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AREA INTEGRALS AND THE EXPONENTIAL SQUARE THEOREM FOR ELLIPTIC OPERATORS WITH COEFFICIENTS SUPPORTED IN WHITNEY TYPE CUBES

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Abstract. We provide a direct proof of a result comparing the area functions of solutions of two second order linear elliptic operators, when the discrepancy between their main coefficients is supported on Whitney type cubes of the unit ball of n dimensional Euclidean space. Our arguments are specialized to this type of operators, and the vanishing Carleson condition that we adopt is inspired by work of C. Sweezy. The comparison between area functions implies the preservation of the so called exponential square theorem assuming the aforementioned discrepancy of the coefficients.

1. Background definitions and results

Harmonic measure. Consider elliptic operators of the form

$$Lu = \operatorname{div} A\nabla u, \tag{1.1}$$

where $A(X) = (a_{ij}(X))$ is a symmetric $n\times n$ matrix of bounded measurable functions satisfying the *ellipticity condition*

$$\lambda_1 \sum_i \xi_i^2 < \sum_j \sum_i a_{ij}(X)\xi_i\xi_j < \lambda_2 \sum_i \xi_i^2, \quad \lambda_1 < \lambda_2, \tag{1.2}$$

for every $\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$ and every $X \in \mathbb{R}^n$. The constants λ_1 and λ_2 are called *ellipticity constants* of L. Solutions u of $Lu = 0$ are understood in the weak sense.

Let $D = \{X \in \mathbb{R}^n : |X| < 1\}$ be the unit ball in $\mathbb{R}^n$. It is well known that for each continuous function $f : \partial D \to \mathbb{R}$ there exists a unique function u_f defined on D, continuous in $\overline{D}$, and such that

$$Lu_f = 0 \text{ on } D, \quad u_f = f \text{ on } \partial D. \tag{1.3}$$

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This implies the existence of the *harmonic measure* $\omega^X(\cdot)$ associated to L, for $X \in D$. This is a Borel measure defined on ∂D that represents the solution u_f of (1.3) in the sense that

$$u_f(X) = \int_{\partial D} f(Q)\, d\omega^X(Q), \quad X \in D,$$

by means of Riesz representation theorem and the maximum principle.

We call $\omega \equiv \omega^{\vec{0}}$ *the* harmonic measure associated to L. The notation is justified, since by Harnack inequality ω^X is mutually absolutely continuous with respect to $\omega^{\vec{0}}$ (see e.g. [4, Lemma 1.2.7]).

Define the *surface ball centered at* $Q \in \partial D$ *and radius* $0 < r < \pi/4$ as $\Delta_r(Q) = \mathscr{B}_r(Q) \cap \partial D$ where the Euclidean ball is defined as $\mathscr{B}_r(Q) = \{X \in \mathbb{R}^n : |Q - X| < r\}$. Assuming $Q \in \partial D$, in spherical coordinates given by $Q = (1, \theta)$, and $0 < r < \pi/4$, $\theta \in S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, we associate with $\Delta_r(Q)$ the *corkscrew point* $A_r(Q) = (1 - (7/8)r, \theta)$. This way, for any surface ball Δ the notation $A(\Delta)$ makes sense without referring to neither its center nor its radius. The point $A_r(Q)$ is useful when considering local estimates of harmonic measure close to $\Delta_r(Q)$, where one in fact works with $\omega^{A_r(Q)}$. In this instance, sometimes is referred to as the *pole* for the harmonic measure.

Denoting by $\sigma(F)$ the surface measure of a Borel set $F \subset \partial D$, we say that $\omega \in A_\infty(d\sigma)$ if given $\varepsilon > 0$ there exists an η such that, for every surface ball $\Delta \subset \partial D$ and any subset $F \subset \Delta$, whenever $\sigma(F)/\sigma(\Delta) < \eta$ one has $\omega(F)/\omega(\Delta) < \varepsilon$. For shortness' sake we call this the A_∞ *property of harmonic measure*.

For more properties, estimates and results on harmonic measure, including comparison estimates with Green's functions, we refer the reader to the first few sections of [4]. As far as the A_∞ property concerns, the amount of references related to the A_∞ property of harmonic measure, and its connection with solvability of so called L^p, $1 < p < \infty$, and BMO Dirichlet problems is very large. Although the reference [4] provides a very complete scenario of the *state of art* around the early-to-mid 1990's, some more recent developments have provided a better understanding of the matter.

Indeed, more recently the work [6] pioneered in the realization that the comparability of L^2 norms of the area integral and non-tangential maximal function is actually equivalent with the A_∞ property of harmonic measure. See also the survey article [7]. It turned out that the A_∞ property of harmonic measure was actually proved to be equivalent to the solvability of the so called BMO Dirichlet problem [2]. This result has recently been improved with a very interesting tecnical achievement in [5], and there is even sharp results on the exponent $1 < p < \infty$ for the L^p solvability [10].

The exponential square class and exponential square theorem. In spherical coordinates one may view the unit ball in $\mathbb{R}^n$ as $D = \{(\theta, r) : \theta \in S^{n-1},\ 0 < r < 1\}$, and we can define the *non-tangential approach region* as the truncated cone

$$\Gamma_\alpha(\theta) = \left\{(\rho, \sigma) : 1/4 < \rho < 1,\ |\theta - \sigma| < \alpha(1-\rho)\},\ \theta \in S^{n-1}\right\},$$

for any choice of $\alpha > 0$ such that $\Gamma_\alpha(\theta) \subset D$. This way, the *area function* given by

$$S_\alpha u(\theta) = \left(\iint_{\Gamma_\alpha(\theta)} |\nabla u(r,\tau)|^2 (1-r)^{2-n} dr d\tau \right)^{1/2}$$

is a well defined object for solutions to $Lu=0$ on D for any L as defined above.

A function $f\in L^1(\partial D)$ is in the *exponential square class of* ∂D, and write $f\in \exp L^2(\partial D)$, if there exist two constants $c_1,c_2>0$ such that the following holds: If Δ denotes any surface cube in ∂D and

$$f_\Delta=\frac{1}{\sigma(\Delta)}\int_\Delta f(Q)\,d\sigma(Q)$$

then the estimate

$$\frac{1}{\sigma(\Delta)}\int_\Delta \exp\left(c_1|f(Q)-f_\Delta|^2\right)d\sigma(Q)<c_2$$

holds uniformly for Δ.

Following [11], given u the harmonic extension of a function $f\in L^1(\partial D)$ we say that the *exponential square theorem holds on D for u with aperture* $\alpha>0$ if $S_\alpha u\in L^\infty(\partial D)$ implies $f\in\exp L^2(\partial D)$.

Having introduced this terminology, we recall that this concept was introduced in [1], where it is proved that the exponential square theorem holds with certain aperture $\alpha>0$, on the semispace $\mathbb{R}^n_+$ for harmonic functions. More recently, in [11, Section I] it is proved that the exponential square theorem holds with certain aperture $\alpha>0$ on D for harmonic functions.

Preliminary description of the main theorem. Let L_0 and L_1 denote two elliptic operators as described through (1.1) and (1.2), and let ω_i denote the harmonic measure on D associated to L_i, $i=0,1$, resp. Define

$$\mathscr{E}(X)=A_1(X)-A_0(X)\quad\text{and}\quad a(X)=\sup\left\{|\mathscr{E}_{i,j}(Y)|:Y\in B_{\delta(X)/2}(X),\,1\leqslant i,j\leqslant n\right\}$$

where $\delta(Y)=\operatorname{dist}(Y,\partial D)$ for $Y\in D$. The *Carleson region* is defined as $T_r(Q)=\{(\theta,\rho):\theta\in\Delta_r(Q),\,1-r<\rho<1\}$, using spherical coordinates. Also, for $Y\in D$ denote by $G_0(Y)=G_0(0,Y)$ the Green's function (with pole at 0) for L_0 on D.

The main motivation of this work is [11], where it is considered the problem of determining sufficient conditions for the exponential square theorem to be preserved under suitable perturbation of the coefficients of two operators as (1.1). It was proved that it was enough to consider a *discrepancy* satisfying a very particular *vanishing Carleson measure*, similar to one contained in [3]. More precisely, the following result is proved.

THEOREM A. [11, Theorem 4] *Let L_0 and L_1 denote two elliptic operators as described through* (1.1) *and* (1.2)*, whose harmonic measures are denoted by ω_0 and ω_1 respectively. Suppose that for every $Q\in\partial D$, $0<r<r_0$ there exists a function $\varepsilon(r)$ satisfying*

$$\frac{1}{\omega_0(\Delta_r(Q))}\int_{T_r(Q)}\frac{a(Y)^2}{\delta(Y)^2}G_0(Y)\,dY\leqslant\varepsilon(r)\tag{1.4}$$

and $\varepsilon(2^jr)\leqslant 2^{\gamma j}r^\gamma$ where $0<\gamma<1$ is suitable small, depending only on ellipticity constants and dimension n (in particular $\varepsilon(r)\to 0$ as $r\to 0$). Assume that the exponential square theorem holds for solutions to $L_0u=0$ on D with aperture α. Then *there exist* $\beta > \alpha > 0$ *such that the exponential square theorem holds for solutions to* $L_1 u = 0$ *on* D *with aperture* β.

On the other hand, in a more recent work [9], it has been established that the membership of ω_0 to $A_\infty(d\sigma)$ is ineherited to ω_1, provided that the coefficients of A_0 and A_1 are supported on *contractions by a factor 1/2* of certain *Whitney-type cubes* of D, and the *discrepancy* between them is uniformly small.

In this note we explore a version of Theorem A in a situation similar to that in [9], and the main result of this paper is a direct proof of such a result, without invoking Theorem A in any form. In order to state precisely this result we must introduce more notation. This is done in the next section, where a precise statement of our main result is provided.

2. Statement of the main result

A Whitney type decomposition of the unit ball. Consider now $\mathscr{R} \subset \mathbb{R}^n$ the rectangle given by

$$\mathscr{R} = [0, \pi/2)^{n-1} \times (0, 1/2),$$

and define $\rho : \mathscr{R} \to D$ as $\rho(x,t) = (x, 1-t)$, where the n-tuple $(x, 1-t)$ must be *interpreted using spherical coordinates*. Namely, the $(n-1)$ coordinates of x indicate the direction angles of a point in ∂D still denoted by x, and $1/2 < (1-t) < 1$ indicates the radii covered through the mapping ρ.

Identifying points on a portion of the $(n-1)$-dimensional unit sphere S^{n-1} in $\mathbb{R}^n$ centered at the origin, with points in $[0, \pi/2)^{n-1}$ (*using spherical coordinates* as explained above), we may think of ρ as a mapping from $\mathscr{R}$ to a sector of D. In this way one may associate the union of the 2^n rectangles on $\mathbb{R}^{n-1}$ congruent to $\mathscr{R}$, to the annulus $D \setminus \mathscr{B}_{1/2}(\vec{0})$ through the mapping ρ.

Let Λ_k a family of cubes in $\mathbb{R}^{n-1}$, given by the product of intervals of the form $\left[\frac{j}{2^k}\frac{\pi}{2}, \frac{j+1}{2^k}\frac{\pi}{2}\right)$ for $j \in \{0, 1, \ldots, 2^k - 1\}$. The family of cubes

$$\Lambda \equiv \bigcup_{k \geqslant 0} \Lambda_k$$

will be referred to as the *dyadic cubes* of $\mathscr{R}_0 \equiv [0, \pi/2)^{n-1}$. If $I \in \Lambda$, there exists a minimal k_0 such that $I \in \Lambda_{k_0}$, and so we define $\ell(I) = \frac{1}{2^{k_0}}\frac{\pi}{2}$, the *length of the interval* I. The *rectangle associated to* I is defined by

$$R_I = I \times \left[\frac{1}{2^{k_0+2}}, \frac{1}{2^{k_0+1}}\right), \tag{2.1}$$

Notice that

$$\mathscr{R} = \bigcup_{I \subset \Lambda} R_I.$$

We call the family $\{R_I : I \in \Lambda\}$ a *decomposition of Whitney type for* $\mathscr{R}$.

Armed with the previous notions, we can define a decomposition of Whitney type for D based on the decomposition of Whitney-type for $\mathscr{R}$, through the mapping ρ. The *generation zero Whitney cube* is defined to be $R_0 = \mathscr{B}_{1/2}(\vec{0})$. Then $D \setminus \mathscr{B}_{1/2}(\vec{0})$ is separated as the union of 2^n copies of the Whitney type decompositions of the form $\{\rho(R_I) : I \in \Lambda\}$. Finally, $\mathscr{W}(D) = \rho(\mathscr{R}) \cup R_0$ is the decomposition of Whitney type for D.

From now on we use the same notations for objects either in $\mathscr{R}$ or in their images under ρ in B. Also, if R is any rectangle we can refer to the $I \in \Lambda$ such that $R = R_I$ as the *radial projection of* R *on* ∂D, and often write $I = \Pi(R)$. This projection also makes sense for any $A \subset D \setminus \mathscr{B}_{1/2}(\vec{0})$.

Description of the main theorem. Let L_0 and L_1 denote two elliptic operators as described through (1.1) and (1.2), and such that their corresponding matrices A_0 and A_1 coincide everywhere except in $(1/2)R_I$ for every $R_I : I \in \Lambda$, a decomposition of Whitney type, as described in the previous paragraph. Here $(1/2)R_I$ denotes the *concentric contraction of* R_I *by a factor* $1/2$. Let ω_i denote the harmonic measure on D associated to the operator L_i, $i = 0,1$. Define the *discrepancy between* A_0 *and* A_1 *within* R by

$$\mathscr{A}(R) = \|\mathscr{E}\|_{L^\infty(R)}, \quad R \in \mathscr{W}(D) \qquad \text{where as before} \quad \mathscr{E}(X) = A_1(X) - A_0(X).$$

In the following theorem we keep all of the notations and definitions introduced before.

THEOREM 1. *Suppose that the harmonic measure for* L_0 *is in* $A_\infty(d\sigma)$ *and that the exponential square theorem holds on* D *for solutions to* $L_0 u = 0$, *and for certain aperture* $\alpha > 0$. *Assume that* $\mathscr{A}(R) \leqslant \varepsilon_j$ *whenever* $R = R_I$ *with* $I \in \Lambda_j$, *where* $\varepsilon_j = 2^{-j\eta}$, $j = 0,1,\ldots,$ *for certain* $0 < \eta < 1/2$ *depending on n and the ellipticity constants in* (1.2). *Then there exist* $\beta > \alpha > 0$ *such that the exponential square theorem holds for solutions to* $L_1 u = 0$ *on* D *with aperture* β.

Note that this theorem is not vacuous. Taking L_0 as the Laplace operator, by the main result in [11, Section I] the theorem is applicable to small perturbations of it, where the discrepancy is supported on Whitney-type cubes.

3. Arguments to prove Theorem 1

From now on, we adopt the notation $C_1 \lesssim C_2$ whenever $C_1 \leqslant kC_2$ for certain constant $k > 0$ that may depend at most on n or λ, the domain D, or a parameter that does not interfere in the essence of the argumentation. Similarly, $C_1 \approx C_2$ means that $C_1 \lesssim C_2$ and $C_2 \lesssim C_1$ hold simultaneously.

Let u_i be the solution to $L_i u_i = 0$, $i = 0,1$, both satisfying $u_0 = u_1 = f$ almost everywhere on ∂D, where $f \in L^1(\partial D)$. The existence u_0 is secured by the assumption that $\omega_0 \in A_\infty(d\sigma)$. As far as the existence of u_1, we can use the main result in [9], which says that elliptic-harmonic measure associated to L_1 is also in the A_∞ class.

For $X \in D$ define $F(X) = u_0(X) - u_1(X)$. By a standard argument (see e.g. [3, p.77]) we actually have

$$F(X) = \int_D \nabla G_0(X,Y)\mathscr{E}(Y)\nabla u_1(Y)dY, \qquad X \in D,$$

where $G_0(X,Y)$ denotes the Green's function for L_0 on D.

Throughout the proof we will not use spherical coordinates, and so we set $\delta(X) = \operatorname{dist}(X, \partial D)$. Accordingly, we adjust the definition of the area function as

$$S_\alpha u(Q) = \left(\int_{\Gamma_\alpha(Q)} |\nabla u(X)|^2 \delta^{2-n}(X)\, dX \right)^{1/2} \quad Q \in \partial D.$$

In order to prove the theorem, we establish that there exists $0 < \alpha < \beta$ such that $\|S_\beta u_1\|_\infty < \infty$ implies $f \in \exp L^2(D)$. For this, we will choose $\beta > 0$ in such a way that $\|S_\alpha u_0\|_\infty < \infty$, which by assumption yields the desired result. In other words, we can focus on proving the comparison between the area functions of u_0 and u_1.

Fix $Q_0 \in \partial D$ and assume that $Q_0 = \rho((\pi/4, \pi/4, \ldots, \pi/4, 0))$ with no loss of generality (this essentially means that $Q_0 \in \partial D$ is *located in the center* of one of the fundamental spherical caps). Call $\mathscr{R}$ the sector on D which is image of $[0, \pi/2)^{n-1} \times (0, 1/2)$ through ρ. Let $\mathfrak{R} = \{R_m\}$ denote the sequence of all the rectangles from the decomposition of Whitney type of D, contained in $\mathscr{R}$ which intersect $\Gamma_\alpha(Q)$.

Now, since

$$\int_{R_j} |\nabla u_0(X)|^2 \delta(X)^{2-n} dX \lesssim \int_{R_j} |\nabla F(X)|^2 \delta(X)^{2-n} dX + \int_{R_j} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX, \tag{3.1}$$

the idea in order to obtain the comparison between area function of u_0 and u_1 is to add over $\{R_j\}$ in (3.1). This way, we can focus on the estimate of $\int_{R_j} |\nabla F(X)|^2 \delta(X)^{2-n} dX$.

Now we proceed with the statement of two lemmata which provide the main building blocks of the proof of Theorem 1. The proof of Lemma 1 below proceeds exactly as in [3, p. 87-88].

LEMMA 1. *With the notations introduced above*

$$\begin{aligned}
&\int_{R_j} |\nabla F(X)|^2 \delta(X)^{2-n} dX \\
&\quad \lesssim \Bigg\{ \left[\int_{R_j^*} |F(X)|^2 \delta(X)^{-n} dX \right]^{1/2} \left[\int_{R_j^*} |\nabla F(X)|^2 \delta(X)^{2-n} dX \right]^{1/2} \qquad (3.2) \\
&\qquad + \varepsilon_j \left[\int_{R_j^*} |F(X)|^2 \delta(X)^{-n} dX \right]^{1/2} \left[\int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX \right]^{1/2} \\
&\qquad + \varepsilon_j \left[\int_{R_j^*} |\nabla F(X)|^2 \delta(X)^{2-n} dX \right]^{1/2} \left[\int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX \right]^{1/2} \Bigg\}.
\end{aligned}$$

Here R_j^* denotes a dilation of R_j by a small factor slightly greater than 1.

Notice that the R_j^* have bounded overlap. At this point we pick the value of the aperture β, satisfying the following properties:

(i) $R_j^* \subset \Gamma_\beta(Q_0)$ for every j.

(ii) $R_I \subset \Gamma_\beta(P)$ for every $P \in I$ and every $I \in \Lambda$.

The property (i) will allow us to add over j and obtain an estimate comparing integrals over cones with aperture α and β. And property (ii) will become useful in a construction later on (see right after (4.10) below).

For the proof of the following result we adapt some ideas from [9].

LEMMA 2. *With all of the previous notations and definitions, the following estimate holds:*

$$\left[\int_{R_j^*} |F(X)|^2 \delta(X)^{-n} dX\right]^{1/2} \lesssim \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty. \tag{3.3}$$

Assuming these lemmata, along with the choice of $\beta > 0$ and ε_j, we may conclude from (3.2) that

$$\begin{aligned}
&\int_{R_j} |\nabla F(X)|^2 \delta(X)^{2-n} dX \\
&\lesssim \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty \left[\int_{R_j^*} |\nabla F(X)|^2 \delta(X)^{2-n} dX\right]^{1/2} \\
&\quad + \frac{j^2}{2^{2j\eta}} \|S_\beta u_1\|_\infty \left[\int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX\right]^{1/2} \qquad (3.4)\\
&\quad + \frac{j}{2^{j\eta}} \left[\int_{R_j^*} |\nabla F(X)|^2 \delta(X)^{2-n} dX\right]^{1/2} \left[\int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX\right]^{1/2}.
\end{aligned}$$

Plugging the estimate

$$\int_{R_j^*} |\nabla F(X)|^2 \delta(X)^{2-n} dX \lesssim \int_{R_j^*} |\nabla u_0(X)|^2 \delta(X)^{2-n} dX + \int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX$$

in the right hand side of (3.4) we obtain

$$\begin{aligned}
&\int_{R_j} |\nabla F(X)|^2 \delta(X)^{2-n} dX \\
&\lesssim \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty \left[\left(\int_{R_j^*} |\nabla u_0(X)|^2 \delta(X)^{2-n} dX\right)^{1/2} + \left(\int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX\right)^{1/2}\right] \\
&\quad + \frac{j}{2^{j\eta}} \left(\int_{R_j^*} |\nabla u_0(X)|^2 \delta(X)^{2-n} dX\right)^{1/2} \left(\int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX\right)^{1/2} \\
&\quad + \frac{j}{2^{j\eta}} \left(\int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX\right) \equiv I + II + III \qquad (3.5)
\end{aligned}$$

where we have used the elementary inequality $(a+b)^{1/2} \leqslant 2^{1/2}(a^{1/2}+b^{1/2})$, valid for $a,b>0$.

To handle the terms I and II we use the fact that the assumption $\|S_\beta u_1\|_{L^\infty}<\infty$, along with the A_∞ property of the elliptic-harmonic measure of L_0 (thus of L_1), imply that $f\in BMO(\partial D)$ (see [8]). By the main theorem of [2], for $i=0,1$

$$\int_{R_j^*} |\nabla u_i(X)|^2 \delta(X)^{2-n} dX \lesssim \frac{1}{(\operatorname{diam} R_j)^{n-1}} \int_{R_j^*} |\nabla u_i(X)|^2 \delta(X) dX \lesssim \|f\|_*^2, \tag{3.6}$$

where $\|f\|_*$ denotes the BMO norm of f, and where $C>0$ is a constant not depending on R_j or f. Then

$$I, II \lesssim \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty \|f\|_*$$

because $\left(\displaystyle\int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX\right)^{1/2} \leqslant S_\beta u_1(Q_0)$.

By the same token, $III \lesssim j2^{-j\eta}\|S_\beta u_1\|_\infty^2$, and we can plug all of these estimates back in (3.1) to obtain

$$\int_{R_j} |\nabla u_0(X)|^2 \delta(X)^{2-n} dX \lesssim \frac{j}{2^{j\eta}} \left(\|S_\beta u_1\|_\infty + \|S_\beta u_1\|_\infty^2\right) + \int_{R_j^*} |\nabla u_1(X)|^2 \delta(X)^{2-n} dX.$$

When summing over every j that intersect $\Gamma_\alpha(Q)$, we pick up the estimate

$$[S_\alpha u_0(Q)]^2 \lesssim \|S_\beta u_1\|_{L^\infty} + \|S_\beta u_1\|_{L^\infty}^2.$$

This way, under the assumption that $\|S_\beta u_1\|_{L^\infty}<\infty$ we conclude that $\|S_\alpha u_0\|_{L^\infty}<\infty$. Since the exponential square theorem holds on D for soloutions to $L_0 u=0$ for certain aperture $\alpha>0$ we conclude that $f\in \exp L^2(D)$. This means that the exponential square theorem holds for u_1 with the aperture $\beta>0$ chosen above. The proof of Theorem 1 is finished except for the proof of the Lemma 2.

4. Proof of Lemma 2

Recall that

$$\begin{aligned} F(X) &= \int_D \nabla_Y G_0(X,Y)\mathscr{E}(Y)\nabla u(Y)\,dY \\ &= \left(\int_R + \int_{D\setminus R}\right) \nabla_Y G_0(X,Y)\mathscr{E}(Y)\nabla u(Y)\,dY \equiv F_1(X)+F_2(X), \end{aligned}$$

where $R=R_j$ as in the statement of the lemma. Denote simply by $\varepsilon=\varepsilon_j$ the discrepancy of A_0 and A_1 within R, i.e. $\varepsilon = 2^{-j\eta}$.

Recall also that $\Lambda=\bigcup \Lambda_k$ denotes a *dyadic decomposition* of the cap of S^{n-1} which is the part of the closure of the basic *annular sector* $\mathscr{R}$, that lies in ∂D. Actually the Λ_k, $k=1,2,\dots$ denote the *generations* of this dyadic decomposition. Associated to $I \in \Lambda$ we have R_I the *rectangle* within D associated with I; for $Y \in D$ we denote by $\delta(Y)$ the distance from Y to ∂D. Finally recall that $Q_0 \in \partial B$ has been fixed, and that $\mathfrak{R} = \{R_j\}$ denotes the sequence of rectangles that intersect $\Gamma_\alpha(Q_0)$. With these notations, set $\delta(A) = \inf\{\delta(Y) : Y \in A\}$ for $A \subset D$.

For the analysis of F_2 we define auxiliar regions $\mathscr{B}_R$, $\mathscr{G}$ and $\mathscr{H}$ as follows. The *inner band* around ∂D is

$$\mathscr{B}_R \equiv \bigcup\{R_I : I \in \Lambda_k, k > j-1\} = \{X \in D : 0 < \delta(X) \leqslant 4\delta(R)\}.$$

Define also the truncated *conical region* $\mathscr{G} \equiv \bigcup\left\{R_\ell \in \mathfrak{R} : R_\ell^\circ \cap \mathscr{B}_R^\circ = \emptyset\right\}$. Finally set $\mathscr{H} = \mathscr{R} \setminus (\mathscr{B}_R \cup \mathscr{G})$. We now write

$$\begin{aligned} F_2(X) &= \left(\int_{\mathscr{B}_R} + \int_{\mathscr{G}} + \int_{\mathscr{H}} + \int_{D\setminus\mathscr{R}}\right) \nabla G_0(X,Y)\mathscr{E}(Y)\nabla u_1(Y)\,dY \\ &\equiv b(X) + g(X) + h(X) + \psi(X). \end{aligned} \tag{4.1}$$

We will handle separately the term F_1 and each of the terms in (4.1) arising from F_2.

4.1. Estimates for F_1

Note that if $Z \in R_j^*$ by Cauchy's inequality

$$\begin{aligned} |F_1(Z)| &\leqslant \int_R |\nabla_Y G_0(Z,Y)|\,|\mathscr{E}(Y)|\,|\nabla u_1(Y)|\,dY \\ &\leqslant \left(\int_R |\nabla_Y G_0(Z,Y)|^2 |\mathscr{E}(Y)|^2\,dY\right)^{1/2} \left(\int_R |\nabla u_1(Y)|^2\,dY\right)^{1/2} \end{aligned} \tag{4.2}$$

and by the definition of $\mathscr{E}$ and Caccioppoli's inequality

$$\begin{aligned} \int_R |\nabla_Y G_0(Z,Y)|^2 |\mathscr{E}(Y)|^2\,dY &\leqslant \varepsilon^2 \iint_{(1/2)R} |\nabla_Y G_0(Z,Y))|^2\,dY \\ &\lesssim \frac{\varepsilon^2}{\operatorname{diam}(R)^2} \int_{(9/16)R} |G_0(Z,Y)|^2\,dY, \end{aligned} \tag{4.3}$$

where $(9/16)R$ is the concentric dilatation of R by a factor $9/16$. By boundary Harnack and estimates for the Green's function (see [4, Lemmata 1.3.4 and 1.3.3 resp.]) we have

$$G_0(Z,Y) \lesssim G_0(A_R,Y) \lesssim \frac{\omega_0^{A_R}((\Delta_R)}{\operatorname{diam}(R)^{n-2}} \tag{4.4}$$

where $A_R = A(\Delta_R)$ is the pole associated with Δ_R, which in turn denotes the radial projection of R onto S^{n-1} (see pages 312 and 315 for this terminology). This and (4.3) imply that

$$\int_R |\nabla_Y G_0(Z,Y)|^2 |\mathscr{E}(Y)|^2\,dY \lesssim \frac{\varepsilon^2}{\operatorname{diam}(R)^2} \int_{(9/16)R} |G_0(Z,Y)|^2\,dY \lesssim \frac{\varepsilon^2}{\operatorname{diam}(R)^{n-2}}$$

because $\omega_0^{A_R}((\Delta_R) \leqslant 1$. All in all for $Z \in R_j^*$

$$|F_1(Z)| \lesssim \varepsilon \left(\int_R |\nabla u_1(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2}$$

which after integrating and back in (4.2) yields

$$\left(\int_{R_j^*} |F_1(X)|^2 \delta(X)^{-n} dX \right)^{1/2} \lesssim \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty. \tag{4.5}$$

4.2. Estimates for b

For $Z \in R_j^*$

$$\begin{aligned} |b(Z)| &\leqslant \int_{\mathscr{B}_R} |\nabla_Y G_0(Z,Y)| |\mathscr{E}(Y)| |\nabla u_1(Y)| dY \\ &\leqslant \sum_k \int_{Q_k} |\nabla_Y G_0(Z,Y)| |\mathscr{E}(Y)| |\nabla u_1(Y)| dY \\ &\leqslant \sum_k \left(\int_{Q_k} |\nabla_Y G_0(Z,Y)|^2 |\mathscr{E}(Y)|^2 dY \right)^{1/2} \left(\int_{Q_k} |\nabla u_1(Y)|^2 dY, \right)^{1/2} \end{aligned} \tag{4.6}$$

where Q_k are the rectangles contained in $\mathscr{B}_R$. Now we can apply the same idea in (4.3) and (4.4) to get

$$|b(Z)| \lesssim \sum_k \varepsilon \left(\int_{Q_k} |\nabla u_1(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2} \omega_0^{A_R}(\Delta(Q_k)) \tag{4.7}$$

because the discrepancy within each and every Q_k is controlled by ε.

Now, to estimate (4.7) we use a device from [3, p. 84-85]. Define

$$\begin{aligned} O_k &= \left\{ P \in \partial D : S_\beta(u_1)(P) > 2^k \right\}, \\ \tilde{O}_k &= \left\{ P \in \partial D : M_{\omega_0^{A_R}}(\chi_{O_k})(P) > 1/2 \right\} \\ J_k &= \left\{ J \in \Lambda : \omega_0^{A_R}(J \cap O_k) > 1/2\ \omega_0^{A_R}(J) \text{ and } \omega_0^{A_R}(J \cap O_{k+1}) \leqslant 1/2\ \omega_0^{A_R}(J)) \right\}, \end{aligned} \tag{4.8}$$

where $M_{\omega_0^{A_R}}$ denote the Hardy-Littlewood maximal function, with respect to $\omega_0^{A_R}$. By (4.7) we obtain

$$|b(Z)| \lesssim \varepsilon \sum_k \sum_{J \in J_k} \left(\int_{Q_J} |\nabla u_1(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2} \omega_0^{A_R}(J). \tag{4.9}$$

Using the weak (1,1) inequality for $M_{\omega_0^{A_R}}$ we obtain

$$\omega_0^{A_R}\left(\tilde{O}_k \setminus O_{k+1}\right) \leqslant C \omega_0^{A_R}(O_k). \tag{4.10}$$

Also, if $J \in J_k$ then $J \subset \tilde{O}_k \setminus O_{k+1}$ (see details in [3, p. 84-85]).

Now recall the property (ii) of our choice of β (page 317), which in this case means that for every $Q \in J$ we have $Q_J \subset \Gamma_\beta(Q)$. Hence, from (4.7) we get

$$\begin{aligned}
|b(Z)| &\lesssim \varepsilon \sum_k \sum_{J\in J_k} \left(\int_{\Gamma_\beta(Q)} |\nabla u_1(Y)|^2 \delta(Y)^{2-n} dY \right)^{1/2} \omega_0^{A_R}(J) \\
&\lesssim \varepsilon \sum_k \sum_{J\in J_k} S_\beta(u_1)(Q)\omega_0^{A_R}(J) \qquad \text{(for any } Q \in J) \\
&\lesssim \varepsilon \sum_k \sum_{J\in J_k} \int_J 2^{k+1} d\omega_0^{A_R}(J) \\
&\lesssim \varepsilon \sum_k \int_{\tilde{O}_k \setminus O_{k+1}} 2^k d\omega_0^{A_R}(Q) \lesssim \varepsilon \sum_k 2^k \omega_0^{A_R}(\tilde{O}_k \setminus O_{k+1}) \\
&\lesssim \varepsilon \sum_k 2^k \omega_0^{A_R}(O_k) \lesssim \varepsilon \int_{\partial B} S_\beta(u_1)(P)\omega^{A_R}(P) \leqslant \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty. \qquad (4.11)
\end{aligned}$$

Thus we obtain

$$\left(\int_{R_j^*} |b(X)|^2 \delta(X)^{-n} dX \right)^{1/2} \lesssim \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty. \qquad (4.12)$$

4.3. Estimates for g

We start separating $\mathscr{G}$ according to the different $j-1$ generations to which each of its rectangles belong. That is, since $R_\ell \in \mathscr{G}$ whenever $R^\circ \cap \mathscr{B}_R^\circ = \emptyset$, and it is assumed that $R = R_j$ is in the jth generation, then

$$\mathscr{G} = \bigcup_{k=1}^{j-2} \mathscr{G}_k, \quad \mathscr{G}_k = \left\{ X \in \mathscr{G} : 2^k \frac{1}{2^j} < \delta(X) \leqslant 2^{k+1} \frac{1}{2^j} \right\}.$$

In this way

$$\begin{aligned}
g(X) &= \int_{\mathscr{G}} \nabla G_0(X,Y)\mathscr{E}(Y)\nabla u_1(Y)dY \\
&= \sum_{k=1}^{j-2} \int_{\mathscr{G}_k} \nabla G_0(X,Y)\mathscr{E}(Y)\nabla u_1(Y)dY \equiv \sum_{k=1}^{j-2} g_k(X).
\end{aligned}$$

We will prove first that

$$\left[\int_{R^*} |g_k(X)|^2 \delta(X)^{-n} dX \right]^{1/2} \lesssim \frac{1}{2^{j\eta}} \|S_\beta u_1\|_\infty, \qquad (4.13)$$

for $k = 1, \ldots, j-2$. For this purpose observe that if $Z \in R^*$

$$g_k(Z) = \int_{\mathscr{G}_k} \nabla G_0(Z,Y)\mathscr{E}(Y)\nabla u_1(Y)dY = \int_{\widetilde{\mathscr{G}_k}} \nabla G_0(Z,Y)\mathscr{E}(Y)\nabla u_1(Y)dY,$$

because $\mathscr{E}$ is supported on the $(1/2)R_\ell$, and where $\widetilde{\mathscr{G}_k}$ denotes a small concentric contraction of $\mathscr{G}_k$. Therefore

$$|g_k(Z)| \leqslant \int_{\widetilde{\mathscr{G}_k}} |\nabla_Y G_0(Z,Y)|\,|\mathscr{E}(Y)|\,|\nabla u_1(Y)|\,dY$$
$$\leqslant \left(\int_{\widetilde{\mathscr{G}_k}} |\nabla_Y G_0(Z,Y)|^2 |\mathscr{E}(Y)|^2 dY\right)^{1/2} \left(\int_{\widetilde{\mathscr{G}_k}} |\nabla u_1(Y)|^2 dY\right)^{1/2} \tag{4.14}$$

Now notice that for $Y \in \widetilde{\mathscr{G}_k}$ one has $|\mathscr{E}(Y)| \approx \varepsilon_{j-k-1}$ for $1 \leqslant k \leqslant j-2$, by the assumption on the size of the discrepancy. By Caccioppoli's inequality

$$\int_{\widetilde{\mathscr{G}_k}} |\nabla_Y G_0(Z,Y)|^2 |\mathscr{E}(Y)|^2 dY \lesssim \frac{(\varepsilon_{j-k-1})^2}{(\operatorname{diam}(\mathscr{G}_k))^2} \int_{\mathscr{G}_k} |G_0(Z,Y)|^2 dY. \tag{4.15}$$

Applying Hölder continuity on the boundary to $G_0(\cdot,Y)$ (see [4, Corollary 1.1.24]) we obtain

$$G_0(Z,Y) \lesssim \left(\frac{|Z-Q_0|}{2^{-j+k+1}}\right)^{\gamma} G_0(A_k,Y) \lesssim \frac{1}{2^{(k+1)\gamma}} G_0(A_k,Y) \tag{4.16}$$

for certain $0<\gamma<1$ depending only on n and the ellipticity constants in (1.2). Here A_k is the pole associated to the projection of $\mathscr{G}_k$ over S^{n-1}. Then from estimates for the Green's function, as in (4.4), we get

$$G_0(A_k,Y) \lesssim \frac{\omega_0^{A_k}((\Pi(\mathscr{G}_k))}{(\operatorname{diam}(\mathscr{G}_k))^{n-2}} \lesssim \frac{1}{(\operatorname{diam}(\mathscr{G}_k))^{n-2}}.$$

This along with (4.14) and (4.15) imply that for $Z \in R_j^*$

$$|g_k(Z)| \lesssim \frac{\varepsilon_{j-k-1}}{2^{(k+1)\gamma}} \left(\int_{\widetilde{\mathscr{G}_k}} |\nabla u_1(Y)|^2 \delta(Y)^{2-n} dY\right)^{1/2}$$

By the choice on the decay of ε_j we have for $k=1,\dots,j-2$

$$\frac{\varepsilon_{j-k-1}}{2^{(k+1)\gamma}} = \left(\frac{2^{k+1}}{2^j}\right)^{\eta} \frac{1}{2^{(k+1)\gamma}} = \frac{1}{2^{j\eta}} 2^{(k+1)(\eta-\gamma)} < \frac{1}{2^{j\eta}}, \tag{4.17}$$

when choosing $0<\eta<\gamma<1$. This already implies (4.13).

Finally, after summing all of the $j-2$ terms and applying triangular inequality

$$\left[\int_{R^*} |g(X)|^2 \delta(X)^{-n} dX\right]^{1/2} \lesssim \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty. \tag{4.18}$$

4.4. Estimates for h

Start by defining the *kth slice of* $\mathscr{H}$ as $\widetilde{\mathscr{H}^k} = \left\{Y \in \mathscr{H} : 2^k \frac{1}{2^j} < \delta(Y) \leqslant 2^{k+1}\frac{1}{2^j}\right\}$. Observe that $\widetilde{\mathscr{H}^k}$ is a union of rectangles whose distance is roughly 2^{k-j}. We introduce two more notions regarding cubes in a Whitney type decomposition. Two rectangles R_{I_1} and R_{I_2}, with $I_1, I_2 \in \Lambda_k$ are called *siblings* if there exists $J \in \Lambda_{k-1}$ such that $I_1, I_2 \subset J$. Also define the *descendants of a rectangle* R_I as the family of rectangles R_J such that $J \subset I$.

Given a rectangle R_I, consider the property (*) $\overline{R_I} \cap \overline{\mathscr{G}_k} \neq \emptyset$. Define $\widehat{\mathscr{H}^k}$ as the collection of rectangles $R_I \in \widetilde{\mathscr{H}^k}$ such that either R_I satisfies property (*), or it is a sibling of a rectangle satisfying (*). Notice that

$$\mathscr{H} = \bigcup_{k=1}^{j-2} \mathscr{H}^k \quad \text{where} \quad \mathscr{H}^k = \left\{ R_J \subset \mathscr{H} : R_J \text{ is a descendant of an element in } \widehat{\mathscr{H}^k} \right\}.$$

The point of these constructions is that $\mathscr{H}^k$ is a sort of *Carleson box* where we can apply estimates for Green's functions, boundary Harnack inequality or boundary Hölder continuity of positive solutions to $L_i u = 0$, $i = 0, 1$.

Our task is now to estimate each summand in the following expression:

$$\begin{aligned} h(X) &= \int_{\mathscr{H}} \nabla G_0(X,Y) \mathscr{E}(Y) \nabla u_1(Y)\, dY \\ &= \sum_{k=1}^{j-2} \int_{\mathscr{H}^k} \nabla G_0(X,Y) \mathscr{E}(Y) \nabla u_1(Y)\, dY \equiv \sum_{k=1}^{j-2} h_k(X). \end{aligned}$$

For $k = 1, \ldots, j-2$ and $Z \in R^*$

$$\begin{aligned} |h_k(Z)| &\leqslant \int_{\mathscr{H}^k} |\nabla_Y G_0(Z,Y)|\, |\mathscr{E}(Y)|\, |\nabla u_1(Y)|\, dY \\ &\leqslant \sum_{\ell} \int_{Q_\ell} |\nabla_Y G_0(Z,Y)|\, |\mathscr{E}(Y)|\, |\nabla u_1(Y)|\, dY \\ &\leqslant \sum_{\ell} \left(\int_{Q_\ell} |\nabla_Y G_0(Z,Y)|^2 |\mathscr{E}(Y)|^2 dY \right)^{1/2} \left(\int_{Q_\ell} |\nabla u_1(Y)|^2 dY, \right)^{1/2} \end{aligned} \tag{4.19}$$

where Q_ℓ are the rectangles in $\mathscr{H}^k$. We can proceed now as in (4.14)–(4.16). And again from estimates for the Green's function we obtain

$$G_0(A_\ell, Y) \lesssim \frac{\omega_0^{A_k}(\Pi(Q_\ell))}{(\operatorname{diam}(Q_\ell))^{n-2}}$$

With the same method we used in §4.2, only this time with

$$O_m = \left\{ P \in \Pi(\mathscr{H}^k) : S(u_1)(P) > 2^m \right\},$$

we can obtain

$$|h_k(X)| \lesssim \frac{\varepsilon_{j-k-1}}{2^{(k+1)\gamma}} \|S_\beta u_1\|_\infty \leqslant \frac{1}{2^{j\eta}} \|S_\beta u_1\|_\infty$$

(see (4.11) and (4.17)). In this way, for $k = 1, \dots, j-2$

$$\int_{R^*} |h_k(X)|^2 \delta(X)^{-n} dX \lesssim \left(\frac{1}{2^{j\eta}}\right)^2 \|S_\beta u_1\|_\infty^2.$$

Adding over $k = 1, \dots, j-2$

$$\begin{aligned}\left[\int_{R_j^*} |h(X)|^2 \delta^{-n}(X) dX\right]^{1/2} &\leqslant \sum_{k=1}^{j-2} \left[\int_{R_j^*} |h_k(X)|^2 \delta^{-n}(X) dX\right]^{1/2} \\ &\leqslant \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty. \end{aligned} \tag{4.20}$$

4.5. Estimates for ψ

In this case we follow the same ideas in (4.8)–(4.11), and for $X \in R_j^*$ we obtain first

$$|\psi(X)| \lesssim \varepsilon_0 \int_{\partial D \setminus \mathscr{R}_0} S u_1(Q)\, d\omega^X(Q),$$

where $\varepsilon_0 \approx 1$ is the discrepancy in the *generation zero* Whitney rectangle of D. Let $\Delta_k = \Delta_{2^k 2^{-j}}(Q_0)$, with $\Delta_0 \equiv \mathscr{R}_0$, and with $k \geqslant N+1$, where N is chosen minimal with the property $1/4 < 2^N 2^{-j}$. It is known (see e.g. [4, Lemma 1.3.12]) that

$$\sup_{Q \in \Delta_k \setminus \Delta_{k-1}} K(X,Q) \lesssim \frac{2^{-\gamma k}}{\omega(\Delta_k)},$$

where $K(X,Q) = d\omega^X / d\omega(Q)$ is the Radon-Nikodym derivative, also called *kernel function*. Using this we get

$$\begin{aligned} |\psi(X)| &\lesssim \varepsilon_0 \int_{\partial D \setminus \mathscr{R}_0} S_\beta u_1(Q) d\omega^X(Q) \lesssim \varepsilon_0 \sum_{k=N+1}^{\infty} \int_{\Delta_k \setminus \Delta_{k-1}} S_\beta u_1(Q) d\omega^X(Q) \\ &\lesssim \varepsilon_0 \sum_{k=N+1}^{\infty} \int_{\Delta_k \setminus \Delta_{k-1}} S_\beta u_1(Q) K(X,Q) d\omega(Q) \\ &\leqslant \varepsilon_0 \sum_{k=N+1}^{\infty} \int_{\Delta_k \setminus \Delta_{k-1}} S_\beta u_1(Q) \frac{2^{-\gamma k}}{\omega(\Delta_k)} d\omega(Q) \lesssim \varepsilon_0 \sum_{k=N+1}^{\infty} 2^{-\gamma k} M_\omega(S_\beta u_1)(Q_0) \\ &\lesssim \varepsilon_0 \|S_\beta u_1\|_\infty \sum_{k=N+1}^{\infty} 2^{-\gamma k} \leqslant \|S_\beta u_1\|_\infty \frac{1}{2^{(N+1)\gamma}} \\ &\lesssim \frac{j}{2^{j\gamma}} \|S_\beta u_1\|_\infty \lesssim \frac{j}{2^{j\eta}} \|S_\beta u_1\|_\infty \end{aligned}$$

since $0 < \eta < \gamma < 1$ and by the choice of N. Hence

$$\left[\int_{R^*} |\psi(X)|^2 \delta(X)^{-n} dX\right]^{1/2} \lesssim \frac{j}{2^j} \|S_\beta u_1\|_\infty. \tag{4.21}$$

From (4.12), (4.18), (4.20) and (4.21) we obtain the desired inequality.

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