that $d_\psi(T(X_t),T(Y_t)) \leqslant C_7 C_{X_t,Y_t}$. We may conclude that $d_\psi(T(X_t),T(Y_t)) \leqslant C_7 d_\psi(X_t,Y_t)$ for any $X_t,Y_t \in L^p_{ad}([a,b],\Omega)$. By assumption $(A6)$, the mapping T is strictly contractive on the metric space $(L^p_{ad}([a,b],\Omega),d_\psi)$. Thus, by the Banach fixed point principle, Equation (1) has a unique solution.

Let X_t be a solution of Inequation (3) and let U_t be the solution of Equation (1). By the triangle inequality, we have

$$\begin{aligned} d_\psi(X_t,U_t) &\leqslant d_\psi(X_t,T(X_t)) + d_\psi(T(X_t),U_t) = d_\psi(X_t,T(X_t)) + d_\psi(T(X_t),T(U_t)) \leqslant \\ &\leqslant \beta_\phi + C_7 d(X_t,U_t) \end{aligned}$$

which implies that

$$d_\psi(X_t,U_t) \leqslant \frac{\beta_\phi}{1-C_7}.$$

Hence,

$$\|X_t - U_t\|_p \leqslant \frac{\beta_\phi}{1-C_7}\psi(t) \leqslant M_\phi\phi(t),$$

where $M_\phi = \dfrac{\beta_\phi}{\alpha_\phi(1-C_7)}$. It means that Equation (1) has the Ulam-Hyers-Rassias stability. The proof of the theorem thus is complete.

5. Examples

In this section, we consider the case $p = 2$, $[a,b] \equiv [0,1]$. Remark that $\phi(t)$, $t \in [0,1]$, is a function satisfying the condition $(A5)$ in Theorem 4 and the condition $(A5)$ in Theorem 6.

Consider the following stochastic integral equation (see Example 10.1.8. in [13])

$$X_t = 1 + \int_0^t X_s^3 ds + \int_0^t X_s^2 dW_s. \tag{5}$$

Here, ξ and the functions A, B are given by

$$\xi = 1, \ \ A(t,x) = x^3, \ \ B(t,x) = x^2$$

satisfying all the hypotheses of Theorem 4. Hence, Equation (5) has Ulam-Hyer-Rasiass stability and its solution is given by

$$X_t = \frac{1}{1 - W_t}.$$

In order to illustrate Theorem 5 and Theorem 6, we continue considering the Langevin equation (see Example 10.1.1. in [13])

$$X_t = x_0 - \int_0^t \alpha X_s ds + \int_0^t \beta dW_s, \tag{6}$$

where α, β are constants.

In the case $p = 2$, $[a,b] \equiv [0,1]$, the condition $(A5)$ in Theorem 5 is equivalent to $K < \frac{1}{2}$. Clearly, the functions $A(t,x) = -\alpha x$ and $B(t,x) = \beta$ satisfy Lipschitz condition in x with Lipschitz constant $K = |\alpha|$. So that, with $|\alpha| < \frac{1}{2}$, all the assumptions of Theorem 5 are satisfied.

In addition, with choosing $\phi(t) = t$ and $N_\phi = \frac{1}{2}$, the condition $(A6)$ in Theorem 6 becomes $K < \frac{1}{\sqrt{2}}$. In the case $K = |\alpha| < \frac{1}{\sqrt{2}}$, all the hypotheses of Theorem 6 are satisfied. Thus, Equation (6) has Ulam-Hyer-Rasiass stability with respect to $\phi(t) = t$ and its solution is an Ornstein-Uhlenbeck process given by

$$X_t = e^{-\alpha t} x_0 + \beta \int_0^t e^{-\alpha(t-s)} dW_s.$$

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