

## ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF IMPULSIVE NEUTRAL DIFFERENTIAL EQUATIONS WITH CONSTANT JUMPS

CHOLTICHA NUCHPONG, SOTIRIS K. NTOUYAS,  
PHOLLAKRIT THIRAMANUS AND JESSADA TARIBOON

(Communicated by Jurang Yan)

*Abstract.* In this paper, we investigate the asymptotic behavior of solutions for a class of mixed type impulsive neutral delay differential equations with constant jumps. Sufficient conditions are given to guarantee that every non-oscillatory solution of the system tends to zero as  $t \rightarrow \infty$ . An example illustrating the result is also presented.

### 1. Introduction

The asymptotic behavior of solutions of neutral delay differential equations has been studied by two basic methods, by construction of Lyapunov functionals, see [1]–[5] and by considering the asymptotic behavior of non-oscillatory and oscillatory solutions respectively, for example, see [6, 7] and the references therein.

The theory of impulsive differential equations is not only richer than the corresponding theory of differential equations but also represents a more natural framework for mathematical modeling of many real world phenomena, see the monographs [8, 9, 10].

In [11] Jiang and Shen investigated the following nonlinear neutral delay differential equation with constant impulsive jumps and forced term

$$\begin{cases} [x(t) - px(t - \tau)]' + \sum_{i=1}^n q_i(t)f(x(t - \sigma_i)) = h(t), & t \neq t_k, \\ x(t_k^+) - x(t_k^-) = \alpha_k, & k \in \mathbb{Z}_+, \end{cases} \quad (1.1)$$

and derived that every non-oscillatory/oscillatory solution tends to zero as  $t \rightarrow \infty$ . These results were improved in [12].

In [13] Jiang and Sun considered the asymptotic behavior of every non-oscillatory/oscillatory solution for the following forced nonlinear neutral differential equation in first-order Euler form with constant impulsive jumps and unbounded delay

$$\begin{cases} [x(t) - C(t)x(\gamma(t))] + \sum_{i=1}^n \frac{P_i(t)}{t} f(x(\beta_i t)) = h(t), & t \neq t_k, \\ x(t_k^+) - x(t_k^-) = \alpha_k, & k \in \mathbb{Z}_+, \end{cases} \quad (1.2)$$

*Mathematics subject classification* (2010): 34K15, 34K20.

*Keywords and phrases:* Asymptotic behavior, neutral differential equation, impulsive differential equation.

and proved that every non-oscillatory/oscillatory solution tends to zero as  $t \rightarrow \infty$ .

The aim of this paper is to investigate the asymptotic behavior of solutions of the following mixed type impulsive neutral differential equation with constant jumps:

$$\begin{cases} [x(t) - bx(t - \tau) - C(t)x(\gamma(t))]' \\ \quad + \sum_{i=1}^n \left\{ q_i(t)f(x(t - \sigma_i)) + \frac{P_i(t)}{t}g(x(\beta_{it})) \right\} = h(t), \quad t \neq t_k, \\ x(t_k^+) - x(t_k^-) = \alpha_k, \quad k = 1, 2, 3, \dots, \end{cases} \quad (1.3)$$

where  $b, \tau, \sigma_i$  are given constants such that  $\tau > 0, 0 < \sigma_1 < \sigma_2 < \dots < \sigma_n, \gamma$  is monotone increasing for  $t > t_0$  and  $\gamma(t) \leq t, 0 < \beta_i < 1$  satisfying  $\beta_1 < \beta_2 < \dots < \beta_n, i \in \Lambda; C, q_i, P_i, h \in PC([t_0, \infty), \mathbb{R})$  where  $\Lambda = \{1, 2, \dots, n\}, t_0 > 0, \mathbb{R}$  denotes the set of real numbers, for  $J \subset \mathbb{R}, PC(J, \mathbb{R})$  denotes the set of all functions  $\varphi : J \rightarrow \mathbb{R}$  such that  $\varphi$  is continuous everywhere except at some points  $t_k, k \in \mathbb{Z}_+$  and the limits  $\varphi(t_k^+) = \lim_{t \rightarrow t_k^+} \varphi(t), \varphi(t_k^-) = \lim_{t \rightarrow t_k^-} \varphi(t)$  exist with  $\varphi(t_k) = \varphi(t_k^-)$ , the sequence  $\{t_k\}, k \in \mathbb{Z}_+$  is impulsive points satisfying  $0 < t_0 < t_1 < \dots < t_k < t_{k+1} < \dots \rightarrow \infty$  as  $k \rightarrow \infty$ , the notation  $\{\alpha_k\}, k \in \mathbb{Z}_+$  is a constant impulsive sequence, and  $\mathbb{Z}_+$  denotes the set of positive integers. Notice that problem (1.3) reduces to the problem (1.1) for  $C = 0, P_i = 0$  and to problem (1.2) for  $b = 0, q_i = 0$ .

In this paper we derive sufficient conditions such that every non-oscillatory solution of system (1.3) tends to zero as  $t \rightarrow \infty$ . The rest of the paper is organized as follows. In the next section, we present some preliminaries. In Section 3, we give and prove our main result by a technique of construction. Finally, in Section 4, as an application of our results, we present an example to illustrate the usefulness of the obtained results.

### 2. Preliminaries

Before going to prove our main result, we would like to state the hypotheses. Let  $f, g : \mathbb{R} \rightarrow \mathbb{R}$  be continuous functions. Assume that:

(H<sub>1</sub>) There exists two constants  $M > 0$  and  $N > 0$  such that

$$|f(x)| \leq M|x| \quad \text{for } x \in \mathbb{R}; \quad xf(x) > 0 \quad \text{for } x \neq 0,$$

and

$$|g(x)| \leq N|x| \quad \text{for } x \in \mathbb{R}; \quad xg(x) > 0 \quad \text{for } x \neq 0.$$

(H<sub>2</sub>) For all  $0 < t_0 \leq t$ , the integral

$$G(t) = \int_t^\infty h(s)ds \quad \text{is convergent.}$$

(H<sub>3</sub>)  $t_k - \tau, \gamma(t_k)$  are not impulsive points for all  $k \in \mathbb{Z}_+$  and the limit  $\lim_{t \rightarrow \infty} \alpha_k^+ = 0$  where  $\alpha_k^+ = \max\{\alpha_k, 0\}$ .

To set the initial function, we define  $\rho_1 = \max\{\tau, \sigma_n\}$ ,  $\rho_2 = \min\left\{\frac{\gamma(t_0)}{t_0}, \beta_1\right\}$ ,  $0 < \rho = \min\{t_0 - \rho_1, \rho_2 t_0\}$ . Also, we define an initial value function

$$x(t) = \varphi(t), \quad t \in [\rho, t_0], \quad (2.1)$$

where  $\varphi \in PC([\rho, t_0], \mathbb{R}) = \{\varphi : [\rho, t_0] \rightarrow \mathbb{R} : \varphi \text{ is continuous everywhere except at some points } t_k, k \in \mathbb{Z}_+ \text{ and } \varphi(t_k^-) = \lim_{t \rightarrow t_k^-} \varphi(t), \varphi(t_k^+) = \lim_{t \rightarrow t_k^+} \varphi(t) \text{ exist with } \varphi(t_k^-) = \varphi(t_k)\}$ .

The solution of problem (1.3) is defined as follows.

DEFINITION 1. A function  $x(t)$  is said to be a solution of system (1.3) satisfying the initial value condition (2.1) if

- (1).  $x(t) = \varphi(t)$  for  $0 < \rho \leq t \leq t_0$  and  $x(t)$  is continuous for  $t \geq t_0$ ,  $t \neq t_k$ ,  $k \in \mathbb{Z}_+$ ;
- (2).  $x(t) - bx(t - \tau) - C(t)x(\gamma(t))$  is continuously differentiable for  $t > t_0$ ,  $t \neq t_k$ ,  $k \in \mathbb{Z}_+$  and satisfies equation (1.3);
- (3).  $x(t_k^+)$  and  $x(t_k^-)$  exist with  $x(t_k^-) = x(t_k)$  for all  $k \in \mathbb{Z}_+$  and satisfies equation (1.3).

The oscillatory and non-oscillatory solutions of system (1.3) are defined as follows.

DEFINITION 2. A solution  $x(t)$  of system (1.3) is said to be eventually positive (negative) if it is positive (negative) for all sufficiently large  $t$ . It is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be non-oscillatory.

Throughout this paper, we introduce the function  $H(t)$  defined by

$$H(t) = \begin{cases} \int_t^\infty h(s)ds, & t \in (t_k, t_{k+1}], \\ \int_t^\infty h(s)ds + \alpha_{k-1}^+, & t = t_k, k \in \mathbb{Z}_+, \end{cases} \quad (2.2)$$

where  $\alpha_k^+ = \max\{\alpha_k, 0\}$ ,  $k \in \mathbb{Z}_+ \cup \{0\}$  and  $\alpha_0 = 0$ .

### 3. Main result

THEOREM 1. Let the conditions  $(H_1)$ – $(H_3)$  hold. Assume that for some  $\xi_1, \xi_2 > 0$ , there exist two constants  $\theta_1, \theta_2 > 0$  such that

$$|f(x)| \geq \theta_1, |x| \geq \xi_1 \quad \text{and} \quad |g(x)| \geq \theta_2, |x| \geq \xi_2. \quad (3.1)$$

Suppose that

$$|b| = B < 1, \quad \lim_{t \rightarrow \infty} |C(t)| = C < 1 \text{ such that } B + C < 1, \quad (3.2)$$

and

$$\sum_{i=1}^n q_i(t + \sigma_i) \geq 0, \quad \int_{t_0}^{\infty} \sum_{i=1}^n q_i(s + \sigma_i) ds = \infty, \tag{3.3}$$

$$\sum_{i=1}^n \frac{P_i(t/\beta_i)}{t} \geq 0, \quad \int_{t_0}^{\infty} \sum_{i=1}^n \frac{P_i(s/\beta_i)}{s} ds = \infty. \tag{3.4}$$

In addition, for sufficiently large  $t$ , assume that there exist constants  $\lambda_1 > 0$  and  $\lambda_2 > 0$  such that

$$\sum_{\sigma_i < r} \int_{t-r}^{t-\sigma_i} q_i^-(s + \sigma_i) ds + \sum_{\sigma_i > r} \int_{t-\sigma_i}^{t-r} q_i^+(s + \sigma_i) ds \leq \lambda_1, \tag{3.5}$$

$$\sum_{\beta_i < u} \int_{\beta_i t}^{ut} \left( \frac{P_i(s/\beta_i)}{s} \right)^+ ds + \sum_{\beta_i > u} \int_{ut}^{\beta_i t} \left( \frac{P_i(s/\beta_i)}{s} \right)^- ds \leq \lambda_2, \tag{3.6}$$

where a fixed constant  $u \in (0, \beta_n]$ ,  $\lambda < (1 - |b| - C)/(M + N)$  and  $\lambda = \max\{\lambda_1, \lambda_2\}$ ,  $r \in [0, \sigma_n]$ ,  $q_i^+(s) = \max\{q_i(s), 0\}$ ,  $q_i^-(s) = \max\{-q_i(s), 0\}$  and

$$((P_i(s/\beta_i))/s)^+ = \max\{(P_i(s/\beta_i))/s, 0\},$$

$$((P_i(s/\beta_i))/s)^- = \max\{(-P_i(s/\beta_i))/s, 0\}.$$

Then every non-oscillatory solution of equation (1.3) tends to zero as  $t \rightarrow \infty$ .

*Proof.* Firstly, we choose a positive integer  $N$  sufficiently large enough such that there exists a positive integer  $m$  large enough satisfying  $\gamma(t_m)$ ,  $t - \tau > t_N$  and (3.5)–(3.6) hold for  $t \geq t_N$ , where  $N$  is the largest subscript satisfying  $\gamma(t_m)$ ,  $t - \tau > t_N$ . Let  $x(t)$  be a non-oscillatory solution of equation (1.3). Without loss of generality, we will assume that  $x(t)$  is eventually positive solution. For the case  $x(t)$  is eventually negative, the proof is similar and we omit it. Let  $x(t) > 0$  for  $t \geq t_N$ . For all  $t \geq t_N$ , we set

$$\alpha(t) = \begin{cases} \alpha_{N'}^+; & t > t_{N+1}, \\ 0; & t \in [t_N, t_{N+1}], \end{cases} \tag{3.7}$$

where  $N'$  corresponds to the largest subscript of impulsive points in the interval  $t \geq t_N$ .

Next, we define

$$\begin{aligned} y(t) &= x(t) - bx(t - \tau) - C(t)x(\gamma(t)) \\ &\quad - \sum_{i=1}^n \left[ \int_{t-\sigma_i}^{t-r} q_i(s + \sigma_i) f(x(s)) ds + \int_{\beta_i t}^{ut} \frac{P_i(s/\beta_i)}{s} g(x(s)) ds \right] \\ &\quad + H(t) - \alpha(t), \end{aligned} \tag{3.8}$$

where  $H(t)$  is as in (2.2). Now, we derive the derivative of a function  $\alpha(t)$ . For  $t \neq t_k$ , we choose  $\Delta t$  sufficiently small such that there is no impulsive point in the interval  $(t, t + \Delta t)$ . Then we have

$$\alpha'(t) = \lim_{\Delta t \rightarrow 0} \frac{\alpha(t + \Delta t) - \alpha t}{\Delta t} = 0, \quad t \neq t_k.$$

From (3.8) and  $(H_2)$ – $(H_3)$ , we get that for  $t \neq t_k$ ,  $t \neq t_k + \sigma_i$ ,  $t \neq t_k/\beta_i$ ,  $i \in \Lambda$ ,  $k \in \mathbb{Z}_+$ ,

$$\begin{aligned}
 y'(t) &= [x(t) - bx(t - \tau) - C(t)x(\gamma(t))] - \sum_{i=1}^n \left[ q_i(t - r + \sigma_i)f(x(t - r)) \right. \\
 &\quad \left. - q_i(t)f(x(t - \sigma_i)) + \frac{P_i(ut/\beta_i)}{rt}g(x(ut))r - \frac{P_i(t)}{\beta_it}g(x(\beta_it))\beta_i \right] - h(t) \\
 &= [x(t) - bx(t - \tau) - C(t)x(\gamma(t))] + \sum_{i=1}^n \left[ q_i(t)f(x(t - \sigma_i)) + \frac{P_i(t)}{t}g(x(\beta_it)) \right] \\
 &\quad - \sum_{i=1}^n \left[ q_i(t - r + \sigma_i)f(x(t - r)) + \frac{P_i(ut/\beta_i)}{t}g(x(ut)) \right] - h(t) \\
 &= - \sum_{i=1}^n \left[ q_i(t - r + \sigma_i)f(x(t - r)) + \frac{P_i(ut/\beta_i)}{t}g(x(ut)) \right] \leq 0. \tag{3.9}
 \end{aligned}$$

For  $t = t_k$ ,  $k = N + 1, N + 2, \dots$ , we have

$$H(t_k^+) - H(t_k) = -\alpha_{k-1}^+. \tag{3.10}$$

In addition, for  $t = t_k$ ,  $k = N + 1, N + 2, \dots$ , we obtain

$$\begin{aligned}
 y(t_k^+) - y(t_k) &= x(t_k^+) - bx(t_k^+ - \tau) - C(t_k^+)x(\gamma(t_k^+)) \\
 &\quad - \sum_{i=1}^n \left[ \int_{t_k^+ - \sigma_i}^{t_k^+ - r} q_i(s + \sigma_i)f(x(s))ds + \int_{\beta_it_k^+}^{rt_k^+} \frac{P_i(s/\beta_i)}{s}g(x(s))ds \right] \\
 &\quad + H(t_k^+) - \alpha(t_k^+) - x(t_k) + bx(t_k - \tau) + C(t_k)x(\gamma(t_k)) \\
 &\quad + \sum_{i=1}^n \left[ \int_{t_k - \sigma_i}^{t_k - r} q_i(s + \sigma_i)f(x(s))ds + \int_{\beta_it_k}^{rt_k} \frac{P_i(s/\beta_i)}{s}g(x(s))ds \right] \\
 &\quad - H(t_k) + \alpha(t_k) \\
 &= x(t_k^+) - x(t_k) + H(t_k^+) - H(t_k) - \alpha(t_k^+) + \alpha(t_k) \\
 &= \alpha_k - \alpha_{k-1}^+ - \alpha_k^+ + \alpha_{k-1}^+ = \alpha_k - \alpha_k^+ \leq 0.
 \end{aligned}$$

From (3.9) and the above inequality, we have  $y(t)$  is nonincreasing on  $[\frac{t_N}{u} + r, \infty)$ .

Now, we will claim that  $y(t)$  is convergence. Let  $L = \lim_{t \rightarrow \infty} y(t)$ , we will show that  $L \in \mathbb{R}$ . Otherwise,  $L = -\infty$ , then  $x(t)$  is unbounded. Indeed, if  $x(t)$  is bounded

then it follows from (3.8) and  $(H_1)$  that for some constants  $D > 0, E > 0,$

$$\begin{aligned}
 y(t) &= x(t) - bx(t - \tau) - C(t)x(\gamma(t)) \\
 &\quad - \sum_{i=1}^n \left[ \int_{t-\sigma_i}^{t-r} q_i(s + \sigma_i) f(x(s)) ds + \int_{\beta_{it}}^{ut} \frac{P_i(s/\beta_i)}{s} g(x(s)) ds \right] + H(t) - \alpha(t) \\
 &\geq x(t) - bx(t - \tau) - C(t)x(\gamma(t)) \\
 &\quad - D \left[ \sum_{\sigma_i < r} \int_{t-r}^{t-\sigma_i} q_i^-(s - \sigma_i) ds + \sum_{\sigma_i > r} \int_{t-\sigma_i}^{t-r} q_i^+(s + \sigma_i) ds \right] \\
 &\quad - E \left[ \sum_{\beta_i < u} \int_{\beta_{it}}^{ut} \left( \frac{P_i(s/\beta_i)}{s} \right)^+ ds + \sum_{\beta_i > u} \int_{ut}^{\beta_{it}} \left( \frac{P_i(s/\beta_i)}{s} \right)^- ds \right] + H(t) - \alpha(t).
 \end{aligned}$$

From the conditions  $(H_2)$ – $(H_3)$  and (3.5)–(3.6), we have that  $L = -\infty \geq K$ . This is a contradiction and then  $x(t)$  is unbounded.

On the other hand, from  $x(t)$  is unbounded and  $\lim_{t \rightarrow \infty} y(t) = -\infty,$  we can choose  $t^* \geq \max\{t_N + \sigma_n, t_N + \tau, t_N/\beta_1, \gamma(t_m)\}$  such that  $y(t^*) - H(t^*) + \alpha(t^*) < 0$  and  $x(t^*) = \max\{x(t) : t_N \leq t \leq t^*\}$ . Therefore, it follows from (3.5)–(3.6) that

$$\begin{aligned}
 0 &> y(t^*) - H(t^*) + \alpha(t^*) \\
 &= x(t^*) - bx(t^* - \tau) - C(t^*)x(\gamma(t)) \\
 &\quad - \sum_{i=1}^n \left[ \int_{t^*-\sigma_i}^{t^*-r} q_i(s + \sigma_i) f(x(s)) ds + \int_{\beta_{it^*}}^{ut^*} \frac{P_i(s/\beta_i)}{s} g(x(s)) ds \right] \\
 &\quad + H(t^*) - \alpha(t^*) - H(t^*) + \alpha(t^*) \\
 &\geq x(t^*) - |b|x(t^*) - Cx(t^*) \\
 &\quad - Mx(t^*) \left[ \sum_{\sigma_i < r} \int_{t^*-r}^{t^*-\sigma_i} q_i^-(s - \sigma_i) ds + \sum_{\sigma_i > r} \int_{t^*-\sigma_i}^{t^*-r} q_i^+(s + \sigma_i) ds \right] \\
 &\quad - Nx(t^*) \left[ \sum_{\beta_i < u} \int_{\beta_{it^*}}^{ut^*} \left( \frac{P_i(s/\beta_i)}{s} \right)^+ ds + \sum_{\beta_i > u} \int_{ut^*}^{\beta_{it^*}} \left( \frac{P_i(s/\beta_i)}{s} \right)^- ds \right] \\
 &\geq x(t^*) \{1 - |b| - C - M\lambda_1 - N\lambda_2\} \\
 &\geq x(t^*) \{1 - |b| - C - \lambda(M + N)\} > 0,
 \end{aligned}$$

which is a contradiction and therefore  $L \in \mathbb{R}.$

Integrating both sides of (3.9) from  $t_N/\beta_1 + \sigma_n$  to  $t$ , we obtain

$$\begin{aligned} & \int_{\frac{t_N}{\beta_1} + \sigma_n}^t \left[ \sum_{i=1}^n q_i(s-r + \sigma_i) f(x(s-r)) + \sum_{i=1}^n \frac{P_i(us/\beta_i)}{s} g(x(us)) \right] ds \\ &= - \int_{\frac{t_N}{\beta_1} + \sigma_n}^t y'(s) ds \\ &= -y(t) + y\left(\frac{t_N}{\beta_1} + \sigma_n\right) + \sum_{\frac{t_N}{\beta_1} + \sigma_n < t_k \leq t} [y(t_k^+) - y(t_k)] \\ &< y\left(\frac{t_N}{\beta_1} + \sigma_n\right) - L. \end{aligned} \quad (3.11)$$

From (3.3), (3.4) and (3.11), we have

$$f(x(t)) \in L^1\left(\left[\frac{t_N}{\beta_1} + \sigma_n, \infty\right), \mathbb{R}\right) \quad \text{and} \quad g(x(t)) \in L^1\left(\left[\frac{t_N}{\beta_1} + \sigma_n, \infty\right), \mathbb{R}\right).$$

Hence,  $\liminf_{t \rightarrow \infty} f(x(t)) = 0$  and  $\liminf_{t \rightarrow \infty} g(x(t)) = 0$ .

Now, we claim that

$$\liminf_{t \rightarrow \infty} x(t) = 0. \quad (3.12)$$

Let  $\{S_m\}$  be a sequence such that  $S_m \rightarrow \infty$  as  $m \rightarrow \infty$  with  $\lim_{m \rightarrow \infty} f(x(S_m)) = 0$  and  $\lim_{m \rightarrow \infty} g(x(S_m)) = 0$ . We must show that  $\liminf_{m \rightarrow \infty} x(S_m) = c = 0$ . If  $c > 0$ , then there exists a subsequence  $\{S_{m_k}\}$  of  $\{S_m\}$  such that  $x(S_{m_k}) \geq c/2$  for some  $k$  sufficiently large. From (3.1), we have  $f(x(S_{m_k})) \geq \theta_{1c}$  and  $g(x(S_{m_k})) \geq \theta_{2c}$  for some  $\theta_{1c}, \theta_{2c} > 0$  and sufficiently large  $k$ . Then it is a contradiction because  $\lim_{k \rightarrow \infty} f(x(S_{m_k})) = 0$  and  $\lim_{k \rightarrow \infty} g(x(S_{m_k})) = 0$ . Thus, (3.12) holds.

Observe that (3.11) implies

$$\int_{t_0}^{\infty} \sum_{i=1}^n q_i(s-r + \sigma_i) f(x(s-r)) ds + \int_{t_0}^{\infty} \sum_{i=1}^n \frac{P_i(rs/\beta_i)}{s} f(x(sr)) ds < \infty. \quad (3.13)$$

Set

$$z(t) = y(t) + \sum_{i=1}^n \left[ \int_{t-\sigma_i}^{t-r} q_i(s + \sigma_i) f(x(s)) ds + \int_{\beta_i t}^{ut} \frac{P_i(s/\beta_i)}{s} f(x(s)) \right] ds - H(t) + \alpha(t).$$

From  $(H_2)$ – $(H_3)$  and (3.13), we have

$$\lim_{t \rightarrow \infty} z(t) = \mu \quad \text{exists.}$$

Then, from (3.8), we obtain

$$\lim_{t \rightarrow \infty} [x(t) - bx(t-\tau) - C(t)x(\gamma(t))] = \mu. \quad (3.14)$$

Next, we will show that  $\lim_{t \rightarrow \infty} x(t) = 0$ . From the condition (3.2), we can choose a sufficiently large  $T_1$  such that  $|b| + |C(t)| < 1$  for  $t > T_1$ . Set

$$\eta = \limsup_{t \rightarrow \infty} x(t).$$

By  $\liminf_{t \rightarrow \infty} x(t) = 0$ , we get that there exist two sequences  $\{u_n\}$  and  $\{v_n\}$  with  $u_n \rightarrow \infty, v_n \rightarrow \infty$  as  $n \rightarrow \infty$  such that

$$\lim_{t \rightarrow \infty} x(u_n) = 0, \quad \lim_{t \rightarrow \infty} x(v_n) = \eta.$$

For all  $t > T_1$ , we divide the following nine possible cases to discuss.

*Case 1.* If  $b = 0$  and  $\lim_{t \rightarrow \infty} C(t) = 0$  for  $t > T_1$ , then we get

$$\lim_{t \rightarrow \infty} x(t) = \mu = 0.$$

Since  $\lim_{t \rightarrow \infty} x(t)$  exists and  $\liminf_{t \rightarrow \infty} x(t) = 0$ .

*Case 2.* If  $b = 0$  and  $-1 < C(t) < 0$  for  $t > T_1$ , then we have

$$\mu = \lim_{n \rightarrow \infty} [x(u_n) - C(u_n)x(\gamma(u_n))] \leq C\eta,$$

and

$$\mu = \lim_{n \rightarrow \infty} [x(v_n) - C(v_n)x(\gamma(v_n))] \geq \eta,$$

which imply that  $\eta \leq C\eta$ . It follows from  $\eta \geq 0$  and  $0 < C < 1$  that  $\eta = 0$ . This shows  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Case 3.* If  $b = 0$  and  $0 < C(t) < 1$  for  $t > T_1$ , then we obtain

$$\mu = \lim_{n \rightarrow \infty} [x(u_n) - C(u_n)x(\gamma(u_n))] \leq 0,$$

and

$$\mu = \lim_{n \rightarrow \infty} [x(v_n) - C(v_n)x(\gamma(v_n))] \geq \eta - C\eta,$$

which imply that  $\eta(1 - C) \leq 0$ . It follows that  $\eta \geq 0$  and  $0 < C < 1$  which imply  $\eta = 0$ . This shows  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Case 4.* If  $\lim_{t \rightarrow \infty} C(t) = 0$  and  $-1 < b < 0$  for  $t > T_1$ , then we get

$$\mu = \lim_{n \rightarrow \infty} [x(u_n) - bx(u_n - \tau)] \leq B\eta,$$

and

$$\mu = \lim_{n \rightarrow \infty} [x(v_n) - bx(v_n - \tau)] \geq \eta,$$

which imply that  $\eta \leq B\eta$ . It follows that  $\eta \geq 0$  and  $0 < B < 1$  which imply  $\eta = 0$ . This shows  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Case 5.* If  $\lim_{t \rightarrow \infty} C(t) = 0$  and  $0 < b < 1$  for  $t > T_1$ , then we obtain

$$\mu = \lim_{n \rightarrow \infty} [x(u_n) - bx(u_n - \tau)] \leq 0,$$



and

$$\mu = \lim_{n \rightarrow \infty} [x(v_n) - bx(v_n - \tau)] \geq \eta - B\eta,$$

which imply that  $\eta(1 - B) \leq 0$ . It follows that  $\eta \geq 0$  and  $0 < B < 1$  which yield  $\eta = 0$ . This means  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Case 6.* If  $0 < b < 1$  and  $-1 < C(t) < 0$  for  $t > T_1$ , then we have

$$\mu = \lim_{n \rightarrow \infty} [x(u_n) - bx(u_n - \tau) - C(u_n)x(\gamma(u_n))] \leq C\eta,$$

and

$$\mu = \lim_{n \rightarrow \infty} [x(v_n) - bx(v_n - \tau) - C(v_n)x(\gamma(v_n))] \geq \eta - B\eta,$$

which imply that  $\eta[1 - (B + C)] \leq 0$ . It follows that  $\eta \geq 0$  and  $0 < B + C < 1$  which lead to  $\eta = 0$ . This implies  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Case 7.* If  $0 < b < 1$  and  $0 < C(t) < 1$  for  $t > T_1$ , then we obtain

$$\mu = \lim_{n \rightarrow \infty} [x(u_n) - bx(u_n - \tau) - C(u_n)x(\gamma(u_n))] \leq 0,$$

and

$$\mu = \lim_{n \rightarrow \infty} [x(v_n) - bx(v_n - \tau) - C(v_n)x(\gamma(v_n))] \geq \eta - B\eta + C\eta,$$

which imply that  $\eta[1 - (B + C)] \leq 0$ . It follows that  $\eta \geq 0$  and  $0 < B + C < 1$  which imply  $\eta = 0$ . This shows  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Case 8.* If  $-1 < b < 0$  and  $-1 < C(t) < 0$  for  $t > T_1$ , then we get

$$\mu = \lim_{n \rightarrow \infty} [x(u_n) - bx(u_n - \tau) - C(u_n)x(\gamma(u_n))] \leq B\eta + C\eta,$$

and

$$\mu = \lim_{n \rightarrow \infty} [x(v_n) - bx(v_n - \tau) - C(v_n)x(\gamma(v_n))] \geq \eta,$$

which imply that  $\eta[1 - (B + C)] \leq 0$ . It follows that  $\eta \geq 0$  and  $0 < B + C < 1$ . Thus  $\eta = 0$ . This shows  $\lim_{t \rightarrow \infty} x(t) = 0$ .

*Case 9.* If  $-1 < b < 0$  and  $0 < C(t) < 1$  for  $t > T_1$ , then we have

$$\mu = \lim_{n \rightarrow \infty} [x(u_n) - bx(u_n - \tau) - C(u_n)x(\gamma(u_n))] \leq B\eta,$$

and

$$\mu = \lim_{n \rightarrow \infty} [x(v_n) - bx(v_n - \tau) - C(v_n)x(\gamma(v_n))] \geq \eta - C\eta,$$

which imply that  $\eta[1 - (B + C)] \leq 0$ . It follows that  $\eta \geq 0$  and  $0 < B + C < 1$ . Thus  $\eta = 0$ . This shows  $\lim_{t \rightarrow \infty} x(t) = 0$ .

Therefore, we conclude that  $\lim_{t \rightarrow \infty} x(t) = 0$ , and so the proof is completed.  $\square$

### 4. An Example

EXAMPLE 1. Consider the following mixed type neutral differential equation with impulsive perturbations

$$\left\{ \begin{aligned} & \left[ x(t) - \frac{1}{4}x\left(t - \frac{1}{3}\right) - C(t)x\left(\frac{2t}{e}\right) \right]' + \left(\frac{3}{t+2}\right) \left(\frac{2x(t-1)}{1+(x(t-1))^2}\right) \\ & + \frac{1}{2(\ln(\frac{1}{7}t) - 1)} 3x\left(\frac{t}{7e}\right) + \left(\frac{2}{t+2}\right) \left(\frac{2x(t-2)}{1+(x(t-2))^2}\right) \\ & + \frac{1}{3(\ln(\frac{1}{5}t) - 1)} 3x\left(\frac{t}{5e}\right) + \left(\frac{1}{t+2}\right) \left(\frac{2x(t-3)}{1+(x(t-3))^2}\right) \\ & + \frac{1}{4(\ln(\frac{1}{3}t) - 1)} 3x\left(\frac{t}{3e}\right) = \frac{1}{t^3}, \quad t \geq t_0 = e, \quad t \neq t_k, \\ & x(t_k^+) - x(t_k) = (-1)^k \frac{2}{k}, \quad t_k = k + 2, \quad k \in \mathbb{Z}_+, \end{aligned} \right. \tag{4.1}$$

where

$$C(t) = \frac{(k+2)\lfloor t \rfloor}{2k^2 + 2k + 4}, \quad t \in (k, k + 1], \quad k = 2, 3, 4, \dots$$

Here  $b = 1/4$ ,  $\tau = 1/3$ ,  $\gamma(t) = 2t/e$ ,  $f(x) = 2x/(1+x^2)$ ,  $g(x) = 3x$ ,  $h(t) = 1/t^3$ ,  $q_1(t) = 3/(t+2)$ ,  $q_2(t) = 2/(t+2)$ ,  $q_3(t) = 1/(t+2)$ ,  $P_1(t) = 1/(2(\ln(\frac{1}{7}t) - 1))$ ,  $P_2(t) = 1/(3(\ln(\frac{1}{5}t) - 1))$ ,  $P_3(t) = 1/(4(\ln(\frac{1}{3}t) - 1))$ ,  $\sigma_1 = 1$ ,  $\sigma_2 = 2$ ,  $\sigma_3 = 3$ ,  $\beta_1 = 1/(7e)$ ,  $\beta_2 = 1/(5e)$ ,  $\beta_3 = 1/(3e)$ , when we choose  $M = 2$ ,  $N = 3$ ,  $r = 5/2 \in [0, 3]$ ,  $u = 1/(6e) \in (0, 1/(3e)]$ . We can find that

$$(i) \quad |f(x)| = \left| \frac{2x}{1+x^2} \right| \leq 2|x|, \quad x \in \mathbb{R}, \quad x\left(\frac{2x}{1+x^2}\right) > 0 \text{ for } x \neq 0 \text{ and}$$

$$|g(x)| = |3x| \leq 3|x|, \quad x \in \mathbb{R}, \quad x(3x) > 0 \text{ for } x \neq 0;$$

$$(ii) \quad G(t) = \int_t^\infty \frac{1}{s^3} ds = \frac{1}{2t^2} \text{ is convergent for } t \geq e;$$

$$(iii) \quad t_k - (1/3) \text{ and } (2/e)t_k \text{ are not impulsive points for all } k \in \mathbb{Z}_+ \text{ and } \lim_{k \rightarrow \infty} \alpha_k^+ = \lim_{k \rightarrow \infty} \frac{1}{k} = 0;$$

$$(iv) \quad |b| = \frac{1}{4} = B < 1, \quad \lim_{t \rightarrow \infty} |C(t)| = \frac{1}{2} = \mu < 1 \quad \text{with} \quad B + C = \frac{3}{4} < 1;$$

$$(v) \quad \sum_{i=1}^3 q_i(t + \sigma_i) = \frac{3}{t+3} + \frac{2}{t+4} + \frac{1}{t+5} \geq 0 \text{ for } t \geq e \text{ and}$$

$$\int_e^\infty q_i(s + \sigma_i) ds = \int_e^\infty \left[ \frac{3}{s+3} + \frac{2}{s+4} + \frac{1}{s+5} \right] ds = \infty;$$

$$(vi) \quad \sum_{i=1}^3 \frac{P_i(t/\beta_i)}{t} = \frac{13}{12t \ln t} \geq 0 \text{ for } t \geq e \text{ and}$$

$$\int_e^\infty \sum_{i=1}^3 \frac{P_i(s/\beta_i)}{s} ds = \int_e^\infty \left[ \frac{13}{12s \ln s} \right] ds = \infty;$$

(vii) For large enough  $t$ , there exist constants  $\lambda_1 > 0$  and  $\lambda_2 > 0$  such that

$$\begin{aligned} \sum_{\sigma_i < r} \int_{t-r}^{t-\sigma_i} q_i^-(s + \sigma_i) ds + \sum_{\sigma_i > r} \int_{t-\sigma_i}^{t-r} q_i^+(s + \sigma_i) ds &= \int_{t-3}^{t-5/2} \frac{1}{s+5} ds \\ &= \ln(s+5) \Big|_{t-3}^{t-5/2} \rightarrow 0; \end{aligned}$$

and

$$\begin{aligned} \sum_{\beta_i < u} \int_{\beta_i t}^{ut} \left( \frac{P_i(s/\beta_i)}{s} \right)^+ ds + \sum_{\beta_i > u} \int_{ut}^{\beta_i t} \left( \frac{P_i(s/\beta_i)}{s} \right)^- ds &= \int_{t/(7e)}^{t/(6e)} \frac{1}{2s \ln s} ds \\ &= \frac{1}{2} \ln \ln(s) \Big|_{t/(7e)}^{t/(6e)} \rightarrow 0; \end{aligned}$$

by L'Hôpital's rule. Hence, by (i)–(vii) all assumptions of Theorem 1 are satisfied. Therefore, we conclude that every non-oscillatory solution of (4.1) tends to zero as  $t \rightarrow \infty$ .

**REMARK 1.** In this paper, by combining the impulsive neutral differential equations with bounded and unbounded delays (1.1) and (1.2), respectively, an asymptotic behavior of non-oscillatory solutions of equation (1.3) is proved. Notice that in [11] and [13] the authors proved the asymptotic behavior of oscillatory solutions by assuming that there exists a critical point  $\xi$  such that  $y'(\xi) = 0$  (on page 11 of [11] and 9911 of [13], respectively), and  $y(\xi)$  is the extremum value for oscillatory function  $y \in PC(\mathbb{R}_+, \mathbb{R})$  which does not satisfy the Definition 2.

*Acknowledgement.* This research was funded by King Mongkut's University of Technology North Bangkok. Contract no. KMUTNB-NEW-59-11.

#### REFERENCES

- [1] K. Z. GUAN AND J. H. SHEN, *Asymptotic behavior of solutions of a first-order impulsive neutral differential equation in Euler form*, Appl. Math. Lett. **24** (2011) 1218–1224.
- [2] J. H. SHEN AND Y. J. LIU, *Asymptotic behavior of solutions of nonlinear neutral differential equations with impulses*, J. Math. Anal. Appl. **322** (2007) 179–189.
- [3] G. P. WEI AND J. H. SHEN, *Asymptotic behavior for a class of nonlinear impulsive neutral delay differential equations*, J. Math. Phys. **30** (2010) 753–763.
- [4] A. ZHAO AND J. YAN, *Asymptotic behavior of solutions of impulsive delay differential equations*, J. Math. Anal. Appl. **201** (1996) 943–954.

- [5] F. F. JIANG AND J. H. SHEN, *Asymptotic behaviors of nonlinear neutral impulsive delay differential equations with forced term*, Kodai Math. J. **35** (2012) 126–137.
- [6] X. Z. LIU AND J. H. SHEN, *Asymptotic behavior of solutions of impulsive neutral differential equations*, Appl. Math. Lett. **12** (1999) 51–58.
- [7] J. H. SHEN AND J. YU, *Asymptotic behavior of solutions of neutral differential equations with positive and negative coefficients*, J. Math. Anal. Appl. **195** (1995) 517–526.
- [8] V. LAKSHMIKANTHAM, D. D. BAINOV AND P. S. SIMEONOV, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
- [9] A. M. SAMOILENKO AND N. A. PERESTYUK, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.
- [10] M. BENCHOHRA, J. HENDERSON AND S. K. NTOUYAS, *Impulsive Differential Equations and Inclusions*, vol. 2. Hindawi Publishing Corporation, New York, 2006.
- [11] F. JIANG AND J. SHEN, *Asymptotic behavior of solutions for a nonlinear differential equation with constant impulsive jumps*, Acta Math. Hungar. **138** (2013) 1–14.
- [12] Y. BALACHANDRAN AND G. PURUSHOTHAMAN, *Asymptotic behavior of solutions for forced nonlinear delay impulsive differential equations*, International Journal of Mathematical Trends and Technology (IJMTT) **9** (2014) 145–147.
- [13] F. JIANG AND J. SUN, *Asymptotic behavior of neutral delay differential equation of Euler form with constant impulsive jumps*, Appl. Math. Comput. **219** (2013) 9906–9913.

(Received December 18, 2016)

Cholticha Nuchpong  
 Department of Social and Applied Science, College  
 of Industrial Technology  
 King Mongkut's University of Technology North Bangkok  
 Bangkok, 10800 Thailand  
 e-mail: cholticha.nuch@gmail.com

Sotiris K. Ntouyas  
 Department of Mathematics  
 University of Ioannina  
 451 10 Ioannina, Greece  
 and

Nonlinear Analysis and Applied Mathematics (NAAM) – Research Group  
 Department of Mathematics  
 Faculty of Science, King Abdulaziz University  
 P. O. Box 80203, Jeddah 21589, Saudi Arabia  
 e-mail: sntouyas@uoi.gr

Phollakrit Thiramanus  
 Nonlinear Dynamic Analysis Research Center  
 Department of Mathematics, Faculty of Applied Science  
 King Mongkut's University of Technology North Bangkok  
 Bangkok 10800, Thailand  
 e-mail: phollakritt@kmutnb.ac.th

Jessada Tariboon  
 Nonlinear Dynamic Analysis Research Center  
 Department of Mathematics, Faculty of Applied Science  
 King Mongkut's University of Technology North Bangkok  
 Bangkok 10800, Thailand  
 e-mail: jessada.t@sci.kmutnb.ac.th