Differential
Equations
& Applications
Volume 9, Number 3 (2017), 285–310
doi:10.7153/dea-09-22

OSCILLATIONS CAUSED BY SEVERAL NON–MONOTONE DEVIATING ARGUMENTS

GEORGE E. CHATZARAKIS

(*Communicated by Leonid Berezansky*)

Abstract. This paper presents new sufficient conditions, involving $\limsup$ and $\liminf$, for the oscillation of all solutions of differential equations with several non-monotone deviating arguments and nonnegative coefficients. Corresponding differential equations of both delay and advanced type are studied. We illustrate the results and the improvement over other known oscillation criteria by examples, numerically solved in MATLAB.

1. Introduction

Consider the differential equation with several variable deviating arguments of either delay (DDE)

$$x'(t)+\sum_{i=1}^{m} p_i(t)x(\tau_i(t))=0, \quad \forall t \geqslant t_0, \tag{E}$$

or advanced type (ADE)

$$x'(t)-\sum_{i=1}^{m} q_i(t)x(\sigma_i(t))=0, \quad t \geqslant t_0, \tag{E$'$}$$

where p_i, q_i, $1 \leqslant i \leqslant m$, are functions of nonnegative real numbers, and τ_i, σ_i, $1 \leqslant i \leqslant m$, are functions of positive real numbers such that

$$\tau_i(t) < t, \quad t \geqslant t_0 \quad \text{and} \quad \lim_{t\to\infty} \tau_i(t) = \infty, \quad 1 \leqslant i \leqslant m \tag{1.1}$$

and

$$\sigma_i(t) > t, \quad t \geqslant t_0, \quad 1 \leqslant i \leqslant m, \tag{1.1$'$}$$

respectively.

In addition, we consider the initial condition for (E)

$$x(t)=\varphi(t), \quad t \leqslant t_0, \tag{1.2}$$

where $\varphi : (-\infty, t_0] \to \mathbb{R}$ is a bounded Borel measurable function.

Mathematics subject classification (2010): 34K11, 34K06.
Keywords and phrases: differential equation; non-monotone argument; oscillatory solution; nonoscillatory solution.

A *solution* of (E), (1.2) is an absolutely continuous on $[t_0,\infty)$ function satisfying (E) for almost all $t \geqslant t_0$ and (1.2) for all $t \leqslant t_0$. By a solution of (E$'$) we mean an absolutely continuous on $[t_0,\infty)$ function satisfying (E$'$) for almost all $t \geqslant t_0$.

A solution of (E) or (E$'$) is *oscillatory*, if it is neither eventually positive nor eventually negative. If there exists an eventually positive or an eventually negative solution, the equation is *nonoscillatory*. An equation is *oscillatory* if all its solutions oscillate.

The problem of establishing sufficient conditions for the oscillation of all solutions of equations (E) or (E$'$) has been the subject of many investigations. The reader is referred to $[1-23]$ and the references cited therein. Most of these papers concern the special case where the arguments are nondecreasing, while a small number of these papers are dealing with the general case where the arguments are not necessarily monotone. See, for example, $[1-4, 12]$ and the references cited therein.

In the present paper we establish new oscillation criteria for the oscillation of all solutions of (E) and (E$'$) when the arguments are not necessarily monotone. Our results essentially improve several known criteria existing in the literature.

1.1. DDEs

By Remark 2.7.3 in [18], it is clear that if $\tau_i(t)$, $1 \leqslant i \leqslant m$ are nondecreasing and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds > 1, \tag{1.3}$$

where $\tau(t) = \max_{1\leqslant i\leqslant m}\{\tau_i(t)\}$, then all solutions of (E) oscillate. This result is similar to Theorem 2.1.3 [18] which is a special case of Ladas, Lakshmikantham and Papadakis's result [15].

In 1978 Ladde [17] and in 1982 Ladas and Stavroulakis [16] proved that if

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s)ds > \frac{1}{e}, \tag{1.4}$$

then all solutions of (E) oscillate.

In 1984, Hunt and Yorke [8] proved that if $t-\tau_i(t) \leqslant \tau_0$, $1 \leqslant i \leqslant m$, and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} p_i(t)\,(t-\tau_i(t)) > \frac{1}{e}, \tag{1.5}$$

then all solutions of (E) oscillate.

Assume that $\tau_i(t)$, $1 \leqslant i \leqslant m$ are not necessarily monotone. Set

$$h_i(t) = \sup_{t_0\leqslant s\leqslant t} \tau_i(s), \quad t \geqslant t_0 \quad \text{and} \quad h(t) = \max_{1\leqslant i\leqslant m} h_i(t), \quad t \geqslant t_0 \tag{1.6}$$

and

$$\begin{aligned} a_1(t,s) &:= \exp\left\{\int_s^t \sum_{i=1}^m p_i(\zeta)d\zeta\right\} \\ a_{r+1}(t,s) &:= \exp\left\{\int_s^t \sum_{i=1}^m p_i(\zeta)a_r(\zeta,\tau_i(\zeta))d\zeta\right\}. \end{aligned} \tag{1.7}$$

Clearly, $h_i(t)$, $h(t)$ are nondecreasing and $\tau_i(t) \leqslant h_i(t) \leqslant h(t) < t$ for all $t \geqslant t_0$.

In 2016, Braverman, Chatzarakis and Stavroulakis [1] proved that if for some $r \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r(h(t), \tau_i(\zeta)) d\zeta > 1, \tag{1.8}$$

or

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r(h(t), \tau_i(\zeta)) d\zeta > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.9}$$

or

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r(h(t), \tau_i(\zeta)) d\zeta > \frac{1}{e}, \tag{1.10}$$

where $\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s) ds$, then all solutions of (E) oscillate.

Recently, Chatzarakis and Péics [4] proved that if

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r(h(\zeta), \tau_i(\zeta)) d\zeta > \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{1.11}$$

where λ_0 is the smaller root of the transcendental equation $e^{\alpha\lambda} = \lambda$, then all solutions of (E) oscillate.

1.2. ADEs

For Eq. (E$'$), the dual condition of (1.3) is

$$\limsup_{t\to\infty} \int_{t}^{\sigma(t)} \sum_{i=1}^{m} q_i(s) ds > 1, \tag{1.12}$$

where $\sigma_i(t)$, $1 \leqslant i \leqslant m$ are nondecreasing and $\sigma(t) = \min_{1\leqslant i\leqslant m}\{\sigma_i(t)\}$. (see [18], paragraph 2.7.)

In 1978 Ladde [17] and in 1982 Ladas and Stavroulakis [16] proved that if

$$\liminf_{t\to\infty} \int_{t}^{\sigma(t)} \sum_{i=1}^{m} q_i(s) ds > \frac{1}{e}, \tag{1.13}$$

then all solutions of (E$'$) oscillate.

In 1990, Zhou [23] proved that if $\sigma_i(t) - t \leqslant \sigma_0$, $1 \leqslant i \leqslant m$, and

$$\liminf_{t\to\infty} \sum_{i=1}^{m} q_i(t) \left(\sigma_i(t) - t\right) > \frac{1}{e}, \tag{1.14}$$

then all solutions of (E$'$) oscillate. (See also [5, Corollary 2.6.12])

Assume that $\sigma_i(t)$, $1 \leqslant i \leqslant m$ are not necessarily monotone. Set

$$\rho_i(t) = \inf_{s\geqslant t} \sigma_i(s), \quad t \geqslant t_0 \quad \text{and} \quad \rho(t) = \min_{1\leqslant i\leqslant m} \rho_i(t), \quad t \geqslant t_0 \tag{1.15}$$

and

$$\begin{aligned} b_1(t,s) &:= \exp\left\{\int_t^s \sum_{i=1}^{m} q_i(\zeta) d\zeta\right\} \\ b_{r+1}(t,s) &:= \exp\left\{\int_t^s \sum_{i=1}^{m} q_i(\zeta) b_r(t, \sigma_i(\zeta)) d\zeta\right\}. \end{aligned} \tag{1.16}$$

Clearly, $\rho_i(t)$, $\rho(t)$ are nondecreasing and $\sigma_i(t) \geqslant \rho_i(t) \geqslant \rho(t) > t$ for all $t \geqslant t_0$.

In 2016, Braverman, Chatzarakis and Stavroulakis [1] proved that if for some $r \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m q_i(\zeta) b_r(\rho(t), \sigma_i(\zeta))\, d\zeta > 1, \tag{1.17}$$

or

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m q_i(\zeta) b_r(\rho(t), \sigma_i(\zeta))\, d\zeta > 1 - \frac{1-\beta-\sqrt{1-2\beta-\beta^2}}{2}, \tag{1.18}$$

or

$$\liminf_{t\to\infty} \int_t^{\rho(t)} \sum_{i=1}^m q_i(\zeta) b_r(\rho(t), \sigma_i(\zeta))\, d\zeta > \frac{1}{e}, \tag{1.19}$$

where $\beta = \liminf_{t\to\infty} \int_t^{\sigma(t)} \sum_{i=1}^m q_i(s)ds$, then all solutions of (E$'$) oscillate.

2. Main results

2.1. DDEs

We further study (E) and derive new sufficient oscillation conditions, involving lim sup and lim inf, which essentially improve all known results in the literature.

THEOREM 1. *Assume that $h(t)$ is defined by (1.6) and for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_{h(t)}^t \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(t)} \overline{P}_j(u) du \right) ds > 1, \tag{2.1}$$

where

$$\overline{P}_j(t) = \overline{P}(t) \left[1 + \int_{\tau(t)}^t \overline{P}(s) \exp\left(\int_{\tau(s)}^t \overline{P}(u) \exp\left(\int_{\tau(u)}^u \overline{P}_{j-1}(\xi) d\xi \right) du \right) ds \right], \tag{2.2}$$

with $\overline{P}_0(t) = \overline{P}(t) = \sum_{i=1}^m p_i(t)$. Then all solutions of (E) are oscillatory.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (E). Since $-x(t)$ is also a solution of (E), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists $t_1 > t_0$ such that $x(t)$, $x(\tau_i(t)) > 0$, $1 \leqslant i \leqslant m$ for all $t \geqslant t_1$. Thus, from (E) we have

$$x'(t) = -\sum_{i=1}^m p_i(t) x(\tau_i(t)) \leqslant 0, \quad \text{for all } t \geqslant t_1,$$

which means that $x(t)$ is an eventually nonincreasing function of positive numbers. In view of this, and taking into accout the fact that $\tau_i(t) < t$, (E) implies

$$x'(t) + \left(\sum_{i=1}^m p_i(t) \right) x(t) \leqslant x'(t) + \sum_{i=1}^m p_i(t) x(\tau_i(t)) = 0, \quad \text{for all } t \geqslant t_1,$$

or

$$x'(t) + \overline{P}(t) x(t) \leqslant 0. \tag{2.3}$$

Dividing the last inequality by $x(t)>0$ and integrating on $[s,t]$ we obtain

$$\int_s^t \frac{x'(u)}{x(u)}du+\int_s^t \overline{P}(\xi)d\xi \leqslant 0,$$

or

$$x(s)\geqslant x(t)\exp\left(\int_s^t \overline{P}(\xi)d\xi\right), \quad t_1\leqslant \tau(s)<s\leqslant t. \tag{2.4}$$

Now we divide (E) by $x(t)>0$ and integrate on $[s,t]$, so

$$\begin{aligned}-\int_s^t \frac{x'(u)}{x(u)}du &= \int_s^t \sum\nolimits_{i=1}^m p_i(u)\frac{x(\tau_i(u))}{x(u)}du\\ &\geqslant \int_s^t \left(\sum\nolimits_{i=1}^m p_i(u)\right)\frac{x(\tau(u))}{x(u)}du\\ &= \int_s^t \overline{P}(u)\frac{x(\tau(u))}{x(u)}du\end{aligned}$$

or

$$\ln\frac{x(s)}{x(t)}\geqslant \int_s^t \overline{P}(u)\frac{x(\tau(u))}{x(u)}du. \tag{2.5}$$

Since $\tau(u)<u$, setting $u=t$, $s=\tau(u)$ in (2.4) we take

$$x(\tau(u))\geqslant x(u)\exp\left(\int_{\tau(u)}^u \overline{P}(\xi)d\xi\right). \tag{2.6}$$

Combining (2.5) and (2.6) we obtain, for sufficiently large t

$$\ln\frac{x(s)}{x(t)}\geqslant \int_s^t \overline{P}(u)\exp\left(\int_{\tau(u)}^u \overline{P}(\xi)d\xi\right)du$$

or

$$x(s)\geqslant x(t)\exp\left(\int_s^t \overline{P}(u)\exp\left(\int_{\tau(u)}^u \overline{P}(\xi)d\xi\right)du\right). \tag{2.7}$$

Integrating (E) from $\tau(t)$ to t, we have

$$x(t)-x(\tau(t))+\int_{\tau(t)}^t \sum\nolimits_{i=1}^m p_i(s)x(\tau_i(s))\,ds=0,$$

or

$$x(t)-x(\tau(t))+\int_{\tau(t)}^t \left(\sum\nolimits_{i=1}^m p_i(s)\right)x(\tau(s))\,ds\leqslant 0,$$

i.e.,

$$x(t)-x(\tau(t))+\int_{\tau(t)}^t \overline{P}(s)x(\tau(s))\,ds\leqslant 0. \tag{2.8}$$

In view of (2.7) the last inequality gives

$$x(t)-x(\tau(t))+x(t)\int_{\tau(t)}^t \overline{P}(s)\exp\left(\int_{\tau(s)}^t \overline{P}(u)\exp\left(\int_{\tau(u)}^u \overline{P}(\xi)d\xi\right)du\right)ds\leqslant 0.$$

Multiplying the last inequality by $\overline{P}(t)$, we find

$$\begin{aligned}&\overline{P}(t)x(t)-\overline{P}(t)x(\tau(t))\\&\qquad+\overline{P}(t)x(t)\int_{\tau(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}(\xi)d\xi\right)du\right)ds\leqslant 0.\end{aligned}\tag{2.9}$$

Furthermore,

$$x'(t)=-\sum\nolimits_{i=1}^{m}p_i(t)x(\tau_i(t))\leqslant -x(\tau(t))\sum\nolimits_{i=1}^{m}p_i(t)=-\overline{P}(t)x(\tau(t)).\tag{2.10}$$

Combining the inequalities (2.9) and (2.10), we have

$$x'(t)+\overline{P}(t)x(t)+\overline{P}(t)x(t)\int_{\tau(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}(\xi)d\xi\right)du\right)ds\leqslant 0.$$

Hence,

$$x'(t)+\overline{P}(t)\left[1+\int_{\tau(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}(\xi)d\xi\right)du\right)ds\right]x(t)\leqslant 0,$$

or

$$x'(t)+\overline{P}_1(t)x(t)\leqslant 0,\tag{2.11}$$

where

$$\overline{P}_1(t)=\overline{P}(t)\left[1+\int_{\tau(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}(\xi)d\xi\right)du\right)ds\right].$$

Clearly (2.11) resembles (2.3) with $\overline{P}$ replaced by $\overline{P}_1$, so an integration of (2.11) on $[s,t]$ leads to

$$x(s)\geqslant x(t)\exp\left(\int_{s}^{t}\overline{P}_1(\xi)d\xi\right).\tag{2.12}$$

Taking the steps starting from (2.3) to (2.6) we may see that x satisfies the inequality

$$x(\tau(u))\geqslant x(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}_1(\xi)d\xi\right).\tag{2.13}$$

Combining now (2.5) and (2.13), we obtain

$$\ln\frac{x(s)}{x(t)}\geqslant\int_{s}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}_1(\xi)d\xi\right)du$$

or

$$x(s)\geqslant x(t)\exp\left(\int_{s}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}_1(\xi)d\xi\right)du\right),$$

from which we take

$$x(\tau(s))\geqslant x(t)\exp\left(\int_{\tau(s)}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}_1(\xi)d\xi\right)du\right).\tag{2.14}$$

By (2.8) and (2.14) we have

$$x(t)-x(\tau(t))+x(t)\int_{\tau(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}_1(\xi)d\xi\right)du\right)ds\leqslant 0.$$

Multiplying the last inequality by $\overline{P}(t)$, as before, we find

$$x'(t)+\overline{P}(t)\left[1+\int_{\tau(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}_1(\xi)d\xi\right)du\right)ds\right]x(t)\leqslant 0.$$

Therefore, for sufficiently large t

$$x'(t)+\overline{P}_2(t)x(t)\leqslant 0, \tag{2.15}$$

where

$$\overline{P}_2(t)=\overline{P}(t)\left[1+\int_{\tau(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}_1(\xi)d\xi\right)du\right)ds\right].$$

Repeating the above procedure, it follows by induction that for sufficiently large t

$$x'(t)+\overline{P}_j(t)x(t)\leqslant 0, \quad (j\in\mathbb{N}),$$

where

$$\overline{P}_j(t)=\overline{P}(t)\left[1+\int_{\tau(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{t}\overline{P}(u)\exp\left(\int_{\tau(u)}^{u}\overline{P}_{j-1}(\xi)d\xi\right)du\right)ds\right].$$

Moreover, since $\tau(s)\leqslant h(s)\leqslant h(t)$ from (2.13) we have

$$x(\tau(s))\geqslant x(h(t))\exp\left(\int_{\tau(s)}^{h(t)}\overline{P}_j(u)\,du\right). \tag{2.16}$$

Integrating (E) from $h(t)$ to t and using the above inequality, we obtain

$$\begin{aligned}
0&=x(t)-x(h(t))+\int_{h(t)}^{t}\sum\nolimits_{i=1}^{m}p_i(s)x(\tau_i(s))\,ds\\
&\geqslant x(t)-x(h(t))+\int_{h(t)}^{t}\left(\sum\nolimits_{i=1}^{m}p_i(s)\right)x(\tau(s))\,ds\\
&=x(t)-x(h(t))+\int_{h(t)}^{t}\overline{P}(s)x(\tau(s))\,ds\\
&\geqslant x(t)-x(h(t))+x(h(t))\int_{h(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{h(t)}\overline{P}_j(u)du\right)ds
\end{aligned}$$

or

$$x(t)-x(h(t))+x(h(t))\int_{h(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{h(t)}\overline{P}_j(u)du\right)ds\leqslant 0. \tag{2.17}$$

The strict inequality is valid if we omit $x(t) > 0$ in the left-hand side:

$$0 > -x(h(t)) + x(h(t)) \int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(t)} \overline{P}_j(u) du \right) ds,$$

or

$$x(h(t)) \left[\int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(t)} \overline{P}_j(u) du \right) ds - 1 \right] < 0,$$

i.e.,

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(t)} \overline{P}_j(u) du \right) ds \leqslant 1.$$

This contradicts (2.1).

The proof of the theorem is complete.

We now cite three lemmas which will be used in the proof of our next results. The proofs of their are similar to the proofs of Lemmas 2.1.1, 2.1.3 and 2.1.2 in [5], respectively.

LEMMA 1 *Assume that* $h(t)$ *is defined by* (1.6). *Then*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s) ds = \liminf_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s) ds. \tag{2.18}$$

Next lemma provides a lower estimate for the ratio $x(t)/x(h(t))$ in terms of the smaller root of the equation $\xi^2 - (1-\alpha)\xi + \alpha^2/2 = 0$, where α is given by

$$0 < \alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{m} p_i(s) ds \leqslant \frac{1}{e}. \tag{2.19}$$

LEMMA 2 *Assume that* x *is an eventually positive solution of (E),* $h(t)$ *is defined by* (1.6) *and* α *by* (2.19). *Then*

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \geqslant \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}. \tag{2.20}$$

The last lemma provides a lower estimate for the ratio $x(h(t))/x(t)$ in terms of the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

LEMMA 3 *Assume that* $h(t)$ *is defined by* (1.6), x *is a positive solution of (E) and* α *is defined by* (2.19). *Then*

$$\liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \geqslant \lambda_0, \tag{2.21}$$

where λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\alpha\lambda}$.

Based on the above lemmas, we establish the following three theorems.

THEOREM 2. *Assume that* α *is defined by* (2.19), $h(t)$ *by* (1.6) *and for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{h(t)}\overline{P}_j(u)du\right)ds > 1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.22}$$

where $\overline{P}_j$ *is defined by* (2.2). *Then all solutions of (E) are oscillatory.*

Proof. Let x be an eventually positive solution of (E). Then, as in the proof of Theorem 1, (2.17) is satisfied, i.e.,

$$x(t)-x(h(t))+x(h(t))\int_{h(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{h(t)}\overline{P}_j(u)du\right)ds \leqslant 0.$$

That is,

$$\int_{h(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{h(t)}\overline{P}_j(u)du\right)ds \leqslant 1-\frac{x(t)}{x(h(t))},$$

which gives

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{h(t)}\overline{P}_j(u)du\right)ds \leqslant 1-\liminf_{t\to\infty}\frac{x(t)}{x(h(t))}. \tag{2.23}$$

By combining Lemmas 1 and 2, it becomes obvious that inequality (2.20) is fulfilled. So, (2.23) leads to

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{h(t)}\overline{P}_j(u)du\right)ds \leqslant 1-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2},$$

which contradicts (2.22).

The proof of the theorem is complete.

REMARK 1. It is clear that the left-hand sides of both conditions (2.1) and (2.22) are identical, also the right hand side of condition (2.22) reduces to (2.1) in case that $\alpha = 0$. So it seems that Theorem 2 is the same as Theorem 1 when $\alpha = 0$. However, one may notice that condition (2.19) is required in Theorem 2 but not in Theorem 1.

THEOREM 3. *Assume that* α *is defined by* (2.19), $h(t)$ *by* (1.6) *and for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\overline{P}(s)\exp\left(\int_{\tau(s)}^{t}\overline{P}_j(u)du\right)ds > \frac{2}{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}, \tag{2.24}$$

where $\overline{P}_j$ *is defined by* (2.2). *Then all solutions of (E) are oscillatory.*

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution x of (E) and that x is eventually positive. Then, as in the proof of Theorem 1, for sufficiently large t we have

$$x(\tau(s)) \geqslant x(t)\exp\left(\int_{\tau(s)}^{t} \overline{P}_j(\xi)d\xi\right). \tag{2.25}$$

Integrating (E) from $h(t)$ to t, we have

$$x(t) - x(h(t)) + \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(s)x(\tau_i(s))ds = 0,$$

or

$$x(t) - x(h(t)) + \int_{h(t)}^{t} \left(\sum_{i=1}^{m} p_i(s)\right) x(\tau(s))ds \leqslant 0.$$

Thus

$$x(t) - x(h(t)) + \int_{h(t)}^{t} \overline{P}(s)x(\tau(s))ds \leqslant 0.$$

In view of (2.25), the last inequality gives

$$x(t) - x(h(t)) + \int_{h(t)}^{t} \overline{P}(s)x(t)\exp\left(\int_{\tau(s)}^{t} \overline{P}_j(u)du\right) ds \leqslant 0,$$

or

$$x(t) - x(h(t)) + x(h(t))\int_{h(t)}^{t} \overline{P}(s)\frac{x(t)}{x(h(t))}\exp\left(\int_{\tau(s)}^{t} \overline{P}_j(u)du\right) ds \leqslant 0.$$

Since $x(t) > 0$, it is clear that

$$x(h(t))\left[\frac{x(t)}{x(h(t))}\int_{h(t)}^{t} \overline{P}(s)\exp\left(\int_{\tau(s)}^{t} \overline{P}_j(u)du\right) ds - 1\right] < 0.$$

That is, for all sufficiently large t it holds

$$\int_{h(t)}^{t} \overline{P}(s)\exp\left(\int_{\tau(s)}^{t} \overline{P}_j(u)du\right) ds < \frac{x(h(t))}{x(t)}$$

and therefore

$$\limsup_{t\to\infty}\int_{h(t)}^{t} \overline{P}(s)\exp\left(\int_{\tau(s)}^{t} \overline{P}_j(u)du\right) ds \leqslant \limsup_{t\to\infty}\frac{x(h(t))}{x(t)}. \tag{2.26}$$

By combining Lemmas 1 and 2, it becomes obvious that inequality (2.20) is fulfilled. So, (2.26) leads to

$$\limsup_{t\to\infty}\int_{h(t)}^{t} \overline{P}(s)\exp\left(\int_{\tau(s)}^{t} \overline{P}_j(u)du\right) ds \leqslant \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2},$$

which contradicts (2.24).

The proof of the theorem is complete.

THEOREM 4. *Assume that* α *is defined by* (2.19), $h(t)$ *by* (1.6) *and for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du \right) ds > \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}, \tag{2.27}$$

where $\overline{P}_j$ *is defined by* (2.2) *and* λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\alpha\lambda}$. *Then all solutions of (E) are oscillatory.*

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution x of (E) and that x is eventually positive. Then, as in the previous theorems, (2.25) holds.

Observe that (2.21) implies that for each $\varepsilon > 0$ there exists a t_ε such that

$$\lambda_0 - \varepsilon < \frac{x(h(t))}{x(t)} \quad \text{for all } t \geqslant t_\varepsilon\,. \tag{2.28}$$

Noting that by nondecreasing nature of the function $\frac{x(h(t))}{x(s)}$ in s, it holds

$$1 = \frac{x(h(t))}{x(h(t))} \leqslant \frac{x(h(t))}{x(s)} \leqslant \frac{x(h(t))}{x(t)}, \quad t_\varepsilon \leqslant h(t) \leqslant s \leqslant t,$$

in particular for $\varepsilon \in (0, \lambda_0 - 1)$, by continuity we see that there exists a $t^* \in (h(t), t]$ such that

$$1 < \lambda_0 - \varepsilon = \frac{x(h(t))}{x(t^*)}. \tag{2.29}$$

By (2.25), it is obvious that

$$x(\tau(s)) \geqslant x(h(s)) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(\xi)d\xi \right). \tag{2.30}$$

Integrating (E) from t^* to t we have

$$x(t) - x(t^*) + \int_{t^*}^{t} \sum\nolimits_{i=1}^{m} p_i(s) x(\tau_i(s))ds = 0,$$

or

$$x(t) - x(t^*) + \int_{t^*}^{t} \left(\sum\nolimits_{i=1}^{m} p_i(s) \right) x(\tau(s))ds \leqslant 0,$$

i.e.,

$$x(t) - x(t^*) + \int_{t^*}^{t} \overline{P}(s) x(\tau(s))ds \leqslant 0.$$

By using (2.30) along with $h(s) \leqslant h(t)$ in combination with the nonincreasingness of x, we have

$$x(t) - x(t^*) + x(h(t)) \int_{t^*}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du \right) ds \leqslant 0,$$

or

$$\int_{t^*}^{t} \overline{P}(s)\exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du\right) ds \leqslant \frac{x(t^*)}{x(h(t))} - \frac{x(t)}{x(h(t))}.$$

In view of (2.29) and Lemma 2, for the ε considered, there exists $t'_\varepsilon \geqslant t_\varepsilon$ such that

$$\int_{t^*}^{t} \overline{P}(s)\exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du\right) ds < \frac{1}{\lambda_0-\varepsilon} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} + \varepsilon, \tag{2.31}$$

for $t \geqslant t'_\varepsilon$.

Dividing (E) by $x(t)$ and integrating from $h(t)$ to t^* we find

$$\int_{h(t)}^{t^*} \sum\nolimits_{i=1}^{m} p_i(s)\frac{x(\tau_i(s))}{x(s)} ds = -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds,$$

or

$$\int_{h(t)}^{t^*} \left(\sum\nolimits_{i=1}^{m} p_i(s)\right) \frac{x(\tau(s))}{x(s)} ds \leqslant -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds,$$

i.e.,

$$\int_{h(t)}^{t^*} \overline{P}(s)\frac{x(\tau(s))}{x(s)} ds \leqslant -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds,$$

and using (2.30), we find

$$\int_{h(t)}^{t^*} \overline{P}(s)\frac{x(h(s))}{x(s)} \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du\right) ds \leqslant -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds. \tag{2.32}$$

By (2.28), for $s \geqslant h(t) \geqslant t'_\varepsilon$, we have $\frac{x(h(s))}{x(s)} > \lambda_0 - \varepsilon$, so from (2.32) we get

$$(\lambda_0-\varepsilon)\int_{h(t)}^{t^*} \overline{P}(s)\exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du\right) ds < -\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds.$$

Hence, for all sufficiently large t we have

$$\begin{aligned}\int_{h(t)}^{t^*} \overline{P}(s)\exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du\right) ds \\ < -\frac{1}{\lambda_0-\varepsilon}\int_{h(t)}^{t^*} \frac{x'(s)}{x(s)} ds = \frac{1}{\lambda_0-\varepsilon}\ln\frac{x(h(t))}{x(t^*)} = \frac{\ln(\lambda_0-\varepsilon)}{\lambda_0-\varepsilon},\end{aligned}$$

i.e.,

$$\int_{h(t)}^{t^*} \overline{P}(s)\exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du\right) ds < \frac{\ln(\lambda_0-\varepsilon)}{\lambda_0-\varepsilon}. \tag{2.33}$$

Adding (2.31) and (2.33), and then taking the limit as $t \to \infty$, we have

$$\begin{aligned}\limsup_{t\to\infty} \int_{h(t)}^{t} \overline{P}(s)\exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du\right) ds \\ \leqslant \frac{1+\ln(\lambda_0-\varepsilon)}{\lambda_0-\varepsilon} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} + \varepsilon.\end{aligned}$$

Since ε may be taken arbitrarily small, this inequality contradicts (2.27).

The proof of the theorem is complete.

THEOREM 5. *Assume that $h(t)$ is defined by (1.6) and for some $j \in \mathbb{N}$*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du\right) ds > \frac{1}{e}, \tag{2.34}$$

where $\overline{P}_j$ is defined by (2.2). Then all solutions of (E) are oscillatory.

Proof. Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (E). Since $-x(t)$ is also a solution of (E), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists $t_1 > t_0$ such that $x(t),\ x(\tau_i(t)) > 0,\ 1 \leqslant i \leqslant m$ for all $t \geqslant t_1$. Thus, from (E) we have

$$x'(t) = -\sum\nolimits_{i=1}^{m} p_i(t)x(\tau_i(t)) \leqslant 0, \quad \text{for all } t \geqslant t_1,$$

which means that $x(t)$ is an eventually nonincreasing function of positive numbers. Furthermore, as in previous theorem, (2.30) is satisfied.

Dividing (E) by $x(t)$ and integrating from $h(t)$ to t, for some $t_2 \geqslant t_1$, we have

$$\begin{aligned}
\ln\left(\frac{x(h(t))}{x(t)}\right) &= \int_{h(t)}^{t} \sum\nolimits_{i=1}^{m} p_i(s)\frac{x(\tau_i(s))}{x(s)}ds \\
&\geqslant \int_{h(t)}^{t} \left(\sum\nolimits_{i=1}^{m} p_i(s)\right) \frac{x(\tau(s))}{x(s)}ds \\
&= \int_{h(t)}^{t} \overline{P}_j(s)\frac{x(\tau(s))}{x(s)}ds.
\end{aligned} \tag{2.35}$$

Combining the inequalities (2.35) and (2.30) we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geqslant \int_{h(t)}^{t} \overline{P}(s)\frac{x(h(s))}{x(s)} \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)\,du\right) ds.$$

Taking into account that x is nonincreasing and $h(s) < s$, the last inequality becomes

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geqslant \int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)\,du\right) ds. \tag{2.36}$$

From (2.34), it follows that there exists a constant $c > 0$ such that for some $t_3 \geqslant t_2$

$$\int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)\,du\right) ds \geqslant c > \frac{1}{e}, \quad t \geqslant t_3. \tag{2.37}$$

Combining inequalities (2.36) and (2.37), we obtain

$$\ln\left(\frac{x(h(t))}{x(t)}\right) \geqslant c, \quad t \geqslant t_3.$$

Thus

$$\frac{x(h(t))}{x(t)} \geqslant e^c \geqslant ec > 1,$$

which implies for some $t \geqslant t_4 \geqslant t_3$

$$x(h(t)) \geqslant (ec)x(t).$$

Repeating the above procedure, it follows by induction that for any positive integer k,

$$\frac{x(h(t))}{x(t)} \geqslant (ec)^k, \quad \text{for sufficiently large } t.$$

Since $ec > 1$, there is $k \in \mathbb{N}$ satisfying $k > 2(\ln(2) - \ln(c))/(1 + \ln(c))$ such that for t sufficiently large

$$\frac{x(h(t))}{x(t)} \geqslant (ec)^k > \frac{4}{c^2}. \tag{2.38}$$

Next we split the integral in (2.37) into two integrals, each integral being no less than $c/2$:

$$\begin{aligned} &\textstyle\int_{h(t)}^{t_m} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)\,du\right) ds \geqslant \frac{c}{2}, \\ &\textstyle\int_{t_m}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)\,du\right) ds \geqslant \frac{c}{2}. \end{aligned} \tag{2.39}$$

Integrating (E) from t_m to t, gives

$$x(t) - x(t_m) + \int_{t_m}^{t} \sum_{i=1}^{m} p_i(s)x(\tau_i(s)) = 0,$$

or

$$x(t) - x(t_m) + \int_{t_m}^{t} \left(\sum_{i=1}^{m} p_i(s)\right) x(\tau(s)) \leqslant 0.$$

Thus

$$x(t) - x(t_m) + \int_{t_m}^{t} \overline{P}(s)x(\tau(s)) \leqslant 0,$$

which, in view of (2.30), gives

$$x(t) - x(t_m) + x(h(t)) \int_{t_m}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)\,du\right) ds \leqslant 0.$$

The strict inequality is valid if we omit $x(t) > 0$ in the left-hand side:

$$-x(t_m) + x(h(t)) \int_{t_m}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)\,du\right) ds < 0.$$

Together with the second inequality in (2.39), implies that

$$x(t_m) > \frac{c}{2} x(h(t)). \tag{2.40}$$

Similarly, integration of (E) from $h(t)$ to t_m with a later application of (2.30) leads to

$$x(t_m) - x(h(t)) + x(h(t_m)) \int_{h(t)}^{t_m} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)\, du \right) ds \leqslant 0.$$

The strict inequality is valid if we omit $x(t_m) > 0$ in the left-hand side:

$$-x(h(t)) + x(h(t_m)) \int_{h(t)}^{t_m} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)\, du \right) ds < 0.$$

Together with the first inequality in (2.39) implies that

$$x(h(t)) > \frac{c}{2} x(h(t_m)). \tag{2.41}$$

Combining the inequalities (2.40) and (2.41), we obtain

$$x(h(t_m)) < \frac{2}{c} x(h(t)) < \frac{4}{c^2} x(t_m),$$

which contradicts (2.38).

The proof of the theorem is complete.

2.2. Advanced differential equations

Similar oscillation conditions for the (dual) advanced differential equation (E′) can be derived easily. The proofs are omitted, since they are quite similar to the delay equation.

THEOREM 6. *Assume that $\rho(t)$ is defined by (1.15) and for some $j \in \mathbb{N}$*

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \overline{Q}(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} \overline{Q}_j(u) du \right) ds > 1, \tag{2.42}$$

where

$$\overline{Q}_j(t) = \overline{Q}(t) \left[1 + \int_t^{\sigma(t)} \overline{Q}(s) \exp\left(\int_t^{\sigma(s)} \overline{Q}(u) \exp\left(\int_u^{\sigma(u)} \overline{Q}_{j-1}(\xi) d\xi \right) du \right) ds \right], \tag{2.43}$$

with $\overline{Q}_0(t) = \overline{Q}(t) = \sum_{i=1}^m q_i(t)$. Then all solutions of (E') are oscillatory.

THEOREM 7. *Assume that*

$$0 < \beta := \liminf_{t\to\infty} \int_t^{\sigma(t)} \sum_{i=1}^m q_i(s) ds \leqslant \frac{1}{e} \tag{2.44}$$

and for some $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \overline{Q}(s) \exp\left(\int_{\rho(t)}^{\sigma(s)} \overline{Q}_j(u) du \right) ds > 1 - \frac{1 - \beta - \sqrt{1 - 2\beta - \beta^2}}{2}, \tag{2.45}$$

where $\overline{Q}_j$ is defined by (2.43) and $\rho(t)$ by (1.15). Then all solutions of (E') are oscillatory.

THEOREM 8. *Assume that* $\rho(t)$ *is defined by* (1.15), β *by (2.44) and for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \overline{Q}(s) \exp\left(\int_t^{\sigma(s)} \overline{Q}_j(u)du\right) ds > \frac{2}{1-\beta-\sqrt{1-2\beta-\beta^2}}, \tag{2.46}$$

where $\overline{Q}_j$ *is defined by* (2.43). *Then all solutions of (E′) are oscillatory.*

THEOREM 9. *Assume that* $\rho(t)$ *is defined by* (1.15), β *by (2.44) and for some* $j \in \mathbb{N}$

$$\limsup_{t\to\infty} \int_t^{\rho(t)} \overline{Q}(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} \overline{Q}_j(u)du\right) ds > \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-\beta-\sqrt{1-2\beta-\beta^2}}{2}, \tag{2.47}$$

where $\overline{Q}_j$ *is defined by* (2.43) *and* λ_0 *is the smaller root of the transcendental equation* $\lambda = e^{\beta\lambda}$. *Then all solutions of (E′) are oscillatory.*

THEOREM 10. *Assume that* $\rho(t)$ *is defined by* (1.15) *and for some* $j \in \mathbb{N}$

$$\liminf_{t\to\infty} \int_t^{\rho(t)} \overline{Q}(s) \exp\left(\int_{\rho(s)}^{\sigma(s)} \overline{Q}_j(u)du\right) ds > \frac{1}{e}, \tag{2.48}$$

where $\overline{Q}_j$ *is defined by* (2.43). *Then all solutions of (E′) are oscillatory.*

2.3. Differential inequalities

A slight modification in the proofs of Theorems $1-10$ leads to the following results about differential inequalities.

THEOREM 11. *Assume that all the conditions of Theorem 1 [6] or 2 [7] or 3 [8] or 4 [9] or 5 [10] hold. Then*

(i) the delay [advanced] differential inequality

$$x'(t) + \sum_{i=1}^m p_i(t)x(\tau_i(t)) \leqslant 0 \quad \left[x'(t) - \sum_{i=1}^m q_i(t)x(\sigma_i(t)) \geqslant 0\right], \ \forall t \geqslant t_0,$$

has no eventually positive solutions;

(ii) the delay [advanced] differential inequality

$$x'(t) + \sum_{i=1}^m p_i(t)x(\tau_i(t)) \geqslant 0 \quad \left[x'(t) - \sum_{i=1}^m q_i(t)x(\sigma_i(t)) \leqslant 0\right], \ \forall t \geqslant t_0,$$

has no eventually negative solutions.

3. Examples

In this section, examples illustrate cases when the results of the present paper imply oscillation while previously known results fail. The calculations were made by the use of MATLAB software.

EXAMPLE 1. Consider the delay differential equation

$$x'(t) + \frac{4}{25}x(\tau_1(t)) + \frac{2}{25}x(\tau_2(t)) = 0, \quad t \geqslant 0, \tag{3.1}$$

with (see Fig. 1, (a))

$$\tau_1(t) = \begin{cases} 1.5t - 2.5k - 1, & \text{if } t \in [5k, 5k+1] \\ -0.5t + 7.5k + 1, & \text{if } t \in [5k+1, 5k+2] \\ t - 2, & \text{if } t \in [5k+2, 5k+3] \\ -1.5t + 12.5k + 5.5, & \text{if } t \in [5k+3, 5k+4] \\ 4.5t - 17.5k - 18.5, & \text{if } t \in [5k+4, 5k+5] \end{cases} \quad \text{and} \quad \tau_2(t) = \tau_1(t) - 0.5$$

where $k \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

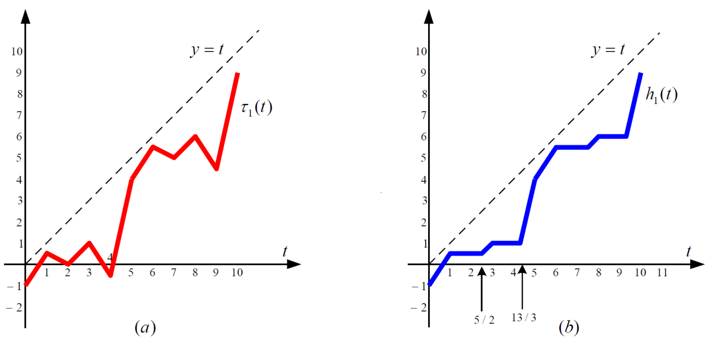

Figure 1: The graphs of $\tau_1(t)$ and $h_1(t)$

By (1.6), we see (Fig. 1, (b)) that

$$h_1(t) = \begin{cases} 1.5t - 2.5k - 1, & \text{if } t \in [5k, 5k+1] \\ 5k + 0.5, & \text{if } t \in [5k+1, 5k+2.5] \\ t - 2, & \text{if } t \in [5k+2.5, 5k+3] \\ 5k + 1, & \text{if } t \in [5k+3, 5k+13/3] \\ 4.5t - 17.5k - 18.5, & \text{if } t \in [5k+13/3, 5k+5] \end{cases} \quad \text{and} \quad h_2(t) = h_1(t) - 0.5$$

and consequently

$$h(t) = \max_{1 \leqslant i \leqslant 2}\{h_i(t)\} = h_1(t) \quad \text{and} \quad \tau(t) = \max_{1 \leqslant i \leqslant 2}\{\tau_i(t)\} = \tau_1(t).$$

Observe that the function $F_j : [0,\infty) \to \mathbb{R}_+$ defined as

$$F_j(t) = \int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(t)} \overline{P}_j(u)du \right) ds$$

attains its maximum at $t = 5k + 13/3$, $k \in \mathbb{N}_0$, for every $j \geqslant 1$. Specifically, by using an algorithm on MATLAB software and taking into account the fact that $\overline{P}(t) = \sum_{i=1}^{2} p_i(t) = \frac{6}{25}$, we obtain

$$\begin{aligned}
F_1(t = 5k + 13/3) &= \int_{5k+1}^{5k+13/3} \overline{P}(s) \exp\left(\int_{\tau(s)}^{5k+1} \overline{P}_1(u)du \right) ds \\
&= \int_{5k+1}^{5k+13/3} \overline{P}(s) \exp\left\{ \int_{\tau(s)}^{5k+1} \overline{P}(u) \left[1 \right.\right. \\
&\quad \left.\left. + \int_{\tau(u)}^{u} \overline{P}(v) \exp\left(\int_{\tau(v)}^{u} \overline{P}(\xi) \exp\left(\int_{\tau(\xi)}^{\xi} \overline{P}_0(z)dz \right) d\xi \right) dv \right] du \right\} ds \\
&\simeq 1.0209.
\end{aligned}$$

Thus

$$\limsup_{t\to\infty} F_1(t) = \limsup_{t\to\infty} \int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(t)} \overline{P}_1(u)du \right) ds \simeq 1.0209 > 1,$$

that is, condition (2.1) of Theorem 1 is satisfied for $j = 1$, and therefore all solutions of (3.1) oscillate.

Observe, however, that

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{2} p_i(s)ds = \limsup_{k\to\infty} \int_{5k+1}^{5k+13/3} \sum_{i=1}^{2} p_i(s)ds = 0.8 < 1,$$

$$\alpha = \liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{2} p_i(s)ds = \liminf_{k\to\infty} \int_{5k+0.5}^{5k+1} \sum_{i=1}^{2} p_i(s)ds = 0.12 < \frac{1}{e},$$

and

$$\begin{aligned}
\liminf_{t\to\infty} \sum_{i=1}^{2} & p_i(t)\,(t - \tau_i(t)) \\
&= \liminf_{t\to\infty} \left[\frac{4}{25}(t - \tau_1(t)) + \frac{2}{25}(t - (\tau_1(t) - 0.5)) \right] \\
&= \liminf_{t\to\infty} \left[\frac{6}{25}(t - \tau_1(t)) + \frac{1}{25} \right] = \liminf_{t\to\infty} \left[\frac{6}{25}(t - \tau_1(t)) \right] + \frac{1}{25} \\
&= \frac{6}{25} \cdot \liminf_{t\to\infty} (t - \tau_1(t)) + \frac{1}{25} = \frac{6}{25} \cdot 0.5 + \frac{1}{25} = 0.16 < \frac{1}{e}.
\end{aligned}$$

Also, observe that the function $G_r : [0,\infty) \to \mathbb{R}_+$ defined as

$$G_r(t) = \int_{h(t)}^{t} \sum_{i=1}^{m} p_i(\zeta) a_r(h(t), \tau_i(\zeta)) d\zeta$$

attains its maximum at $t = 5k+13/3$ and its minimum at $t = 5k+1,\ k \in \mathbb{N}_0$, for every $r \in \mathbb{N}$. Specifically,

$$\begin{aligned}
G_1(t=5k+13/3) &= \int_{5k+1}^{5k+13/3} [p_1(\zeta)a_1(5k+1,\tau_1(\zeta)) + p_2(\zeta)a_1(5k+1,\tau_2(\zeta))]\,d\zeta \\
&= \int_{5k+1}^{5k+2} [p_1(\zeta)a_1(5k+1,\tau_1(\zeta)) + p_2(\zeta)a_1(5k+1,\tau_2(\zeta))]\,d\zeta \\
&\quad + \int_{5k+2}^{5k+3} [p_1(\zeta)a_1(5k+1,\tau_1(\zeta)) + p_2(\zeta)a_1(5k+1,\tau_2(\zeta))]\,d\zeta \\
&\quad + \int_{5k+3}^{5k+4} [p_1(\zeta)a_1(5k+1,\tau_1(\zeta)) + p_2(\zeta)a_1(5k+1,\tau_2(\zeta))]\,d\zeta \\
&\quad + \int_{5k+4}^{5k+13/3} [p_1(\zeta)a_1(5k+1,\tau_1(\zeta)) + p_2(\zeta)a_1(5k+1,\tau_2(\zeta))]\,d\zeta \\
&\simeq 0.9840529
\end{aligned}$$

and

$$\begin{aligned}
G_1(t=5k+1) &= \int_{5k+0.5}^{5k+1} [p_1(\zeta)a_1(5k+0.5,\tau_1(\zeta)) + p_2(\zeta)a_1(5k+0.5,\tau_2(\zeta))]\,d\zeta \\
&\simeq 0.137066.
\end{aligned}$$

Thus

$$\begin{aligned}
&\limsup_{t\to\infty} G_1(t) \simeq 0.9840529 < 1 \\
&\liminf_{t\to\infty} G_1(t) \simeq 0.137066 < 1/e
\end{aligned}$$

and

$$0.9840529 < 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \simeq 0.99174.$$

Also

$$\begin{aligned}
\Phi_1(t=5k+13/3) &= \int_{5k+1}^{5k+13/3} \sum_{i=1}^{2} p_i(\zeta)a_1(h(\zeta),\tau_i(\zeta))\,d\zeta \\
&= \int_{5k+1}^{5k+13/3} [p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta)) + p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))]\,d\zeta \\
&= \int_{5k+1}^{5k+2} [p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta)) + p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))]\,d\zeta \\
&\quad + \int_{5k+2}^{5k+3} [p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta)) + p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))]\,d\zeta \\
&\quad + \int_{5k+3}^{5k+4} [p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta)) + p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))]\,d\zeta \\
&\quad + \int_{5k+4}^{5k+13/3} [p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta)) + p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))]\,d\zeta \\
&\simeq 0.90841.
\end{aligned}$$

Thus

$$\limsup_{t\to\infty}\Phi_1(t)\simeq 0.90841<\frac{1+\ln\lambda_0}{\lambda_0}-\frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2}\simeq 0.98309,$$

where $\lambda_0 = 1.14765$ is the smaller root of $e^{0.12\lambda}=\lambda$.

That is, none of the conditions (1.3), (1.4), (1.5), (1.8) (1.9), (1.10) and (1.11) is satisfied.

Notation. It is worth noting that the improvement of condition (2.1) to the corresponding condition (1.3) is significant, approximately 27.6%, if we compare the values on the left-side of these conditions. Also, observe that conditions (1.8), (1.9), (1.10) and (1.11) do not lead to oscillation for first iteration. On the contrary, condition (2.1) is satisfied from the first iteration. This means that our condition is better and much faster than (1.8), (1.9), (1.10) and (1.11).

EXAMPLE 2. Consider the delay differential equation

$$x'(t)+\frac{47}{250}x(\tau_1(t))+\frac{4}{250}x(\tau_2(t))=0,\quad t\geqslant 0, \tag{3.2}$$

with (see Fig. 2, (a))

$$\tau_1(t)=\begin{cases}6k-2, & \text{if } t\in[6k,6k+1]\\ t-3, & \text{if } t\in[6k+1,6k+2]\\ 2t-6k-5, & \text{if } t\in[6k+2,6k+3]\\ -2t+18k+7, & \text{if } t\in[6k+3,6k+4]\\ 5t-24k-21, & \text{if } t\in[6k+4,6k+5]\\ 6k+4, & \text{if } t\in[6k+5,6k+6]\end{cases}\quad\text{and}\quad \tau_2(t)=\tau_1(t)-1$$

where $k\in\mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

By (1.6), we see (Fig. 2, (b)) that

$$h_1(t)=\begin{cases}6k-2, & \text{if } t\in[6k,6k+1]\\ t-3, & \text{if } t\in[6k+1,6k+2]\\ 2t-6k-5, & \text{if } t\in[6k+2,6k+3]\\ 6k+1, & \text{if } t\in[6k+3,6k+4.4]\\ 5t-24k-21, & \text{if } t\in[6k+4.4,6k+5]\\ 6k+4, & \text{if } t\in[6k+5,6k+6]\end{cases}\quad\text{and}\quad h_2(t)=h_1(t)-1$$

and consequently

$$h(t)=\max_{1\leqslant i\leqslant 2}\{h_i(t)\}=h_1(t)\quad\text{and}\quad \tau(t)=\max_{1\leqslant i\leqslant 2}\{\tau_i(t)\}=\tau_1(t).$$

It is easy to see that

$$\alpha=\liminf_{t\to\infty}\int_{\tau(t)}^{t}\sum\nolimits_{i=1}^{2}p_i(s)ds=\liminf_{k\to\infty}\int_{6k+4}^{6k+5}\frac{51}{250}ds=0.204<\frac{1}{e}.$$

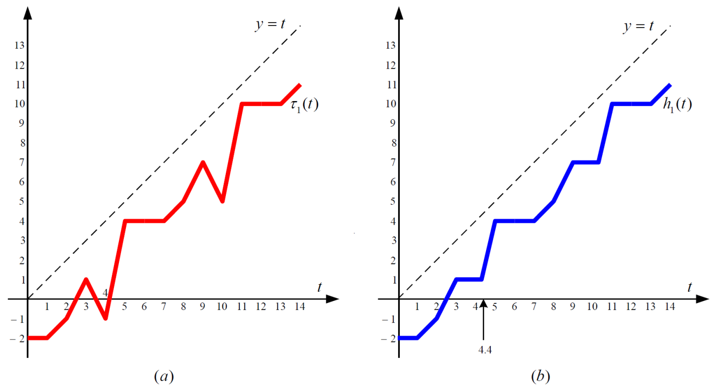

Figure 2: The graphs of $\tau_1(t)$ and $h_1(t)$

Observe that the function $F_j : \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_j(t) = \int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du \right) ds$$

attains its maximum at $t = 6k + 4.4$, $k \in \mathbb{N}_0$, for every $j \in \mathbb{N}$. Specifically, by using an algorithm on MATLAB software and taking into account the fact that $\overline{P}(t) = \sum_{i=1}^{2} p_i(t) = \frac{51}{250}$, we obtain

$$\begin{aligned}
F_1(t = 5k + 13/3) &= \int_{6k+1}^{6k+4.4} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_1(u)du \right) ds \\
&= \int_{6k+1}^{6k+4.4} \overline{P}(s) \exp\left\{ \int_{\tau(s)}^{h(s)} \overline{P}(u) \left[1 \right.\right. \\
&\quad \left.\left. + \int_{\tau(u)}^{u} \overline{P}(v) \exp\left(\int_{\tau(v)}^{u} \overline{P}(\xi) \exp\left(\int_{\tau(\xi)}^{\xi} \overline{P}_0(z)dz \right) d\xi \right) dv \right] du \right\} ds \\
&\simeq 0.9516.
\end{aligned}$$

Thus

$$\limsup_{t\to\infty} F_1(t) = \limsup_{t\to\infty} \int_{h(t)}^{t} \overline{P}(s) \exp\left(\int_{\tau(s)}^{h(s)} \overline{P}_j(u)du \right) ds \simeq 0.9516.$$

Since the smaller root of $e^{0.204\lambda} = \lambda$ is $\lambda_0 = 1.30503$, we have

$$0.9516 > \frac{1 + \ln \lambda_0}{\lambda_0} - \frac{1 - a - \sqrt{1 - 2a - a^2}}{2} \simeq 0.9432.$$

That is, condition (2.25) of Theorem 4 is satisfied for $j = 1$, and therefore all solutions of (3.2) oscillate.

Observe, however, that

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\sum_{i=1}^{2}p_i(s)ds=\frac{51}{250}\cdot 3.4=0.6936<1$$

and

$$\begin{aligned}\liminf_{t\to\infty}\sum_{i=1}^{2}p_i(t)\left(t-\tau_i(t)\right)&=\liminf_{t\to\infty}\left[\frac{47}{250}\left(t-\tau_1(t)\right)+\frac{4}{250}\left(t-\tau_2(t)\right)\right]\\&=\liminf_{t\to\infty}\left[\frac{47}{250}\left(t-\tau_1(t)\right)+\frac{4}{250}\left(t-(\tau_1(t)-1)\right)\right]\\&=\frac{51}{250}\cdot\liminf_{t\to\infty}\left(t-\tau_1(t)\right)+\frac{4}{250}=\frac{51}{250}\cdot 1+\frac{4}{250}=0.22<\frac{1}{e}.\end{aligned}$$

Also, observe that the function $G_r:[0,\infty)\to\mathbb{R}_+$ defined as

$$G_r(t)=\int_{h(t)}^{t}\sum_{i=1}^{2}p_i(\zeta)a_r(h(t),\tau_i(\zeta))d\zeta$$

attains its maximum at $t=6k+4.4$ and its minimum at $t=6k+5,\ k\in\mathbb{N}_0$, for every $r\in\mathbb{N}$. Specifically,

$$\begin{aligned}G_1(t=6k+4.4)&=\int_{6k+1}^{6k+4.4}\left[p_1(\zeta)a_1(6k+1,\tau_1(\zeta))+p_2(\zeta)a_1(6k+1,\tau_2(\zeta))\right]d\zeta\\&=\int_{6k+1}^{6k+2}\left[p_1(\zeta)a_1(6k+1,\tau_1(\zeta))+p_2(\zeta)a_1(6k+1,\tau_2(\zeta))\right]d\zeta\\&\quad+\int_{6k+2}^{6k+3}\left[p_1(\zeta)a_1(6k+1,\tau_1(\zeta))+p_2(\zeta)a_1(6k+1,\tau_2(\zeta))\right]d\zeta\\&\quad+\int_{6k+3}^{6k+4}\left[p_1(\zeta)a_1(6k+1,\tau_1(\zeta))+p_2(\zeta)a_1(6k+1,\tau_2(\zeta))\right]d\zeta\\&\quad+\int_{6k+4}^{6k+4.4}\left[p_1(\zeta)a_1(6k+1,\tau_1(\zeta))+p_2(\zeta)a_1(6k+1,\tau_2(\zeta))\right]d\zeta\\&\simeq 0.96167432\end{aligned}$$

and

$$\begin{aligned}G_1(t=6k+5)&=\int_{6k+4}^{6k+5}\left[p_1(\zeta)a_1(6k+4,\tau_1(\zeta))+p_2(\zeta)a_1(6k+4,\tau_2(\zeta))\right]d\zeta\\&\simeq 0.3609333.\end{aligned}$$

Thus

$$\limsup_{t\to\infty}G_1(t)\simeq 0.96167432<1$$

and

$$\liminf_{t\to\infty}G_1(t)\simeq 0.3609333<\frac{1}{e}.$$

Also

$$
\begin{aligned}
\Phi_1(t=6k+4.4) &= \int_{6k+1}^{6k+4.4} \left[p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta))+p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))\right] d\zeta \\
&= \int_{6k+1}^{6k+2} \left[p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta))+p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))\right] d\zeta \\
&\quad + \int_{6k+2}^{6k+3} \left[p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta))+p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))\right] d\zeta \\
&\quad + \int_{6k+3}^{6k+4} \left[p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta))+p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))\right] d\zeta \\
&\quad + \int_{6k+4}^{6k+4.4} \left[p_1(\zeta)a_1(h(\zeta),\tau_1(\zeta))+p_2(\zeta)a_1(h(\zeta),\tau_2(\zeta))\right] d\zeta \\
&\simeq 0.77416556.
\end{aligned}
$$

Thus

$$
\limsup_{t\to\infty}\Phi_1(t) \simeq 0.77416556 < \frac{1+\ln\lambda_0}{\lambda_0} - \frac{1-a-\sqrt{1-2a-a^2}}{2} \simeq 0.9432.
$$

That is, none of conditions (1.3), (1.4), (1.5), (1.8), (1.10) and (1.11) is satisfied.

Notation. It is worth noting that the improvement of condition (2.25) to the corresponding condition (1.3) is significant, approximately 37.2%, if we compare the values on the left-side of these conditions. Also, observe that conditions (1.8), (1.10) and (1.11) do not lead to oscillation for first iteration. On the contrary, condition (2.25) is satisfied from the first iteration. This means that our condition is better and much faster than (1.8), (1.10) and (1.11).

EXAMPLE 3. Consider the advanced differential equation

$$
x'(t) - \frac{1101}{5000}x(\sigma_1(t)) - \frac{60}{5000}x(\sigma_2(t)) = 0, \quad t \geqslant 0, \tag{3.3}
$$

with (see Fig. 3, (a))

$$
\sigma_1(t) = \begin{cases} 5k+3, & \text{if } t \in [5k,5k+1] \\ 4t-15k-1, & \text{if } t \in [5k+1,5k+2] \\ -3t+20k+13, & \text{if } t \in [5k+2,5k+3] \\ 5t-20k-11, & \text{if } t \in [5k+3,5k+4] \\ -t+10k+13, & \text{if } t \in [5k+4,5k+5] \end{cases} \quad \text{and} \quad \sigma_2(t) = \sigma_1(t)+0.5
$$

where $k \in \mathbb{N}_0$ and $\mathbb{N}_0$ is the set of nonnegative integers.

By (1.15), we see (Fig. 3, (b)) that

$$
\rho_1(t) = \begin{cases} 5k+3, & \text{if } t \in [5k,5k+1] \\ 4t-15k-1, & \text{if } t \in [5k+1,5k+1.25] \\ 5k+4, & \text{if } t \in [5k+1.25,5k+3] \\ 5t-20k-11, & \text{if } t \in [5k+3,5k+3.8] \\ 5k+8, & \text{if } t \in [5k+3.8,5k+5] \end{cases} \quad \text{and} \quad \rho_2(t) = \rho_1(t)+0.5
$$

and consequently

$$\rho(t) = \min_{1\leqslant i\leqslant 2}\{\rho_i(t)\} = \rho_1(t) \quad \text{and} \quad \sigma(t) = \min_{1\leqslant i\leqslant 2}\{\sigma_i(t)\} = \sigma_1(t).$$

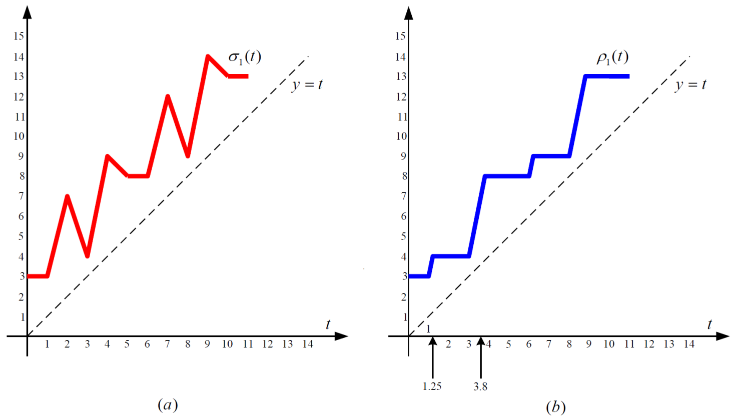

Figure 3: The graphs of $\sigma_1(t)$ and $\rho_1(t)$

Observe, that the function $F_j : \mathbb{R}_0 \to \mathbb{R}_+$ defined as

$$F_j(t) = \int_t^{\rho(t)} \overline{Q}(s)\exp\left(\int_{\rho(s)}^{\sigma(s)} \overline{Q}_j(u)du\right)ds$$

attains its minimum at $t = 5k+3$, $k \in \mathbb{N}_0$, for every $j \in \mathbb{N}$. Specifically, by using an algorithm on MATLAB software and taking into account the fact that $\overline{Q}(t) = \sum_{i=1}^{2} q_i(t) = \frac{1161}{5000}$, we obtain

$$\begin{aligned}
F_1(t = 5k+3) &= \int_{5k+3}^{5k+4} \overline{Q}(s)\exp\left(\int_{\rho(s)}^{\sigma(s)} \overline{Q}_1(u)du\right)ds \\
&= \int_{5k+3}^{5k+4} \overline{Q}(s)\exp\Bigg\{\int_{\rho(s)}^{\sigma(s)} \overline{Q}(u)\Bigg[1 \\
&\quad + \int_u^{\sigma(u)} \overline{Q}(v)\exp\left(\int_u^{\sigma(v)} \overline{Q}(\xi)\exp\left(\int_\xi^{\sigma(\xi)} \overline{Q}_0(z)dz\right)d\xi\right)dv\Bigg]du\Bigg\}ds \\
&= \int_{5k+3}^{5k+4} \frac{1161}{5000}\exp\Bigg\{\int_{\rho(s)}^{\sigma(s)} \frac{1161}{5000}\Bigg[1 \\
&\quad + \int_u^{\sigma(u)} \frac{1161}{5000}\exp\left(\int_u^{\sigma(v)} \frac{1161}{5000}\exp\left(\int_\xi^{\sigma(\xi)} \frac{1161}{5000}dz\right)d\xi\right)dv\Bigg]du\Bigg\}ds \\
&\simeq 0.3689.
\end{aligned}$$

Hence

$$\liminf_{t\to\infty} F_1(t) \simeq 0.3689 > \frac{1}{e}.$$

That is, condition (2.48) of Theorem 10 is satisfied for $j=1$, and therefore all solutions of (3.3) oscillate. Observe, however, that

$$\limsup_{t\to\infty}\int_t^{\rho(t)}\sum_{i=1}^2 q_i(s)ds=\limsup_{k\to\infty}\int_{5k+3.8}^{5k+8}\frac{1161}{5000}ds=0.97524<1,$$

$$\liminf_{t\to\infty}\int_t^{\sigma(t)}\sum_{i=1}^m q_i(s)ds=\liminf_{k\to\infty}\int_{5k+3}^{5k+4}\frac{1161}{5000}ds=0.2322<\frac{1}{e},$$

$$\begin{aligned}\liminf_{t\to\infty}\sum_{i=1}^2 q_i(t)\left(\sigma_i(t)-t\right)&=\liminf_{t\to\infty}\left[\frac{1101}{5000}\left(\sigma_1(t)-t\right)+\frac{60}{5000}\left(\sigma_1(t)+0.5-t\right)\right]\\&=\liminf_{t\to\infty}\left[\frac{1161}{5000}\left(\sigma_1(t)-t\right)+\frac{3}{500}\right]\\&=\liminf_{t\to\infty}\left[\frac{1161}{5000}\left(\sigma_1(t)-t\right)\right]+\frac{3}{500}\\&=\frac{1161}{5000}\cdot 1+\frac{3}{500}=0.2382<\frac{1}{e}.\end{aligned}$$

Therefore none of conditions (1.12), (1.13) and (1.14) is satisfied.

Notation. It is worth noting that the improvement of condition (2.48) to the corresponding condition (1.13) is significant, approximately 58.87%, if we compare the values on the left-side of these conditions.

REMARK 2. Similarly, one can construct examples to illustrate the other main results.

REFERENCES

[1] E. BRAVERMAN, G. E. CHATZARAKIS AND I. P. STAVROULAKIS, *Iterative oscillation tests for differential equations with several non-monotone arguments*, Adv. Difference Equ., 2016, DOI 10.1186/s13662-016-0817-3, pages 18.

[2] E. BRAVERMAN AND B. KARPUZ, *On oscillation of differential and difference equations with non-monotone delays*, Appl. Math. Comput., **218** (2011), 3880–3887.

[3] G. E. CHATZARAKIS, *Differential equations with non-monotone arguments: Iterative Oscillation results*, J. Math. Comput. Sci., **6**, 5 (2016), 953–964.

[4] G. E. CHATZARAKIS AND H. PÉICS, *Differential equations with several non-monotone arguments: An oscillation result*, Appl. Math. Lett., **68** (2017), 20–26.

[5] L. H. ERBE, QINGKAI KONG AND B. G. ZHANG, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York, 1995.

[6] L. H. ERBE AND B. G. ZHANG, *Oscillation of first order linear differential equations with deviating arguments*, Differential Integral Equations, **1** (1988), 305–314.

[7] N. FUKAGAI AND T. KUSANO, *Oscillation theory of first order functional-differential equations with deviating arguments*, Ann. Mat. Pura Appl. **136** (1984), 95–117.

[8] B. R. HUNT AND J. A. YORKE, *When all solutions of $x'(t)=-\sum q_i(t)x(t-T_i(t))$ oscillate*, J. Differential Equations **53** (1984), 139–145.

[9] J. JAROŠ AND I. P. STAVROULAKIS, *Oscillation tests for delay equations*, Rocky Mountain J. Math., **45** (2000), 2989–2997.

[10] M. KON, Y. G. SFICAS AND I. P. STAVROULAKIS, *Oscillation criteria for delay equations*, Proc. Amer. Math. Soc., **128** (1994), 675–685.

[11] R. G. KOPLATADZE AND T. A. CHANTURIJA, *Oscillating and monotone solutions of first-order differential equations with deviating argument*, (Russian), Differentsial'nye Uravneniya **18** (1982), 1463–1465, 1472.

[12] R.G. KOPLATADZE AND G. KVINIKADZE, *On the oscillation of solutions of first order delay differential inequalities and equations*, Georgian Math. J., **3** (1994), 675–685.

[13] T. KUSANO, *On even-order functional-differential equations with advanced and retarded arguments*, J. Differential Equations **45** (1982), 75–84.

[14] M. K. KWONG, *Oscillation of first-order delay equations*, *J. Math. Anal. Appl.* **156** (1991), 274–286.

[15] G. LADAS, V. LAKSHMIKANTHAM AND L. S. PAPADAKIS, *Oscillations of higher-order retarded differential equations generated by the retarded arguments*, Delay and Functional Differential Equations and their Applications, Academic Press, New York, 1972, 219–231.

[16] G. LADAS AND I. P. STAVROULAKIS, *Oscillations caused by several retarded and advanced arguments*, J. Differential Equations **44** (1982), 134–152.

[17] G. S. LADDE, *Oscillations caused by retarded perturbations of first order linear ordinary differential equations*, Atti Acad. Naz. Lincei Rendiconti **63** (1978), 351–359.

[18] G. S. LADDE, V. LAKSHMIKANTHAM, B. G. ZHANG, *Oscillation Theory of Differential Equations with Deviating Arguments*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 110, Marcel Dekker, Inc., New York, 1987.

[19] X. LI AND D. ZHU, *Oscillation and nonoscillation of advanced differential equations with variable coefficients*, J. Math. Anal. Appl. **269** (2002), 462–488.

[20] A. D. MYSHKIS, *Linear homogeneous differential equations of first order with deviating arguments*, Uspekhi Mat. Nauk **5** (1950), 160–162. (Russian)

[21] J. S. YU, Z. C. WANG, B. G. ZHANG AND X. Z. QIAN, *Oscillations of differential equations with deviating arguments*, *Panamer. Math. J.* **2**, 2 (1992), 59–78.

[22] B. G. ZHANG, *Oscillation of solutions of the first-order advanced type differential equations*, Science exploration **2** (1982), 79–82.

[23] D. ZHOU, *On some problems on oscillation of functional differential equations of first order*, J. Shandong University **25** (1990), 434–442.

(Received February 26, 2017)

George E. Chatzarakis
Department of Electrical and Electronic Engineering Educators
School of Pedagogical and Technological Education (ASPETE)
14121, N. Heraklio, Athens, Greece
e-mail: geaxatz@otenet.gr, gea.xatz@aspete.gr