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INFINITELY MANY PERIODIC SOLUTIONS TO A CLASS OF PERTURBED SECOND–ORDER IMPULSIVE HAMILTONIAN SYSTEMS

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Abstract. We investigate the existence of infinitely many periodic solutions to a class of perturbed second-order impulsive Hamiltonian systems. Our approach is based on variational methods and critical point theory.

1. Introduction

The aim of this paper is to investigate the existence of infinitely many periodic solutions to the perturbed impulsive Hamiltonian system with periodic boundary conditions

$$\begin{cases} -\ddot{u}(t)+A(t)u(t)=\lambda\nabla F(t,u(t))+\mu\nabla G(t,u(t))+\nabla H(u(t)), & a.e.\ t\in[0,T], \\ \Delta(\dot{u}_i(t_j))=I_{ij}(u_i(t_j)), \quad i=1,2,\dots,N,\ j=1,2,\dots,p, \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0, \end{cases} \tag{1.1}$$

where $u=(u_1,u_2,\dots,u_N)^{\mathscr{T}}$ (transpose), $N\geqslant 1$, $p>1$, $T>0$, $\lambda>0$ and $\mu\geqslant 0$ are parameters, $0=t_0<t_1<\dots<t_p<t_{p+1}=T$, $A:[0,T]\to\mathbb{R}^{N\times N}$ is a continuous map from the interval $[0,T]$ to the set of $N\times N$ symmetric matrices, and $\Delta(\dot{u}_i(t_j))=\dot{u}_i(t_j^+)-\dot{u}_i(t_j^-)=\lim_{t\to t_j^+}\dot{u}_i(t)-\lim_{t\to t_j^-}\dot{u}_i(t)$. Here, each $I_{ij}:\mathbb{R}\to\mathbb{R}$ satisfies

$$|I_{ij}(s)|\leqslant L_{ij}|s|$$

for every $s\in\mathbb{R}$ and $i=1,2,\dots,N$, $j=1,2,\dots,p$; F, $G:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ are measurable with respect to t for all $u\in\mathbb{R}^N$, continuously differentiable in u for almost every $t\in[0,T]$, and satisfy the summability condition

$$\sup_{|x|\leqslant\alpha}\max\{|F(\cdot,x)|,\ |\nabla F(\cdot,x)|,|G(\cdot,x)|,\ |\nabla G(\cdot,x)|\}\in L^1([0,T]) \tag{1.2}$$

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for any $\alpha > 0$; $F(t,\mathbf{0}) = G(t,\mathbf{0}) = 0$ for all $t \in [0,T]$, where $\mathbf{0} = (0,\ldots,0)$; and $H : \mathbb{R}^N \to \mathbb{R}$ is a continuously differentiable function for which there is a constant $L > 0$ such that

$$|H(x)| \leqslant L|x|^2$$

for every $x \in \mathbb{R}^N$. Note that if ∇F, $\nabla G : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ are continuous, then clearly condition (1.2) is satisfied.

As a special case of dynamical systems, Hamiltonian systems are very important in the study of such areas as fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, and many others. It is now recognized that the theory of Hamiltonian systems is a natural framework for modeling many natural phenomena. For background, theory, and applications of Hamiltonian systems, we refer the reader to [17, 31, 34, 41]. Inspired by the monographs [32, 35], the existence and multiplicity of periodic solutions for Hamiltonian systems using variational methods have been investigated in many papers (see, for example, [3, 5, 8, 9, 12, 14, 15, 16, 18, 19, 23, 24, 26, 42, 43, 44, 46, 47, 48, 49, 51, 52, 54, 55, 56] and the references contained therein) For example, in [43], Tang and Wu obtained existence theorems for periodic solutions of a class of unbounded, nonautonomous, nonconvex, subquadratic, second order Hamiltonian systems by using minimax methods in critical point theory. Cordaro [15] established a multiplicity result for an eigenvalue problem related to second-order Hamiltonian systems, and proved the existence of an open interval of positive eigenvalues in which the problem admits three distinct periodic solutions. Faraci [19] studied multiplicity of solutions of a second order nonautonomous system. He and Wu [24] showed the existence of nontrivial T-periodic solutions to second-order Hamiltonian systems using a mountain pass theorem and a local linking theorem, while Zhang and Tang [52] obtained some new results on T-periodic solutions for the same second-order Hamiltonian systems under weaker assumptions thus generalizing the corresponding results in [24]. In [8], Bonanno and Livrea proved the existence of infinitely many periodic solutions for a class of second-order Hamiltonian systems assuming an oscillating behavior of the nonlinear term. Moreover, they obtained multiplicity of periodic solutions for the system with a coercive potential and also did so in the noncoercive case. Gu and An [23] and Zhang and Liu [51] used a variant of the fountain theorem to show the existence of infinitely many periodic solutions of a class of superquadratic nonautonomous second-order Hamiltonian systems. Zhang and Zhou [56] studied a class of non-autonomous second order Hamiltonian system and obtained new existence theorems by the least action principle.

The theory of impulsive differential equations provides a general framework for mathematically modeling many real world phenomena. For background, theory, and applications of impulsive differential equations, we refer reader to the recent monograph of Graef et al. [22] as well as [4, 6, 27, 37]. There have been many approaches used to study the existence of solutions of impulsive differential equations, such as fixed point theory, topological degree, continuation methods, coincidence degree theory, upper and lower solution methods, and monotone iterative methods; see, for example, [1, 20, 28, 30] and references contained therein. Recently, critical point theory has been used in [2, 7, 33, 45, 50] to obtain existence and multiplicity of solutions of impulsive problems.

Sun et al. [39] studied the existence of multiple solutions to a class of second-order perturbed impulsive Hamiltonian systems and gave some new criteria guaranteeing the existence of at least three solutions by using a variational method and some critical points theorems of Ricceri. In [13], Chen and He used variational methods and critical point theorems of Ricceri to obtain existence of three solutions for second-order impulsive Hamiltonian systems. We also refer the interested reader to [29, 38, 53] in which second order Hamiltonian systems with impulsive effects have been studied.

For a discussion of infinitely many solutions of perturbed problems, we refer the reader to Bonanno and Molica Bisci [10] and Heidarkhani [25].

Motivated by the results in [13, 39], we consider a perturbed form of the equations considered in those papers, and we employ a smooth version of [11, Theorem 2.1], which is a more precise form of Ricceri's Variational Principle [36, Theorem 2.5], to derive sufficient conditions for the existence of an interval about the parameter λ in which the problem (1.1) admits a sequence of periodic solutions. Three examples illustrating the applicability of our results are also included.

2. Preliminaries

Our main tool to investigate the existence of infinitely many periodic solutions of problem (1.1) is a smooth version of Theorem 2.1 in [11]; it is a more precise version of Ricceri's Variational Principle [36, Theorem 2.5] and we state it here.

THEOREM 1. *Let X be a reflexive real Banach space, Φ and $\Psi : X \to \mathbb{R}$ be Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive, and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r]}\Psi(v) - \Psi(u)}{r - \Phi(u)},$$

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \quad \textit{and} \quad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) For every $r > \inf_X \Phi$ and every $\lambda \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(-\infty, r)$ admits a global minimum that is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < +\infty$, then for each $\lambda \in (0, \frac{1}{\gamma})$, either

(b_1) I_λ possesses a global minimum, or

(b_2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$, then for each $\lambda \in (0, \frac{1}{\delta})$, either*

(c_1) *there is a global minimum of Φ which is a local minimum of I_λ, or*

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ that weakly converges to a global minimum of Φ.*

We assume throughout this paper that the matrix A satisfies the following conditions:

(A1) $A(t) = (a_{kl}(t))$, $k = 1,2,\dots,N$, $l = 1,2,\dots,N$, is symmetric with $a_{kl} \in L^\infty[0,T]$ for any $t \in [0,T]$;

(A2) There exists $\kappa > 0$ such that $(A(t)x,x) \geqslant \kappa|x|^2$ for any $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$.

Next, we recall some basic concepts. Let

$$E = \{u : [0,T] \to \mathbb{R}^N \mid u \text{ is absolutely continuous, } u(0) = u(T),\ \dot{u} \in L^2([0,T],\mathbb{R}^N)\}$$

with the inner product

$$\prec u,v \succ_E = \int_0^T [(\dot{u}(t),\dot{v}(t)) + (u(t),v(t))]dt.$$

The corresponding norm is defined by

$$\|u\|_E^2 = \int_0^T (|\dot{u}(t)|^2 + |u(t)|^2)dt, \quad \text{for all } u \in E.$$

For every u, $v \in E$, we define

$$\prec u,v \succ = \int_0^T [(\dot{u}(t),\dot{v}(t)) + (A(t)u(t),v(t))]dt$$

and observe that, by assumptions (A1) and (A2), this defines an inner product in E. Then, E is a separable and reflexive Banach space with the norm

$$\|u\| = \prec u,u \succ^{\frac{1}{2}} \quad \text{for all } u \in E.$$

A simple computation shows that $(A(t)x,x) = \sum_{k,l=1}^N a_{kl}(t)x_kx_l \leqslant \sum_{k,l=1}^N \|a_{kl}\|_\infty |x|^2$ for every $t \in [0,T]$ and $x \in \mathbb{R}^N$, and this along with (A2) implies

$$\sqrt{m}\|u\|_E \leqslant \|u\| \leqslant \sqrt{M}\|u\|_E, \tag{2.1}$$

where $m = \min\{1,\kappa\}$ and $M = \max\{1, \sum_{k,l=1}^N \|a_{kl}\|_\infty\}$, i.e., the norms $\|\cdot\|$ and $\|\cdot\|_E$ are equivalent.

Since $(E,\|\cdot\|)$ is compactly embedded in $C([0,T],\mathbb{R}^N)$ (see [32]), there exists a positive constant c such that

$$\|u\|_\infty \leqslant c \parallel u \parallel, \tag{2.2}$$

where $\|u\|_\infty = \max_{t\in[0,T]} \mid u(t) \mid$ and $c = \sqrt{\frac{2}{m}} \max\{\frac{1}{\sqrt{T}}, \sqrt{T}\}$ (see [13]).

As pointed out in [39], if $u \in E$, then u is absolutely continuous and $\dot{u} \in L^2([0,T], \mathbb{R}^N)$. In this case, $\Delta\dot{u}(t) = \dot{u}(t^+) - \dot{u}(t^-) = 0$ is not necessarily valid for every $t \in (0,T)$, and the derivative $\dot{u}$ may possess some discontinuities that lead to the impulsive effects.

DEFINITION 1. A function $u \in \{u \in E : \dot{u} \in (W^{1,2}(t_j,t_{j+1}))^N, j = 0,1,2,\ldots,p\}$ is said to be a classical solution of the problem (1.1) if u satisfies (1.1).

DEFINITION 2. By a weak solution of the problem (1.1), we mean any $u \in E$ such that

$$\int_0^T \Big[(\dot{u}(t),\dot{v}(t)) + (A(t)u(t),v(t)) - (\nabla H(u(t)),v(t))\Big]dt + \sum_{j=1}^{p}\sum_{i=1}^{N} I_{ij}(u_i(t_j))v_i(t_j)$$
$$-\lambda \int_0^T (\nabla F(t,u(t)),v(t))dt - \mu \int_0^T (\nabla G(t,u(t)),v(t))dt = 0$$

for every $v \in E$.

LEMMA 1. *If $u \in E$ is a weak solution of (1.1), then u is a classical solution of (1.1).*

The scalar case of Lemma 1 for second-order impulsive Sturm-Liouville boundary value problems was proved in [40]. For problem (1.1), the proof is essentially the same. We omit the details.

We will assume throughout that

$$K := c^2(2LT + \sum_{j=1}^{p}\sum_{i=1}^{N} L_{ij}) < 1.$$

A special case of our main result is the following theorem.

THEOREM 2. *Assume that (A1) and (A2) hold and let $F : \mathbb{R}^N \to \mathbb{R}$ be a continuously differentiable function such that*

$$\liminf_{\xi\to+\infty} \frac{\max_{|x|\leqslant\xi} F(x)}{\xi^2} = 0 \quad \textit{and} \quad \limsup_{\xi\to+\infty} \frac{F(\xi\varepsilon)}{\xi^2} = +\infty,$$

where $\varepsilon = (1,0,\cdots,0) \in \mathbb{R}^N$. Then, the problem

$$\begin{cases} -\ddot{u}(t) + A(t)u(t) = \nabla F(u(t)) + \nabla H(u(t)), & a.e.\ t \in [0,T], \\ \Delta(\dot{u}_i(t_j)) = I_{ij}(u_i(t_j)), \quad i = 1,2,\ldots,N,\ j = 1,2,\ldots,p, \\ u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{cases}$$

has an unbounded sequence of classical periodic solutions.

3. Main results

Set

$$D=\frac{(T-t_p)^2}{t_1t_p^2}+\frac{t_1}{3t_p^2}(t_p^2+t_pT+T^2)+(t_p-t_1)+\frac{T-t_p}{t_p^2}+\frac{1}{3t_p^2}(T^3-t_p^3)>0.$$

We formulate our main result as follows.

THEOREM 3. *In addition to conditions (A1) and (A2), assume that:*

(a_1) $F(t,\xi)\geqslant 0$ *for each* $t\in[0,t_1]\cup[t_p,T]$, $|\xi|\in[0,+\infty)$*;*

(a_2) $\liminf_{\xi\to+\infty}\frac{\int_0^T\max_{|x|\leqslant\xi}F(t,x)dt}{\xi^2}<\frac{1-K}{(1+K)DMc^2}\limsup_{\xi\to+\infty}\frac{\int_{t_1}^{t_p}F(t,\xi\varepsilon)dt}{\xi^2}$,

where $\varepsilon=(1,0,\cdots,0)\in\mathbb{R}^N$. *Then, for each* $\lambda\in(\lambda_1,\lambda_2)$ *with*

$$\lambda_1:=\frac{(1+K)DM}{2\limsup\limits_{\xi\to+\infty}\frac{\int_{t_1}^{t_p}F(t,\xi\varepsilon)dt}{\xi^2}}\quad \text{and}\quad \lambda_2:=\frac{(1-K)(\frac{1}{c})^2}{2\liminf\limits_{\xi\to+\infty}\frac{\int_0^T\max_{|x|\leqslant\xi}F(t,x)dt}{\xi^2}},$$

and for every arbitrary non-negative function $G:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ *that is measurable with respect to* t *for all* $x\in\mathbb{R}^N$, *continuously differentiable in* x *for almost every* $t\in[0,T]$, *and satisfies*

$$G_\infty:=\frac{2}{(1-K)(\frac{1}{c})^2}\lim_{\xi\to+\infty}\frac{\int_0^T\max_{|x|\leqslant\xi}G(t,x)dt}{\xi^2}<+\infty,\tag{3.1}$$

and for every $\mu\in[0,\mu_{G,\lambda})$, *where* $\mu_{G,\lambda}:=\frac{1}{G_\infty}\left(1-\frac{\lambda}{\lambda_2}\right)$, *the problem* (1.1) *has an unbounded sequence of classical periodic solutions.*

Proof. Our goal is to apply Theorem 1. Fix $\overline{\lambda}\in(\lambda_1,\lambda_2)$ and let G be a function satisfying condition (3.1). Since, $\overline{\lambda}<\lambda_2$, we have $\mu_{G,\overline{\lambda}}>0$. Fix $\overline{\mu}\in[0,\mu_{G,\overline{\lambda}})$ and set $\nu_1:=\lambda_1$ and $\nu_2:=\frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2G_\infty}$. If $G_\infty=0$, then clearly $\nu_1=\lambda_1$, $\nu_2=\lambda_2$, and $\overline{\lambda}\in(\nu_1,\nu_2)$. If $G_\infty\neq0$, since $\overline{\mu}<\mu_{G,\overline{\lambda}}$, we obtain $\frac{\overline{\lambda}}{\lambda_2}+\overline{\mu}G_\infty<1$, and so $\frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2G_\infty}>\overline{\lambda}$ and $\overline{\lambda}<\nu_2$. Since $\overline{\lambda}>\lambda_1=\nu_1$, we see that $\overline{\lambda}\in(\nu_1,\nu_2)$.

Now set $Q(t,\xi)=F(t,\xi)+\frac{\overline{\mu}}{\overline{\lambda}}G(t,\xi)$ for all $(t,\xi)\in[0,T]\times\mathbb{R}^N$. Take $X=E$ and consider the functionals Φ, $\Psi:X\to\mathbb{R}$ defined by

$$\Phi(u)=\frac{1}{2}\|u\|^2+\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i(t_j)}I_{ij}(s)ds-\int_0^T H(u(t))dt$$

and

$$\Psi(u)=\int_0^T Q(t,u(t))dt$$

for every $u\in X$. It is well known that Ψ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u\in X$ is the functional $\Psi'(u)\in X^*$ given by

$$\Psi'(u)v=\int_0^T\left(\nabla F(t,u(t))+\frac{\overline{\mu}}{\overline{\lambda}}\nabla G(t,u(t)),v(t)\right)dt \tag{3.2}$$

for every $v\in X$. The functional Φ is also Gâteaux differentiable with Gâteaux derivative $\Phi'(u)\in X^*$ at the point $u\in X$ given by

$$\Phi'(u)v=\int_0^T\Big[(\dot u(t),\dot v(t))+(A(t)u(t),v(t))-(\nabla H(u(t)),v(t))\Big]dt \\ +\sum_{j=1}^{p}\sum_{i=1}^{N}I_{ij}(u_i(t_j))v_i(t_j) \tag{3.3}$$

for every $v\in X$.

Now, Φ is sequentially weakly lower semicontinuous. To see this, let $u_n\in X$ with $u_n\to u$ weakly in X, and using the sequential weakly lower semicontinuity of the norm, we have $\liminf_{n\to+\infty}||u_n||\geqslant||u||$ and $u_n\to u$ uniformly on $[0,T]$. Hence, since H is continuous,

$$\liminf_{n\to+\infty}\left(\frac{1}{2}\|u_n\|^2+\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_{n_i}(t_j)}I_{ij}(s)ds-\int_0^T H(u_n(t))dt\right) \\ \geqslant\frac{1}{2}\|u\|^2+\sum_{j=1}^{p}\sum_{i=1}^{N}\int_0^{u_i(t_j)}I_{ij}(s)ds-\int_0^T H(u(t))dt,$$

i.e., $\liminf_{n\to+\infty}\Phi(u_n)\geqslant\Phi(u)$. This implies Φ is sequentially weakly lower semicontinuous.

From the definition of Φ, since $(X,\|\cdot\|)$ is compactly embedded in $C([0,T],\mathbb{R}^N)$, we observe that Φ is strongly continuous. Since $-L|x|^2\leqslant H(x)\leqslant L|x|^2$ for every $x\in\mathbb{R}^N$, and $-L_{ij}|s|\leqslant I_{ij}(s)\leqslant L_{ij}|s|$ for every $s\in\mathbb{R}$ for all $i=1,2,\dots,N$ and $j=1,2,\dots,p$, in view of (2.2), we see that

$$\frac{1}{2}(1-K)\|u\|^2\leqslant\Phi(u)\leqslant\frac{1}{2}(1+K)\|u\|^2. \tag{3.4}$$

This also shows that Φ is coercive. We want to verify that $\gamma<+\infty$, where γ is defined in Theorem 1. Let $\{\xi_n\}$ be a real sequence such that $\xi_n\to+\infty$ as $n\to+\infty$ and

$$\lim_{n\to\infty}\frac{\int_0^T\max_{|x|\leqslant\xi_n}F(t,x)dt}{\xi_n^2}=\liminf_{\xi\to+\infty}\frac{\int_0^T\max_{|x|\leqslant\xi}F(t,x)dt}{\xi^2}.$$

Set $r_n = \frac{1}{2}(1-K)(\frac{\xi_n}{c})^2$ for all $n \in \mathbb{N}$. From inequalities (2.2) and (3.4), for each $u \in X$, we have

$$\begin{aligned}\Phi^{-1}(-\infty, r_n] &= \{u \in X : \ \Phi(u) < r_n\} \\ &\subseteq \left\{u \in X : \ \frac{1}{2}(1-K)\|u\|^2 < r_n\right\} \\ &\subseteq \{u \in X : \ |u(t)| \leqslant \xi_n \text{ for each } t \in [0,T]\}.\end{aligned}$$

Hence, since $\Phi(0) = \Psi(0) = 0$, for large n,

$$\begin{aligned}\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n]}\Psi(v) - \Psi(u)}{r_n - \Phi(u)} \leqslant \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n]}\Psi(v)}{r_n} \\ &\leqslant \frac{\int_0^T \max_{|x|\leqslant\xi_n} Q(t,x)dt}{\frac{1}{2}(1-K)(\frac{\xi_n}{c})^2} \leqslant \frac{\int_0^T \max_{|x|\leqslant\xi_n} F(t,x)dt}{\frac{1}{2}(1-K)(\frac{\xi_n}{c})^2} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_0^T \max_{|x|\leqslant\xi_n} G(t,x)dt}{\frac{1}{2}(1-K)(\frac{\xi_n}{c})^2}.\end{aligned}$$

Moreover, from (a_2), it follows that

$$\liminf_{\xi\to+\infty} \frac{\int_0^T \max_{|x|\leqslant\xi} F(t,x)dt}{\xi^2} < +\infty,$$

so

$$\lim_{n\to\infty} \frac{\int_0^T \max_{|x|\leqslant\xi_n} F(t,x)dt}{\xi_n^2} < +\infty. \tag{3.5}$$

Then, (3.1) and (3.5) imply

$$\lim_{n\to\infty} \frac{\int_0^T \max_{|x|\leqslant\xi_n}[F(t,x) + \frac{\overline{\mu}}{\overline{\lambda}}G(t,x)]dt}{\frac{1}{2}(1-K)(\frac{\xi_n}{c})^2} < +\infty.$$

Therefore,

$$\gamma \leqslant \liminf_{n\to+\infty} \varphi(r_n) \leqslant \lim_{n\to\infty} \frac{\int_0^T \max_{|x|\leqslant\xi_n}[F(t,x) + \frac{\overline{\mu}}{\overline{\lambda}}G(t,x)]dt}{\frac{1}{2}(1-K)(\frac{\xi_n}{c})^2} < +\infty. \tag{3.6}$$

In view of (3.1), we see that

$$\liminf_{\xi\to+\infty} \frac{\int_0^T \max_{|x|\leqslant\xi} Q(t,x)dt}{\frac{1}{2}(1-K)(\frac{\xi}{c})^2} \leqslant \liminf_{\xi\to+\infty} \frac{\int_0^T \max_{|x|\leqslant\xi} F(t,x)dt}{\frac{1}{2}(1-K)(\frac{\xi}{c})^2} + \frac{\overline{\mu}}{\overline{\lambda}} G_\infty. \tag{3.7}$$

Moreover, since G is non-negative, we have

$$\limsup_{\xi\to+\infty} \frac{\int_{t_1}^{t_p} Q(t,\xi\varepsilon)dt}{(1+K)DM\xi^2} \geqslant \limsup_{\xi\to+\infty} \frac{\int_{t_1}^{t_p} F(t,\xi\varepsilon)dt}{(1+K)DM\xi^2}. \tag{3.8}$$

Therefore, from (3.6)–(3.8) and condition (a_2), we have

$$\overline{\lambda} \in (\nu_1, \nu_2) \subseteq \left(\frac{1}{\limsup\limits_{\xi \to +\infty} \frac{\int_{t_1}^{t_p} Q(t, \xi\varepsilon)dt}{\frac{1}{2}(1+K)DM\xi^2}}, \frac{1}{\liminf\limits_{\xi \to +\infty} \frac{\int_0^T \max_{|x| \leqslant \xi} Q(t,x)dt}{\frac{1}{2}(1-K)(\frac{\xi}{c})^2}} \right) \subseteq \left(0, \frac{1}{\gamma}\right).$$

For the fixed $\overline{\lambda}$, the inequality (3.6) implies that condition (b) of Theorem 1 can be applied, and so either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of weak solutions of the problem (1.1) such that $\lim_{n \to +\infty} \|u_n\| = +\infty$.

Next, we show that for the fixed $\overline{\lambda}$, the functional $I_{\overline{\lambda}}$ has no global minimum. To do this, we will show that the functional $I_{\overline{\lambda}}$ is unbounded from below. Since

$$\frac{1}{\overline{\lambda}} < 2 \limsup_{\xi \to +\infty} \frac{\int_{t_1}^{t_p} F(t, \xi\varepsilon)dt}{(1+K)DM\xi^2},$$

we can consider a real sequence $\{d_n\}$ and a positive constant τ such that $d_n \to +\infty$ as $n \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{2\int_{t_1}^{t_p} F(t, d_n\varepsilon)dt}{(1+K)DMd_n^2} \tag{3.9}$$

for large $n \in \mathbb{N}$. Let $\{w_n\}$ be a sequence in X defined by

$$w_n(t) = \begin{cases} (T + \frac{t_p - T}{t_1} t)\frac{d_n\varepsilon}{t_p}, & t \in [0, t_1), \\ d_n\varepsilon, & t \in [t_1, t_p], \\ \frac{d_n\varepsilon}{t_p} t, & t \in (t_p, T]. \end{cases} \tag{3.10}$$

It is clear that $w_n \in X$ for all $n \in \mathbb{N}$, and $\|w_n\|_E^2 = Dd_n^2$. Therefore, from (2.1),

$$Dmd_n^2 \leqslant \|w\|^2 \leqslant DMd_n^2, \tag{3.11}$$

which together with (3.4) gives

$$\Phi(w_n) \leqslant \frac{1}{2}(1+K)DMd_n^2. \tag{3.12}$$

On the other hand, since G is non-negative, from the definition of Ψ and (a_1), we see that

$$\Psi(w_n) \geqslant \int_{t_1}^{t_p} F(t, d_n\varepsilon)dt. \tag{3.13}$$

So (3.9), (3.12), and (3.13) imply

$$\begin{aligned} I_{\overline{\lambda}}(w_n) &= \Phi(w_n) - \overline{\lambda}\Psi(w_n) \\ &\leqslant \frac{1}{2}(1+K)DMd_n^2 - \overline{\lambda} \int_{t_1}^{t_p} F(t, d_n\varepsilon)dt \\ &< (1 - \overline{\lambda}\tau)\frac{1}{2}(1+K)DMd_n^2 \to -\infty \end{aligned}$$

as $n \to \infty$. Hence, the functional $I_{\overline{\lambda}}$ has no global minimum.

Therefore, applying Theorem 1, we conclude that there is a sequence $\{u_n\} \subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to+\infty} \Phi(u_n) = +\infty$, and from (3.4) it follows that $\lim_{n\to+\infty} \|u_n\| = +\infty$. In view of Definition 2, (3.2), and (3.3), we see that weak solutions of the problem (1.1) are exactly the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$. Hence, by Lemma 1, the conclusion of the theorem follows. $\square$

REMARK 1. Under the conditions

$$\liminf_{\xi\to+\infty} \frac{\int_0^T \max_{|x|\leqslant\xi} F(t,x)dt}{\xi^2} = 0$$

and

$$\limsup_{\xi\to+\infty} \frac{\int_{t_1}^{t_p} F(t,\xi\varepsilon)dt}{\xi^2} = +\infty,$$

where $\varepsilon = (1,0,\cdots,0) \in \mathbb{R}^N$, Theorem 3 ensures that for every $\lambda > 0$ and for each $\mu \in [0, \frac{1}{G_\infty})$, the problem (1.1) admits infinitely many classical periodic solutions. Moreover, if $G_\infty = 0$, then the result holds for every $\lambda > 0$ and $\mu \geqslant 0$.

We now exhibit two examples for which the hypotheses of Theorem 3 are satisfied.

EXAMPLE 1. Let $N = 1$, $p = 2$, $T = 3$, $t_1 = 1$, $t_2 = 2$, and define the sequences $\{a_n\}$ and $\{b_n\}$ by $b_1 = 2$, $b_{n+1} = b_n^6$, and $a_n = b_n^4$ for $n \in \mathbb{N}$. Let

$$f(\xi) = \begin{cases} b_1^3\sqrt{1-(1-\xi)^2}+1, & \text{if } \xi \in [0,b_1], \\ (a_n - b_n^3)\sqrt{1-(a_n-1-\xi)^2}+1, & \text{if } \xi \in \cup_{n=1}^{\infty}[a_n-2,a_n], \\ (b_{n+1}^3 - a_n)\sqrt{1-(b_{n+1}-1-\xi)^2}+1, & \text{if } \xi \in \cup_{n=1}^{\infty}[b_{n+1}-2,b_{n+1}], \\ 1, & \text{otherwise,} \end{cases}$$

$h(\xi) = \frac{1}{120}\frac{\xi(2+|\xi|)}{(1+|\xi|)^2}$ for every $\xi \in \mathbb{R}$, and $I_j(x) = \frac{1}{36}x(1+\sin x)$ for $j = 1,2$. Let $F(\xi) = \int_0^\xi f(x)dx$ and $H(\xi) = \int_0^\xi h(x)dx = \frac{1}{120}\frac{\xi^2}{1+|\xi|}$ for all $\xi \in \mathbb{R}$. Then, F is a C^1 function with $F' = f$. From the computation in [21, Example 3.1], we have $F(a_n) = \frac{\pi}{2}a_n + a_n$ and $F(b_n) = \frac{\pi}{2}b_n^3 + b_n$, and so $\lim_{n\to+\infty} \frac{F(a_n)}{a_n^2} = 0$ and $\lim_{n\to+\infty} \frac{F(b_n)}{b_n^2} = +\infty$. Therefore, $\liminf_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = 0$ and $\limsup_{\xi\to+\infty} \frac{F(\xi)}{\xi^2} = +\infty$. Let $g(t,\xi) = e^{t-\xi^+}(\xi^+)^2(3-\xi^+)$, where $\xi^+ = \max\{\xi,0\}$; then we see that $G(t,\xi) = \int_0^\xi g(t,x)dx = e^{t-\xi^+}(\xi^+)^3$ for all $(t,\xi) \in [0,3]\times\mathbb{R}$ and $G_\infty = 0$. We have $m = 1$, $M = 2$, $L = \frac{1}{120}$, $L_j = \frac{1}{18}$ for $j = 1,2$, $c = \sqrt{6}$, and $K = \frac{29}{30}$. Hence, applying Theorem 3, for every $(\lambda,\mu) \in (0,+\infty)\times[0,+\infty)$, the problem

$$\begin{cases} -u''(t) + u(t) = \lambda f(u(t)) + \mu g(t,u(t)) + h(u(t)), & a.e.\ t \in [0,3], \\ \Delta(u'(t_j)) = I_j(u(t_j)), \quad j = 1,2, \\ u(0) - u(3) = u'(0) - u'(3) = 0, \end{cases}$$

admits an unbounded sequence of classical periodic solutions.

EXAMPLE 2. Let $N=3$, $p=2$, $T=3$, $t_1=1$, $t_2=2$, and $A:[0,3]\to\mathbb{R}^{3\times 3}$ be the identity matrix. Let $G:[0,3]\times\mathbb{R}^3\to\mathbb{R}$ be a nonnegative function that is measurable with respect to t for all $x=(x_1,x_2,x_3)\in\mathbb{R}^3$, continuously differentiable in x for almost every $t\in[0,3]$, and satisfies

$$\sup_{|x|\leqslant a}\max\{|G(\cdot,x)|,\ |\nabla G(\cdot,x)|\}\in L^1([0,3])$$

for any $a>0$, and

$$\lim_{\xi\to+\infty}\xi^{-2}\int_0^3\sup_{|x|\leqslant\xi}G(t,x)dx<+\infty.$$

Set $a_k:=\frac{2k!(k+2)!-1}{4(k+1)!}$ and $b_k:=\frac{2k!(k+2)!+1}{4(k+1)!}$ for every $k\in\mathbb{N}$. Define the function f_1 : $\mathbb{R}\to\mathbb{R}$ by

$$f_1(s)=\begin{cases}\frac{32(k+1)!^2[k^3(k+1)!^4-(k-1)^3k!^4]}{\pi}\sqrt{\frac{1}{16(k+1)!^2}-(t-\frac{k!(k+2)}{2})^2}+1, & s\in\bigcup_{k\in\mathbb{N}}[a_k,b_k],\\ 1, & \text{otherwise.}\end{cases}$$

Now let $F(t,x_1,x_2,x_3)=\alpha(t)F_1(x_1)F_2(x_2,x_3)$ for all $(t,x_1,x_2,x_3)\in[0,3]\times\mathbb{R}^3$, where $\alpha\in L^1([0,3])$ is a positive function, $F_1(\xi)=\int_0^\xi f_1(s)ds$ for all $\xi\in\mathbb{R}$, and let $F_2:\mathbb{R}^2\to\mathbb{R}$ be a non-negative bounded and continuously differentiable function with $F_2(0,0)\neq 0$. Take $H(x)=\frac{1}{120}\frac{|x|^2}{1+|x|^2}$ for every $x\in\mathbb{R}^3$, $I_{i1}(t)=\frac{1}{360}t(1+\sin t)$, and $I_{i2}(t)=\arctan\frac{t}{180}$ for every $t\in\mathbb{R}$ and $i=1,2,3$. We then have

$$\lim_{k\to+\infty}\frac{F_1(b_k)F_2(0,0)}{b_k^2}=4F_2(0,0)$$

and

$$\lim_{k\to+\infty}\frac{F_1(a_k)F_2(a_k,a_k)}{a_k^2}=0.$$

Therefore,

$$\liminf_{\xi\to+\infty}\frac{\int_0^3\max_{|x|\leqslant\xi}F(t,x)dt}{\xi^2}=\|\alpha\|_{L^1([0,3])}\liminf_{\xi\to+\infty}\frac{\max_{|x|\leqslant\xi}(F_1(x_1)F_2(x_2,x_3))}{\xi^2}=0$$

and

$$\limsup_{\xi\to+\infty}\frac{\int_{t_1}^{t_p}F(t,\xi\varepsilon)dt}{\xi^2}=\|\alpha\|_{L^1([1,2])}\limsup_{\xi\to+\infty}\frac{F_1(\xi)F_2(0,0)}{\xi^2}=4F_2(0,0)\|\alpha\|_{L^1([1,2])}.$$

We see that $m=1$, $M=3$, $c=\sqrt{6}$, $L=\frac{1}{120}$, $L_{i1}=\frac{1}{180}$ and $L_{i2}=\frac{1}{180}$ for $i=1,2,3$, $D=\frac{7}{2}$, and $K=\frac{1}{2}$. Applying Theorem 3, for every $\lambda>\lambda_1=\frac{7.875}{F_2(0,0)\|\alpha\|_{L^1([1,2])}}$ and for every μ in a convenient interval, the problem (1.1) has an unbounded sequence of classical periodic solutions.

REMARK 2. Assumption (a_2) in Theorem 3 can be replaced by the more general condition

(a_2') there exist two sequence $\{\theta_n\}$ and $\{\eta_n\}$ with $(1-K)(\frac{\eta_n}{c})^2 > (1+K)DM\theta_n^2$ for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty}\eta_n = +\infty$ such that

$$\lim_{n\to+\infty}\frac{\int_0^T \max_{|x|\leqslant\eta_n} F(t,x)dt - \int_{t_1}^{t_p} F(t,\theta_n\varepsilon)dt}{(1-K)(\frac{\eta_n}{c})^2 - (1+K)DM\theta_n^2} < \limsup_{|\xi|\to+\infty}\frac{\int_{t_1}^{t_p} F(t,\xi\varepsilon)dt}{(1+K)DM\xi^2},$$

where $\varepsilon = (1,0,\cdots,0) \in \mathbb{R}^N$.

By choosing $\theta_n = 0$ for all $n \in \mathbb{N}$, (a_2) follows from (a_2'). Moreover, if we assume (a_2') instead of (a_2) and set $r_n = \frac{1}{2}(1-K)(\frac{\eta_n}{c})^2$ for all $n \in \mathbb{N}$, by the same reasoning as in the proof of Theorem 3, we obtain

$$\begin{aligned}\varphi(r_n) &\leqslant \frac{\sup_{v\in\Phi^{-1}(-\infty,r_n]}\Psi(v) - \int_0^T F(t,z_n(t))dt}{r_n - \Phi(z_n)}\\ &\leqslant \frac{\int_0^T \max_{|x|\leqslant\eta_n} Q(t,x)dt - \int_{t_1}^{t_p} F(t,\theta_n\varepsilon)dt}{\frac{1}{2}(1-K)(\frac{\eta_n}{c})^2 - \frac{1}{2}(1+K)DM\theta_n^2},\end{aligned}$$

where

$$z_n(t) = \begin{cases} (T + \frac{t_p-T}{t_1}t)\frac{\theta_n\varepsilon}{t_p}, & t\in[0,t_1),\\ \theta_n\varepsilon, & t\in[t_1,t_p],\\ \frac{\theta_n\varepsilon}{t_p}t, & t\in(t_p,T].\end{cases}$$

We have the same conclusion as in Theorem 3 with the interval (λ_1,λ_2) replaced by (λ_1',λ_2'), where

$$\lambda_1' := \frac{(1+K)DM}{2\limsup\limits_{\xi\to+\infty}\frac{\int_{t_1}^{t_p}F(t,\xi\varepsilon)dt}{\xi^2}} \quad \lambda_2' := \left[\lim_{n\to+\infty}\frac{\int_0^T \max_{|x|\leqslant\eta_n} F(t,x)dt - \int_{t_1}^{t_p} F(t,\theta_n\varepsilon)dt}{\frac{1}{2}(1-K)(\frac{\eta_n}{c})^2 - \frac{1}{2}(1+K)DM\theta_n^2}\right]^{-1}.$$

Next, we point out a simple consequence of Theorem 3.

COROLLARY 1. *Assume that (A1), (A2), and (a_1) hold,*

(b_1) $$\liminf_{\xi\to+\infty}\frac{\int_0^T \max_{|x|\leqslant\xi} F(t,x)dt}{\xi^2} < \frac{1}{2}(1-K)\frac{1}{c^2},$$

and

(b_2) $$\limsup_{\xi\to+\infty}\frac{\int_{t_1}^{t_p} F(t,\xi\varepsilon)dt}{\xi^2} > \frac{1}{2}(1+K)DM,$$

where $\varepsilon = (1,0,\cdots,0) \in \mathbb{R}^N$. *Then, for every arbitrary non-negative function* $G : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ *that is measurable with respect to* t *for all* $x\in\mathbb{R}^N$, *continuously differentiable in* x *for almost every* $t\in[0,T]$, *and satisfies condition* (3.1), *and for every* $\mu\in[0,\mu_{g,1})$, *where*

$$\mu_{g,1} := \frac{1}{G_\infty}\left(1-\frac{1}{\lambda_2}\right)$$

and λ_2 *is given in the statement of Theorem* 3, *the problem*

$$\begin{cases} -\ddot{u}(t)+A(t)u(t) = \nabla F(t,u(t))+\mu\nabla G(t,u(t))+\nabla H(u(t)), & a.e.\ t\in[0,T],\\ \Delta(\dot{u}_i(t_j)) = I_{ij}(u_i(t_j)), \quad i=1,2,\dots,N,\ j=1,2,\dots,p, \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0, \end{cases}$$

has an unbounded sequence of classical periodic solutions.

REMARK 3. Theorem 2 in the previous section is an immediate consequence of Corollary 1 with $\mu = 0$.

Arguing as in the proof of Theorem 3, but using conclusion (c) of Theorem 1 instead of (b), the following result holds.

THEOREM 4. *Let (A1), (A2), and* (a_1) *hold. Assume that*

$$(c_1)\quad \liminf_{\xi\to0^+}\frac{\int_0^T \max_{|x|\leqslant\xi}F(t,x)dt}{\xi^2} < \frac{1-K}{(1+K)DMc^2}\limsup_{\xi\to0^+}\frac{\int_{t_1}^{t_p}F(t,\xi\varepsilon)dt}{\xi^2}$$

where $\varepsilon=(1,0,\cdots,0)\in\mathbb{R}^N$. *Then, for each* $\lambda\in(\lambda_3,\lambda_4)$, *where*

$$\lambda_3 := \frac{(1+K)DM}{2\limsup\limits_{\xi\to0^+}\frac{\int_{t_1}^{t_p}F(t,\xi\varepsilon)dt}{\xi^2}} \quad \text{and} \quad \lambda_4 := \frac{(1-K)(\frac{1}{c})^2}{2\liminf\limits_{\xi\to0^+}\frac{\int_0^T\max_{|x|\leqslant\xi}F(t,x)dt}{\xi^2}},$$

for every arbitrary non-negative function $G:[0,T]\times\mathbb{R}^N\to\mathbb{R}$ *that is measurable with respect to* t *for all* $x\in\mathbb{R}^N$, *continuously differentiable in* x *for almost every* $t\in[0,T]$, *and satisfies*

$$G_0 := \frac{2}{(1-K)(\frac{1}{c})^2}\lim_{\xi\to0^+}\frac{\int_0^T\max_{|x|\leqslant\xi}G(t,x)dt}{\xi^2} < +\infty, \tag{3.14}$$

and for every $\mu\in[0,\mu'_{G,\lambda})$, *where* $\mu'_{G,\lambda} := \frac{1}{G_0}\left(1-\frac{\lambda}{\lambda_4}\right)$, *the problem* (1.1) *has a sequence of classical periodic solutions that converges uniformly to* 0 *in* $[0,T]$.

Proof. Fix $\overline{\lambda}\in(\lambda_3,\lambda_4)$ and let G be a function satisfying condition (3.14). Since $\overline{\lambda}<\lambda_4$, we have

$$\mu'_{G,\overline{\lambda}} := \frac{1}{G_0}\left(1-\frac{\overline{\lambda}}{\lambda_4}\right) > 0.$$

Fix $\overline{\mu} \in [0, \mu'_{G,\overline{\lambda}})$ and set $\nu_3 := \lambda_3$ and $\nu_4 := \frac{\lambda_4}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_4 G_0}$. If $G_0 = 0$, clearly, $\nu_3 = \lambda_3$, $\nu_4 = \lambda_4$, and $\overline{\lambda} \in (\nu_3, \nu_4)$. If $G_0 \neq 0$, since $\overline{\mu} < \mu'_{G,\overline{\lambda}}$, we have $\frac{\overline{\lambda}}{\lambda_4} + \overline{\mu} G_0 < 1$, and so $\frac{\lambda_4}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_4 G_0} > \overline{\lambda}$ and $\overline{\lambda} < \nu_4$. Hence, since $\overline{\lambda} > \lambda_3 = \nu_3$, $\overline{\lambda} \in (\nu_3, \nu_4)$. Now, set $Q(t,\xi) = F(t,\xi) + \frac{\overline{\mu}}{\overline{\lambda}} G(t,\xi)$ for all $(t,\xi) \in [0,T] \times \mathbb{R}^N$. Take X, Φ, Ψ, and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.

To see that $\delta < +\infty$, let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n\to\infty} \frac{\int_0^T \max_{|x|\leqslant \xi_n} F(t,x)dt}{\xi_n^2} < +\infty.$$

Setting $r_n = \frac{1}{2}(1 - 2LTc^2 - c^2 \sum_{j=1}^p \sum_{i=1}^N L_{ij})(\frac{\xi_n}{c})^2$ for all $n \in \mathbb{N}$, and arguing as in the proof of Theorem 3 shows that $\delta < +\infty$.

Since

$$\frac{\int_0^T \max_{|x|\leqslant \xi} Q(t,x)dt}{\xi^2} \leqslant \frac{\int_0^T \max_{|x|\leqslant \xi} F(t,x)dt}{\xi^2} + \frac{\overline{\mu}}{\overline{\lambda}} \frac{\int_0^T \max_{|x|\leqslant \xi} G(t,x)dt}{\xi^2},$$

and (3.14) holds, we have

$$\liminf_{\xi\to 0^+} \frac{\int_0^T \max_{|x|\leqslant \xi} Q(t,x)dt}{\frac{1}{2}(1-K)(\frac{\xi}{c})^2} \leqslant \liminf_{\xi\to 0^+} \frac{\int_0^T \max_{|x|\leqslant \xi} F(t,x)dt}{\frac{1}{2}(1-K)(\frac{\xi}{c})^2} + \frac{\overline{\mu}}{\overline{\lambda}} G_0. \tag{3.15}$$

Moreover, since G is nonnegative, (c_1) implies

$$\limsup_{\xi\to 0^+} \frac{\int_{t_1}^{t_p} Q(t,\xi\varepsilon)dt}{\xi^2} \geqslant \limsup_{\xi\to 0^+} \frac{\int_{t_1}^{t_p} F(t,\xi\varepsilon)dt}{\xi^2}. \tag{3.16}$$

Therefore, from (3.15) and (3.16),

$$\overline{\lambda} \in (\nu_3, \nu_4) \subseteq \left(\frac{(1+K)DM}{2\limsup\limits_{\xi\to 0^+} \frac{\int_{t_1}^{t_p} Q(t,\xi\varepsilon)dt}{\xi^2}}, \frac{(1-K)(\frac{1}{c})^2}{2\liminf\limits_{\xi\to 0^+} \frac{\int_0^T \max_{|x|\leqslant \xi} Q(t,x)dt}{\xi^2}} \right) \subseteq \left(0, \frac{1}{\delta}\right).$$

We need to show that the functional $I_{\overline{\lambda}}$ does not have a local minimum at zero. To do this, let $\{d_n\}$ be a sequence of positive numbers and $\tau > 0$ be such that $d_n \to 0^+$ as $n \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{2\int_{t_1}^{t_p} F(t,d_n\varepsilon)dt}{(1+K)DMd_n^2} \tag{3.17}$$

for large $n \in \mathbb{N}$. Let $\{w_n\}$ be a sequence in X defined as in (3.10). From (3.12), (3.13), and (3.17), we obtain

$$\begin{aligned} I_{\overline{\lambda}}(w_n) &= \Phi(w_n) - \overline{\lambda}\Psi(w_n) \\ &\leqslant \frac{1}{2}(1+K)DMd_n^2 - \overline{\lambda}\int_{t_1}^{t_p} F(t, d_n\varepsilon)dt \\ &< (1-\overline{\lambda}\tau)\frac{1}{2}(1+K)DMd_n^2 \end{aligned}$$

for large $n \in \mathbb{N}$. Since $I_{\overline{\lambda}}(0) = 0$ and (3.17) holds, the functional $I_{\overline{\lambda}}$ does not have a local minimum at zero. Hence, part (c) of Theorem 1 implies there is a sequence $\{u_n\}$ in X of critical points of $I_{\overline{\lambda}}$ such that $||u_n|| \to 0$ as $n \to \infty$. Since the solutions of the equation $\Phi'(u) - \lambda\Psi'(u) = 0$ are exactly the weak solutions of the problem (1.1), by Lemma 1 we have the conclusion of the theorem. □

We conclude this paper with an example for which the hypotheses of Theorem 4 are satisfied.

EXAMPLE 3. Consider the system

$$\begin{cases} -\ddot{u}(t) + A(t)u(t) = \lambda\nabla F(t,u(t)) + \mu\nabla G(t,u(t)) + \nabla H(u(t)), & a.e.\ t \in [0,3], \\ \Delta(\dot{u}_i(t_j)) = I_{ij}(u_i(t_j)), \quad i = 1,2,\ j = 1,2, \\ u(0) - u(3) = \dot{u}(0) - \dot{u}(3) = 0, \end{cases} \tag{3.18}$$

where $A : [0,3] \to \mathbb{R}^{2\times 2}$ is the identity matrix, $N = p = 2$, $t_1 = 1$, $t_2 = 2$, and $F(t,x) = F(x)$ for every $(t,\xi) \in [0,3] \times \mathbb{R}^2$ is defined by

$$F(x) = \begin{cases} |x|^2\Big(1 - \sin(\ln(|x|))\Big), & \text{if } x = (x_1,x_2) \in (\mathbb{R} - \{0\})^2, \\ 0, & \text{if } x = (x_1,x_2) = (0,0). \end{cases}$$

In addition, let $H(x) = \frac{|x|^2}{108(1+|x|^4)}$, $G(t,x) = t|x|^{\frac{5}{4}}$ for $(t,x) \in [0,3] \times \mathbb{R}^2$, and $I_{ij}(s) = \frac{1}{288}s(1+\sin s)$ for $s \in \mathbb{R}$, $i = 1,2$, and $j = 1,2$.

Since $m = 1$, $M = 2$, $c = \sqrt{6}$, $L = \frac{1}{108}$, $Lij = \frac{1}{288}$ for $i = 1,2$, $j = 1,2$, $D = \frac{7}{2}$, $K = \frac{5}{12}$,

$$\liminf_{\xi\to 0^+} \frac{\max_{|x|\leqslant\xi} F(x)}{\xi^2} = 0, \quad \limsup_{\xi\to 0^+} \frac{F(\xi,0)}{\xi^2} = 2,$$

and $G_0 = 0$, we see that all the conditions of Theorem 4 are satisfied. Hence, for every $(\lambda,\mu) \in (\lambda_3, +\infty) \times [0,+\infty)$, where $\lambda_3 \approx 2.47$, system (3.18) has an unbounded sequence of classical periodic solutions that converges uniformly to 0 in E.

REMARK 4. Applying Theorem 4, results similar to Corollary 1 can be obtained. We leave their formulation to the reader.

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