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ON A GENERALIZATION OF THE LIOUVILLE FORMULA

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Abstract. In this article, we obtain a new formula which generalizes the Liouville formula of the linear differential system to nonlinear differential system. We establish the relationship between the Jacobi determinant of the first integral and the trace of Jacobi matrix of the n-dimensional vector field.

1. Introduction

From literatures [1, 2, 3, 4, 5], we know that for linear differential system

$$x' = A(t)x, \quad x \in R^n, \quad A(t) = (a_{ij}(t))_{n\times n}, \tag{1.1}$$

if $\Phi(t)$ is a fundamental matrix of (1.1), $W(t) := \det\Phi(t)$ is the Wronskian determinant of $\Phi(t)$, then

$$W'(t) = W(t)\,\mathrm{tr}A(t), \tag{1.2}$$

this is called *Liouville formula*.

From this formula, even if we can't find out the expression of the fundamental matrix $\Phi(t)$ of (1.1), we also know its geometric properties, such as its monotone, its value, and so on. For the general nonlinear differential system, whether is there such kind of formulas? In the following, we will give a quite new formula for nonlinear differential system, which generalizes the Liouville formula (1.2).

2. Main result

Consider differential system

$$\frac{dx}{dt} = X(t,x), \quad x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}, \quad X(t,x) = \begin{pmatrix} X_1(t,x) \\ X_2(t,x) \\ \cdots \\ X_n(t,x) \end{pmatrix}, \quad t \in R,\ x \in R^n, \tag{2.1}$$

which has a continuously differentiable right-hand side.

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THEOREM 1. *Suppose that*

$$u_i(t,x(t)) = c_i,\ (i=1,2,\ldots,n)$$

are the independent first integrals of (2.1), $c_i\ \ (i=1,2,\ldots,n)$ *are constants,* $U := (u_1,u_2,\ldots,u_n)^T$, $M(t,x) := \det U_x$, *where* $(U_x)_{ij} = u_{ix_j}$. *Then*

$$\frac{\partial M(t,x)}{\partial t} + \sum_{j=1}^{n} \frac{\partial M(t,x)}{\partial x_j} X_j(t,x) + M(t,x)\,\mathrm{tr}\,\frac{\partial X(t,x)}{\partial x} = 0, \tag{2.2}$$

where $\mathrm{tr}\,\frac{\partial X(t,x)}{\partial x} = \sum_{j=1}^{n} \frac{\partial X_j}{\partial x_j}$.

Proof. As $u_i(t,x_1(t),x_2(t),\ldots,x_n(t)) = c_i,\ (i=1,2,\ldots,n)$ are the first integrals of (2.1), then

$$u_{it}(t,x) + \sum_{j=1}^{n} u_{ix_j}(t,x)X_j(t,x) = 0, \quad (i=1,2,\ldots,n). \tag{2.3}$$

In equation (2.3), taking a derivative with respect to x_k, we get

$$u_{itx_k}(t,x) + \sum_{j=1}^{n} u_{ix_jx_k}(t,x)X_j(t,x) = -\sum_{j=1}^{n} u_{ix_j}X_{jx_k}, \quad (i,k=1,2,\ldots,n) \tag{2.4}$$

On the other hand,

$$\begin{aligned}
&\frac{\partial M}{\partial t} + \sum_{j=1}^{n} \frac{\partial M}{\partial x_j} X_j \\
&= \begin{vmatrix} u_{1x_1t} & u_{1x_2t} & \ldots & u_{1x_nt} \\ u_{2x_1} & u_{2x_2} & \ldots & u_{2x_n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{nx_1} & u_{nx_2} & \ldots & u_{nx_n} \end{vmatrix} + \begin{vmatrix} u_{1x_1} & u_{1x_2} & \ldots & u_{1x_n} \\ u_{2x_1t} & u_{2x_2t} & \ldots & u_{2x_nt} \\ \ldots & \ldots & \ldots & \ldots \\ u_{nx_1} & u_{nx_2} & \ldots & u_{nx_n} \end{vmatrix} + \ldots + \begin{vmatrix} u_{1x_1} & u_{1x_2} & \ldots & u_{1x_n} \\ u_{2x_1} & u_{2x_2} & \ldots & u_{2x_n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{nx_1t} & u_{nx_2t} & \ldots & u_{nx_nt} \end{vmatrix} \\
&\quad + \sum_{j=1}^{n} X_j \left[\begin{vmatrix} u_{1x_1x_j} & u_{1x_2x_j} & \ldots & u_{1x_nx_j} \\ u_{2x_1} & u_{2x_2} & \ldots & u_{2x_n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{nx_1} & u_{nx_2} & \ldots & u_{nx_n} \end{vmatrix} + \begin{vmatrix} u_{1x_1} & u_{1x_2} & \ldots & u_{1x_n} \\ u_{2x_1x_j} & u_{2x_2x_j} & \ldots & u_{2x_nx_j} \\ \ldots & \ldots & \ldots & \ldots \\ u_{nx_1} & u_{nx_2} & \ldots & u_{nx_n} \end{vmatrix} + \ldots \right. \\
&\quad \left. \ldots + \begin{vmatrix} u_{1x_1} & u_{1x_2} & \ldots & u_{1x_n} \\ u_{2x_1} & u_{2x_2} & \ldots & u_{2x_n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{nx_1x_j} & u_{nx_2x_j} & \ldots & u_{nx_nx_j} \end{vmatrix} \right] \\
&= \begin{vmatrix} u_{1x_1t} + \sum_{j=1}^{n} X_j u_{1x_1x_j} & u_{1x_2t} + \sum_{j=1}^{n} X_j u_{1x_2x_j} & \ldots & u_{1x_nt} + \sum_{j=1}^{n} X_j u_{1x_nx_j} \\ u_{2x_1} & u_{2x_2} & \ldots & u_{2x_n} \\ \ldots & \ldots & \ldots & \ldots \\ u_{nx_1} & u_{nx_2} & \ldots & u_{nx_n} \end{vmatrix}
\end{aligned}$$

$$+\begin{vmatrix} u_{1x_1} & u_{1x_2} & \dots & u_{1x_n} \\ u_{2x_1t}+\sum_{j=1}^n X_j u_{2x_1x_j} & u_{2x_2t}+\sum_{j=1}^n X_j u_{2x_2x_j} & \dots & u_{2x_nt}+\sum_{j=1}^n X_j u_{2x_nx_j} \\ \dots & \dots & \dots & \dots \\ u_{nx_1} & u_{nx_2} & \dots & u_{nx_n} \end{vmatrix}+\dots$$

$$\dots+\begin{vmatrix} u_{1x_1} & u_{1x_2} & \dots & u_{1x_n} \\ u_{2x_1} & u_{2x_2} & \dots & u_{2x_n} \\ \dots & \dots & \dots & \dots \\ u_{nx_1t}+\sum_{j=1}^n X_j u_{nx_1x_j} & u_{nx_2t}+\sum_{j=1}^n X_j u_{nx_2x_j} & \dots & u_{nx_nt}+\sum_{j=1}^n X_j u_{nx_nx_j} \end{vmatrix}. \quad (2.5)$$

Substituting (2.4) into (2.5) we get

$$\frac{\partial M}{\partial t}+\sum_{j=1}^n \frac{\partial M}{\partial x_j} X_j$$

$$=\begin{vmatrix} -\sum_{j=1}^n X_{jx_1}u_{1x_j} & -\sum_{j=1}^n X_{jx_2}u_{1x_j} & \dots & -\sum_{j=1}^n X_{jx_n}u_{1x_j} \\ u_{2x_1} & u_{2x_2} & \dots & u_{2x_n} \\ \dots & \dots & \dots & \dots \\ u_{nx_1} & u_{nx_2} & \dots & u_{nx_n} \end{vmatrix}$$

$$+\begin{vmatrix} u_{1x_1} & u_{1x_2} & \dots & u_{1x_n} \\ -\sum_{j=1}^n X_{jx_1}u_{2x_j} & -\sum_{j=1}^n X_{jx_2}u_{2x_j} & \dots & -\sum_{j=1}^n X_{jx_n}u_{2x_j} \\ \dots & \dots & \dots & \dots \\ u_{nx_1} & u_{nx_2} & \dots & u_{nx_n} \end{vmatrix}+\dots$$

$$\dots+\begin{vmatrix} u_{1x_1} & u_{1x_2} & \dots & u_{1x_n} \\ u_{2x_1} & u_{2x_2} & \dots & u_{2x_n} \\ \dots & \dots & \dots & \dots \\ -\sum_{j=1}^n X_{jx_1}u_{nx_j} & -\sum_{j=1}^n X_{jx_2}u_{nx_j} & \dots & -\sum_{j=1}^n X_{jx_n}u_{nx_j} \end{vmatrix}$$

$$=-\sum_{j=1}^n u_{1x_j}\begin{vmatrix} X_{jx_1} & X_{jx_2} & \dots & X_{jx_n} \\ u_{2x_1} & u_{2x_2} & \dots & u_{2x_n} \\ \dots & \dots & \dots & \dots \\ u_{nx_1} & u_{nx_2} & \dots & u_{nx_n} \end{vmatrix}-\sum_{j=1}^n u_{2x_j}\begin{vmatrix} u_{1x_1} & u_{1x_2} & \dots & u_{1x_n} \\ X_{jx_1} & X_{jx_2} & \dots & X_{jx_n} \\ \dots & \dots & \dots & \dots \\ u_{nx_1} & u_{nx_2} & \dots & u_{nx_n} \end{vmatrix}-\dots$$

$$\dots-\sum_{j=1}^n u_{nx_j}\begin{vmatrix} u_{1x_1} & u_{1x_2} & \dots & u_{1x_n} \\ u_{2x_1} & u_{2x_2} & \dots & u_{2x_n} \\ \dots & \dots & \dots & \dots \\ X_{jx_1} & X_{jx_2} & \dots & X_{jx_n} \end{vmatrix}$$

$$=-\sum_{j=1}^n \left[X_{jx_1}\begin{vmatrix} u_{1x_j} & u_{1x_2} & \dots & u_{1x_n} \\ u_{2x_j} & u_{2x_2} & \dots & u_{2x_n} \\ \dots & \dots & \dots & \dots \\ u_{nx_j} & u_{nx_2} & \dots & u_{nx_n} \end{vmatrix}+X_{jx_2}\begin{vmatrix} u_{1x_1} & u_{1x_j} & \dots & u_{1x_n} \\ u_{2x_1} & u_{2x_j} & \dots & u_{2x_n} \\ \dots & \dots & \dots & \dots \\ u_{nx_1} & u_{nx_j} & \dots & u_{nx_n} \end{vmatrix}+\dots\right.$$

$$\left.\dots+X_{jx_n}\begin{vmatrix} u_{1x_1} & u_{1x_2} & \dots & u_{1x_j} \\ u_{2x_1} & u_{2x_2} & \dots & u_{2x_j} \\ \dots & \dots & \dots & \dots \\ u_{nx_1} & u_{nx_2} & \dots & u_{nx_j} \end{vmatrix}\right]$$

$$= -\sum_{j=1}^{n} X_{jx_j} \begin{vmatrix} u_{1x_1} & u_{1x_2} & \cdots & u_{1x_n} \\ u_{2x_1} & u_{2x_2} & \cdots & u_{2x_n} \\ \cdots & \cdots & \cdots & \cdots \\ u_{nx_1} & u_{nx_2} & \cdots & u_{nx_n} \end{vmatrix} = -\sum_{j=1}^{n} X_{jx_j} M$$
$$= -M \operatorname{tr} \frac{\partial X(t,x)}{\partial x}.$$

Thus, the proof of the present theorem is finished. □

REMARK 1. By this theorem, we know if $\operatorname{tr}\frac{\partial X(t,x)}{\partial x} = 0$, then $M(t,x(t)) \equiv c$ (c is a constant, $x(t)$ is an arbitrary solution of (2.1)), i.e., $M(t,x) = c$ is a first integral of (2.1).

COROLLARY 1. *For the linear system (1.1), the equation (1.2) holds.*

Proof. As $\Phi(t)$ is the fundamental matrix of (1.1), then

$$U(t,x) = \Phi^{-1}(t)x = (u_1,u_2,\ldots,u_n)^T = (c_1,c_2,\ldots,c_n)^T$$

is the general solution of (1.1), $u_i = c_i$ $(i = 1,2,\ldots,n)$ are the independent first integrals of (1.1). By the above theorem, we get $M = \det U_x = \det \Phi^{-1}$, which satisfies

$$M' = -M \operatorname{tr} \frac{\partial X(t,x)}{\partial x} = -M \operatorname{tr} A(t),$$

i.e.,

$$(W)^{-2} W'(t) = W^{-1} \operatorname{tr} A(t),$$

it implies $W' = W \operatorname{tr} A(t)$ holds. Therefore, our new formula (2.2) generalizes the Liouville formula (1.2). □

REMARK 2. Solving the equation (2.2), we get

$$M(t,x(t)) = M(t_0,x(t_0)) e^{-\int_{t_0}^{t} \operatorname{tr} \frac{\partial X(\tau,x)}{\partial x} d\tau},$$

where $x(t)$ is an arbitrary solution of (2.1). It implies that, if $\operatorname{tr}\frac{\partial X(t,x)}{\partial x} = 0$, then $M(t,x)$ moves along the direction of the vector field X always staying at the same high surface. If $\operatorname{tr}\frac{\partial X(t,x)}{\partial x} > 0$, then $M(t,x(t))$ is monotonically decreasing, if $\operatorname{tr}\frac{\partial X(t,x)}{\partial x} < 0$ then $M(t,x(t))$ is monotonically increasing. In particular, if the function $M(t,x)$ is a positive definite function, when $\operatorname{tr}\frac{\partial X(t,x)}{\partial x} > 0$, the vector field X has a stable singular point, when $\operatorname{tr}\frac{\partial X(t,x)}{\partial x} < 0$, the vector field X has a unstable singular point [6] In short, by using the new formula (2.2), we can discuss the geometric properties of the Jacobi determinant of the first integral of the nonlinear differential system (2.1) and the qualitative behavior of the vector field $X(t,x)$.

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