

VIRIAL IDENTITIES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH A CRITICAL COEFFICIENT INVERSE-SQUARE POTENTIAL

TOSHIYUKI SUZUKI

(Communicated by Pavel I. Naumkin)

Abstract. Virial identities for nonlinear Schrödinger equations with some strongly singular potential $(a|x|^{-2})$ are established. Here if $a = a(N) := -(N-2)^2/4$, then $P_{a(N)} := -\Delta + a(N)|x|^{-2}$ is nonnegative selfadjoint in the sense of Friedrichs extension. But the energy class $D((1 + P_{a(N)})^{1/2})$ does not coincide with $H^1(\mathbb{R}^N)$. Thus justification of the virial identities has a lot of difficulties. The identities can be applicable for showing blow-up in finite time and for proving the existence of scattering states.

1. Introduction and main results

In this article we consider the following Cauchy problems for nonlinear Schrödinger equations with inverse-square potentials

$$\begin{cases} i \frac{\partial u}{\partial t} = \left(-\Delta + \frac{a}{|x|^2}\right)u + g(u) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u(0, x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \quad (\mathbf{CP})_a$$

where $i = \sqrt{-1}$, $N \geq 3$ and

$$a \geq a(N) := -\frac{(N-2)^2}{4}. \quad (1.1)$$

Here (1.1) is based on the nonnegative selfadjointness of $P_a := -\Delta + a|x|^{-2}$ in $L^2(\mathbb{R}^N)$ in the sense of Friedrichs extension, which is followed by the usual *Hardy inequality*

$$\| |x|^{-1}u \|_{L^2} \leq \frac{2}{N-2} \| \nabla u \|_{L^2} \quad \forall f \in H^1(\mathbb{R}^N), N \geq 3. \quad (1.2)$$

By virtue of $a > a(N)$ we see that $D((1 + P_a)^{1/2})$ coincides with $H^1(\mathbb{R}^N)$. Thus a lot of studies of $(\mathbf{CP})_a$ are available. Specifically for nonlinear problems, Okazawa–Suzuki–Yokota [9] showed the global unique existence of $(\mathbf{CP})_a$ with power-type nonlinearities under unsatisfactory conditions of a via the contraction methods. The worse

Mathematics subject classification (2010): 35Q55, 35Q40, 81Q15.

Keywords and phrases: virial identities, nonlinear Schrödinger equations, Hartree equations, inverse-square potentials, blow-up in finite time, scattering states, Mellin transform.

assumption of a is removed in Okazawa–Suzuki–Yokota [10] by applying the abstract energy methods. For Hartree type nonlinearities (in general non-local nonlinearities with non-convolution) see Suzuki [13]. Moreover, a finite time blow-up for $(\mathbf{CP})_a$ is shown in Suzuki [14]. On the other hand, the scattering problems of Hartree equations ($g(u) := u(|x|^{-\gamma} * |u|^2)$, $1 < \gamma < \max\{N, 4\}$) are considered in Suzuki [16] in the weighted energy space $\Sigma^1(\mathbb{R}^N) := H^1(\mathbb{R}^N) \cap D(|x|)$. Whereas Zhang–Zheng [17] studied the scattering problems (especially construction of the wave operators in $H^1(\mathbb{R}^N)$) of power type ($g(u) := |u|^{p-1}u$, $1 + 4/N < p < 1 + 4/(N - 2)$) under the unsatisfactory condition of a . Thus we cannot apply their methods to scattering problems for the critical case $a = a(N)$.

On the other hand, we remark if $a = a(N)$, then we see that the energy class $D((1 + P_{a(N)})^{1/2})$ does not coincide with $H^1(\mathbb{R}^N)$. Now we write down the energy space $D((1 + P_{a(N)})^{1/2})$ as $X^1(\mathbb{R}^N)$. The well-posedness for $(\mathbf{CP})_a$ in $X^1(\mathbb{R}^N)$ is shown in Suzuki [15] and he analyzed $X^1(\mathbb{R}^N)$ spaces.

To observe the blow-up in finite time and the scattering problem we usually need the *virial identity*. For example, if $g(u) = \lambda|u|^{p-1}u$ ($\lambda \in \mathbb{R}$), then we can calculate formally

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8\|P_a^{1/2}u(t)\|_{L^2}^2 + \frac{4N\lambda(p-1)}{p+1} \|u(t)\|_{L^{p+1}}^{p+1}.$$

If $a > a(N)$, then Suzuki [14] justified the virial identity for $(\mathbf{CP})_a$. But if $a = a(N)$, then we have never justified owing to the solvability that has not shown in $H^1(\mathbb{R}^N)$. Thus we need to try the case $a = a(N)$. Here we know $H^1(\mathbb{R}^N) \subset X^1(\mathbb{R}^N) \subset H^s(\mathbb{R}^N)$ ($0 < s < 1$). Thus the approximated argument as in Suzuki [14] can be applicable even in $a = a(N)$. Here when we prove the convergence $a \rightarrow a(N) + 0$, we need to prepare for the Mellin transform argument as in Suzuki [15] (see Lemma 3.1 and (3.7) for useful results in this article).

This paper is divided into four sections. In Section 2 we give some preliminary results. Notations are prepared in Section 2.1. Sections 2.2 is devoted to the linear operator $-\Delta + a|x|^{-2}$. The virial identities for $(\mathbf{CP})_a$ with $a = a(N)$ is justified in Section 3. We give some typical example for the virial identities in Section 3.1. In Section 4 we apply the virial identities to two problems: blow-up in finite time (Section 4.1) and existence of scattering states (Section 4.2).

2. Notations and preliminaries

2.1. Notations

To simplify the notation, we write

$$P_a := -\Delta + a|x|^{-2}, \quad a \geq a(N) = -(N - 2)^2/4. \tag{2.1}$$

Also we use the notation $A(u) \lesssim B(u)$; an abbreviation of $A(u) \leq CB(u)$, where C is independent of u .

$L^p(\mathbb{R}^N)$ is the usual Lebesgue space with norm

$$\|u\|_{L^p} := \left(\int_{\mathbb{R}^N} |u(x)|^p dx \right)^{1/p}, \quad u \in L^p(\mathbb{R}^N) \quad (1 \leq p < \infty),$$

$$\|u\|_{L^\infty} := \text{esssup} |u(x)|, \quad u \in L^\infty(\mathbb{R}^N).$$

Let $p \in [1, \infty]$. Then $p' \in [1, \infty]$ denotes the Hölder conjugate $p' := p/(p-1)$. $H^1(\mathbb{R}^N)$ is the usual L^2 -type Sobolev space with the norm

$$\|u\|_{H^1} := (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{1/2}, \quad u \in H^1(\mathbb{R}^N).$$

On the other hand, $H^{-1}(\mathbb{R}^N)$ is the dual of $H^1(\mathbb{R}^N)$. Note that we have a usual triplet

$$H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N),$$

where the inclusions are continuous and dense. In particular, we have the Sobolev embeddings

$$H^1(\mathbb{R}^N) \subset L^q(\mathbb{R}^N), \quad L^{q'}(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N), \quad 2 \leq q \leq \frac{2N}{N-2}, \quad N \geq 3.$$

$H^1(\mathbb{R}^N)$ coincides with the energy space $D((1+P_a)^{1/2})$ ($a > a(N)$). Here we denote the energy space $D((1+P_{a(N)})^{1/2})$ as $X^1(\mathbb{R}^N)$. $X^{-1}(\mathbb{R}^N)$ is the dual of $X^1(\mathbb{R}^N)$. As we see

$$H^1(\mathbb{R}^N) \subset X^1(\mathbb{R}^N) \subset H^s(\mathbb{R}^N) \quad (s < 1).$$

In particular, applying fractional Sobolev inequality and [15, Theorem 3.2] we see that for all $u \in X^1(\mathbb{R}^N)$ and $0 \leq s < 1$

$$\|u\|_{L^{2N/(N-2s)}} \leq C_{N,s} \|(-\Delta)^{s/2} u\|_{L^2} \leq C_{N,s} C_s \|P_{a(N)}^{s/2} u\|_{L^2}.$$

Note that we also obtain for all $u \in X^1(\mathbb{R}^N)$ and $0 \leq s < 1$

$$\|u\|_{L^{2N/(N-2s)}} \leq C \|(1+P_{a(N)})^{1/2} u\|_{L^2}^s \|u\|_{L^2}^{1-s}.$$

Define $D(|x|) := \{u \in L^2(\mathbb{R}^N); |x|u \in L^2(\mathbb{R}^N)\}$. Then we denote the weighted energy spaces as

$$\Sigma^1(\mathbb{R}^N) := H^1(\mathbb{R}^N) \cap D(|x|), \quad \Sigma_*^1(\mathbb{R}^N) := X^1(\mathbb{R}^N) \cap D(|x|).$$

Let $I \subset \mathbb{R}$ be an open interval and Y be a Banach space. Then $C(\bar{I}; Y)$ is a family of the continuous Y -valued function on \bar{I} . On the other hand, the vector-valued Lebesgue space $L^p(I; Y)$ is equipped with norm

$$\|u\|_{L^p(I; Y)} := \|\|u(\cdot)\|\|_{L^p(I)} < \infty.$$

Moreover the vector-valued Sobolev space $W^{1,p}(I;Y)$ is equipped with norm

$$\|u\|_{W^{1,p}(I;Y)} := \|u\|_{L^p(I;Y)} + \|u'\|_{L^p(I;Y)} < \infty.$$

Here u' denotes the weak derivative of u respect to time variable $t \in I$. Then it is well-known that $W^{1,p}(I;Y) \subset C(\bar{I};Y)$ for $p > 1$.

Strichartz estimates for $\exp(-itP_{a(N)})$ are proved in Suzuki [15, Proposition 4.8].

LEMMA 2.1. *Let $N \geq 3$ and (p_j, q_j) be Schrödinger admissible pairs ($j = 0, 1, 2$), i.e.,*

$$\frac{2}{p_j} + \frac{N}{q_j} = \frac{N}{2}, \quad p_j > 2, q_j \geq 2.$$

Then the following inequalities hold for $\varphi \in L^2(\mathbb{R}^N)$ and $\Phi \in L^{p'_1}(\mathbb{R};L^{q'_1}(\mathbb{R}^N))$:

$$\|\exp(-itP_{a(N)})\varphi\|_{L^{p_0}(\mathbb{R};L^{q_0})} \leq C \|\varphi\|_{L^2}, \tag{2.2}$$

$$\left\| \int_0^t \exp(-i(t-s)P_{a(N)})\Phi(s,x) ds \right\|_{L^{p_2}(\mathbb{R};L^{q_2})} \leq C' \|\Phi\|_{L^{p'_1}(\mathbb{R};L^{q'_1})}. \tag{2.3}$$

Here the end point $(\tau, \rho) = (2, 2N/(N-2))$ is open.

2.2. Spherical harmonics decomposition

Next we consider the spherical harmonics decomposition; see [12, Chapter IV] for details. A function $Q : \mathbb{R}^N \rightarrow \mathbb{C}$ is said to be ℓ -th solid harmonic if Q is harmonic (i.e., $\Delta Q = 0$) and a homogeneous polynomial of degree ℓ (i.e., $Q(x) = |x|^\ell Q(x/|x|)$). Let \mathcal{Q}_ℓ be a family of ℓ -th solid harmonic functions. Then \mathcal{Q}_ℓ is a finite dimensional vector space. Moreover, \mathcal{Q}_ℓ has an orthogonal normal system $\{Y_{\ell,k}\}_k$:

$$\mathcal{Q}_\ell = \text{Span}\{Y_{\ell,k}\}_k, \quad \int_{S^{N-1}} Y_{\ell,k_1}(x') \overline{Y_{\ell,k_2}(x')} dS(x') = \begin{cases} 1 & k_1 = k_2, \\ 0 & k_1 \neq k_2. \end{cases}$$

Let $Q \in \mathcal{Q}_\ell$ and $\tilde{f} \in L^2(0, \infty)$. Then

$$f(x) := |x|^{-\ell-(N-1)/2} \tilde{f}(|x|) Q(x) \in L^2(\mathbb{R}^N).$$

In fact, we can calculate as follows:

$$\int_{\mathbb{R}^N} |f(x)|^2 dx = \left(\int_{|x'|=1} |Q(x')|^2 dS \right) \left(\int_0^\infty |\tilde{f}(r)|^2 dr \right) < \infty. \tag{2.4}$$

Thus we define the subspace of $L^2(\mathbb{R}^N)$ as

$$L^2_{=\ell}(\mathbb{R}^N) := \left\{ \sum_k |x|^{-\ell-(N-1)/2} \tilde{f}_k(|x|) Y_{\ell,k}(x); \tilde{f}_k \in L^2(0, \infty), Y_{\ell,k} \in \mathcal{Q}_\ell \right\}.$$

In particular, $L^2_{=0}(\mathbb{R}^N) = L^2_{\text{rad}}(\mathbb{R}^N)$, the family of radially symmetric functions. Also we define

$$L^2_{\geq d}(\mathbb{R}^N) := \bigoplus_{\ell \geq d} L^2_{=\ell}(\mathbb{R}^N).$$

Now we have the following (see e.g. [12, Lemma IV.2.18]).

PROPOSITION 2.2. *Let ℓ, ℓ_1, ℓ_2 be nonnegative integers. Then one has*

(i) $L^2_{=\ell}(\mathbb{R}^N)$ is a closed subspace of $L^2(\mathbb{R}^N)$;

(ii) $L^2_{=\ell_1}(\mathbb{R}^N) \perp L^2_{=\ell_2}(\mathbb{R}^N)$ if $\ell_1 \neq \ell_2$, i.e.,

$$\int_{\mathbb{R}^N} f_1(x) \overline{f_2(x)} dx = 0 \quad \forall f_1 \in L^2_{=\ell_1}(\mathbb{R}^N), \forall f_2 \in L^2_{=\ell_2}(\mathbb{R}^N);$$

(iii) $L^2(\mathbb{R}^N) = \bigoplus_{\ell=0}^{\infty} L^2_{=\ell}(\mathbb{R}^N)$, i.e., for every $f \in L^2(\mathbb{R}^N)$ there uniquely exists $\{f_\ell\}_\ell \subset L^2(\mathbb{R}^N)$ such that $f_\ell \in L^2_{=\ell}(\mathbb{R}^N)$ for $\ell \in \mathbb{N} \cup \{0\}$ and

$$f = \sum_{\ell=0}^{\infty} f_\ell \quad (\text{spherical harmonics decomposition}).$$

As seen in Suzuki [15], we have

$$-\Delta f = A_{\mu(\ell)} f \quad \forall f \in L^2_{=\ell}(\mathbb{R}^N), \tag{2.5}$$

$$P_a f = A_{\nu(\ell)} f \quad \forall f \in L^2_{=\ell}(\mathbb{R}^N), \tag{2.6}$$

where $\lambda = (N - 2)/2$ and

$$A_\nu \tilde{f} := -\partial_r^2 \tilde{f} - \frac{N-1}{r} \partial_r \tilde{f} + \frac{\nu^2 - \lambda^2}{r^2} \tilde{f}, \tag{2.7}$$

$$\mu(\ell) := \mu_\ell = \lambda + \ell. \tag{2.8}$$

$$\nu(\ell) := \nu_\ell = [(\lambda + \ell)^2 + a]^{\frac{1}{2}}. \tag{2.9}$$

Next we introduce the Mellin transform.

DEFINITION 2.1. Let f be a complex-valued measurable function such that $x^{\gamma-1} f(x) \in L^1(0, \infty; \mathbb{C})$ for some $\gamma \in \mathbb{R}$. Then the Mellin transform of f is defined as

$$\mathcal{M}[f(r)](z) := \int_0^\infty r^{z-1} f(r) dr.$$

In general, let $f \in L^2(\mathbb{R}^N)$ ($N \geq 2$). Then the Mellin transform of f is defined as

$$\mathcal{M}[f(x)](z) := \int_0^\infty r^{z-1} f\left(\frac{rx}{|x|}\right) dr.$$

Note that $\mathcal{M}[r^\alpha f(r)](z) = \mathcal{M}[f(r)](z + \alpha)$ for $\alpha \in \mathbb{R}$. Next we see that

$$\mathcal{M}[\partial_r f](z) = (1 - z)\mathcal{M}[f](z - 1). \tag{2.10}$$

As in Suzuki [15, (2.20)] (see also [11]) we have for $\nu \geq 0$ and $\sigma \geq 0$

$$\mathcal{M}[A_\nu^{\sigma/2} f](z) = 2^\sigma \frac{\Gamma((z - \lambda + \nu)/2)\Gamma(1 - (z - \sigma - \lambda - \nu)/2)}{\Gamma((z - \sigma - \lambda + \nu)/2)\Gamma(1 - (z - \lambda - \nu)/2)} \mathcal{M}[f](z - \sigma), \tag{2.11}$$

where $\lambda = (N - 2)/2$.

Here we have the Plancherel type equality

$$\int_0^\infty f(s)\overline{g(s)}s^{N-1}ds = \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{M}[f]\left(\frac{N}{2} + iy\right) \overline{\mathcal{M}[g]\left(\frac{N}{2} + iy\right)} dy \tag{2.12}$$

(see Suzuki [15, Lemma 2.5]).

3. Proof of the virial identities

First we consider the key inequalities.

LEMMA 3.1. *Let $a \geq a(N)$. Assume that φ is real-valued and radially symmetric. Then*

$$\left| \text{Im} \int_{\mathbb{R}^N} \overline{x\varphi u} \cdot \nabla u dx \right| \leq \|(1 + P_a)^{1/2} u\|_{L^2} \|x\varphi u\|_{L^2} \tag{3.1}$$

Proof. Let $u = \sum_{\ell,k} u_{\ell,k}(r)Y_{\ell,k}$ be a spherical harmonics decomposition. Then we see that

$$\int_{\mathbb{R}^N} \overline{\varphi u x} \cdot \nabla u dx = \sum_{\ell,k} \int_0^\infty \overline{\varphi u_{\ell,k}(r)} r (\partial_r u_{\ell,k})(r) r^{N-1} dr. \tag{3.2}$$

By using (2.12) and (2.10) we obtain that

$$\begin{aligned} & \int_0^\infty \overline{r\varphi u_{\ell,k}(r)} (\partial_r u_{\ell,k})(r) r^{N-1} dr \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{M}[\partial_r u_{\ell,k}]\left(\frac{N}{2} + iy\right) \overline{\mathcal{M}[r\varphi u_{\ell,k}]\left(\frac{N}{2} + iy\right)} dy \\ &= \frac{N}{2} \frac{1}{2\pi} \int_{-\infty}^\infty \mathcal{M}[u_{\ell,k}]\left(\frac{N}{2} - 1 + iy\right) \overline{\mathcal{M}[\varphi u_{\ell,k}]\left(\frac{N}{2} + 1 + iy\right)} dy \\ &+ \frac{i}{2\pi} \int_{-\infty}^\infty y \mathcal{M}[u_{\ell,k}]\left(\frac{N}{2} - 1 + iy\right) \overline{\mathcal{M}[\varphi u_{\ell,k}]\left(\frac{N}{2} + 1 + iy\right)} dy \\ &=: \frac{N}{2} I_1 + I_2. \end{aligned} \tag{3.3}$$

Here I_1 is calculated

$$I_1 = \int_0^\infty r^{-1} u_{\ell,k} \overline{r\varphi u_{\ell,k}} r^{N-1} dr = \int_0^\infty \varphi(r) |u_{\ell,k}(r)|^2 r^{N-1} dr \in \mathbb{R}.$$

Thus we have

$$\begin{aligned} & \operatorname{Im} \int_0^\infty \overline{r\varphi u_{\ell,k}(r)} (\partial_r u_{\ell,k})(r) r^{N-1} dr = \operatorname{Im} I_2 \\ & = \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^\infty y \mathcal{M}[u_{\ell,k}] \left(\frac{N}{2} - 1 + iy \right) \overline{\mathcal{M}[\varphi u_{\ell,k}] \left(\frac{N}{2} + 1 + iy \right)} dy. \end{aligned} \tag{3.4}$$

On the other hand, we see from (2.11) that

$$\begin{aligned} & \mathcal{M}[A_v^{1/2} f] \left(\frac{N}{2} + iy \right) \\ & = (v - iy) \frac{\Gamma((v - iy)/2) \Gamma((v + 1 + iy)/2)}{\Gamma((v + iy)/2) \Gamma((v + 1 - iy)/2)} \mathcal{M}[f] \left(\frac{N}{2} - 1 + iy \right). \end{aligned}$$

Using $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ we obtain

$$\int_0^\infty |A_v^{1/2} f|^2 r^{N-1} dr = \frac{1}{2\pi} \int_{-\infty}^\infty (v^2 + y^2) \left| \mathcal{M}[f] \left(\frac{N}{2} - 1 + iy \right) \right|^2 dy. \tag{3.5}$$

Applying (3.5) and (2.12) we calculate

$$\begin{aligned} & \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^\infty y \mathcal{M}[u_{\ell,k}] \left(\frac{N}{2} - 1 + iy \right) \overline{\mathcal{M}[\varphi u_{\ell,k}] \left(\frac{N}{2} + 1 + iy \right)} dy \\ & \leq \frac{1}{2\pi} \left[\int_{-\infty}^\infty \left| y \mathcal{M}[u_{\ell,k}] \left(\frac{N}{2} - 1 + iy \right) \right|^2 dy \right]^{1/2} \left[\int_{-\infty}^\infty \left| \mathcal{M}[\varphi u_{\ell,k}] \left(\frac{N}{2} + 1 + iy \right) \right|^2 dy \right]^{1/2} \\ & = \left[\int_0^\infty |A_{v(\ell)}^{1/2} f|^2 r^{N-1} dr \right]^{1/2} \left[\int_0^\infty |r\varphi u_{\ell,k}|^2 r^{N-1} dr \right]^{1/2}. \end{aligned}$$

Therefore we have

$$\begin{aligned} & \left| \operatorname{Im} \int_0^\infty \overline{r\varphi u_{\ell,k}(r)} (\partial_r u_{\ell,k})(r) r^{N-1} dr \right| \\ & \leq \left[\int_0^\infty |A_{v(\ell)}^{1/2} f|^2 r^{N-1} dr \right]^{1/2} \left[\int_0^\infty |r\varphi u_{\ell,k}|^2 r^{N-1} dr \right]^{1/2} \\ & = \|P_a^{1/2}(u_{\ell,k} Y_{\ell,k})\|_{L^2} \|x\varphi u_{\ell,k} Y_{\ell,k}\|_{L^2}. \end{aligned} \tag{3.6}$$

Summing (3.6) over k and ℓ we conclude (3.1) from (3.2). \square

REMARK 3.1. Let $\varphi(x) \equiv 1$. In a way similar to Lemma 3.1 we can conclude that for all $u, v \in \Sigma_*^1(\mathbb{R}^N)$

$$\left| \int_{\mathbb{R}^N} \overline{xu} \cdot \nabla v dx \right| \leq \|xu\|_{L^2} \|P_{a(N)}^{1/2} v\|_{L^2} + \frac{N}{2} \|u\|_{L^2} \|v\|_{L^2}. \tag{3.7}$$

Now we show the virial identities for $(\mathbf{CP})_a$ with $a = a(N)$. To end this we consider approximated problems for $(\mathbf{CP})_a$:

$$\begin{cases} i \frac{\partial u_{\varepsilon,a}}{\partial t} = \left(-\Delta + \frac{a}{|x|^2} \right) u_{\varepsilon,a} + g_\varepsilon(u_{\varepsilon,a}) & \text{in } \mathbb{R} \times \mathbb{R}^N, \\ u_{\varepsilon,a}(0, x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{CP}_{\varepsilon,a}$$

Here g_ε is approximation of g . g and g_ε satisfy weak closedness (see [10, (G5)]):

$$\begin{aligned} v_m(t) &\rightarrow v(t) \quad (m \rightarrow \infty) \text{ weakly in } L^\infty(-T, T; X_S), \\ g(v_m(t)) &\rightarrow f(t) \quad (m \rightarrow \infty) \text{ weakly}^* \text{ in } L^\infty(-T, T; X_S^*) \\ \Rightarrow 0 &= \lim_{m \rightarrow \infty} \operatorname{Im} \int_{\mathbb{R}^N} g(v_m(t)) \overline{v_m(t)} dx = \operatorname{Im} \int_{\mathbb{R}^N} f(t) \overline{v(t)} dx, \end{aligned}$$

where $X_S := H^1(\mathbb{R}^N)$ ($a > a(N)$) or $X^1(\mathbb{R}^N)$ ($a = a(N)$). Assume further that $v_m(t) \rightarrow v(t)$ in $C([-T, T]; L^2(\mathbb{R}^N))$. Then $f(t) = g(v(t))$.

We give three types of nonlinearities g and their approximations as we can consider in this article.

EXAMPLE 3.1. Let $g : \mathbb{C} \rightarrow \mathbb{C}$ be power type nonlinearities so that

(N1) $g(0) = 0$ and there exist $p \in [1, (N + 2)/(N - 2))$ and $K \geq 0$ such that

$$|g(z_1) - g(z_2)| \leq K(1 + |z_1|^{p-1} + |z_2|^{p-1})|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{C};$$

(N2) $g(x) \in \mathbb{R}$ ($x > 0$) and $g(e^{i\theta}z) = e^{i\theta}g(z)$ ($z \in \mathbb{C}$, $\theta \in \mathbb{R}$).

In such a case we define $g_\varepsilon(u) := \rho_\varepsilon * [g(\rho_\varepsilon * u)]$, where ρ_ε is the Friedrichs mollifier. Moreover, the energy functionals of g and g_ε are

$$G(u) := \int_{\mathbb{R}^N} F(|u(x)|) dx, \quad G_\varepsilon(u) := \int_{\mathbb{R}^N} F(|\rho_\varepsilon * u(x)|) dx,$$

where F is the primitive integral of g :

$$F(x) := \int_0^x g(s) ds \quad \forall x > 0.$$

EXAMPLE 3.2. Let

$$g(u) := \lambda |x|^{-r} |u|^{p-1} u \quad (\lambda \in \mathbb{R}, 0 < r < 2, 1 \leq p < (N + 2 - 2r)/(N - 2)).$$

In such a case we define

$$g_\varepsilon(u) := \lambda \rho_\varepsilon * [(|x|^2 + \varepsilon)^{-r/2} |\rho_\varepsilon * u|^{p-1} (\rho_\varepsilon * u)].$$

Moreover, the energy functionals of g and g_ε are

$$G(u) := \lambda \int_{\mathbb{R}^N} \frac{|u(x)|^{p+1}}{|x|^r} dx, \quad G_\varepsilon(u) := \lambda \int_{\mathbb{R}^N} \frac{|\rho_\varepsilon * u(x)|^{p+1}}{(|x|^2 + \varepsilon)^{r/2}} dx.$$

EXAMPLE 3.3. Let $g(u) := uK[k](|u|^2)$, where

$$K[k](f) := \int_{\mathbb{R}^N} k(x, y) f(y) dy.$$

Here k satisfies three conditions:

(K1) k is a symmetric real-valued function, that is, $k(x, y) = k(y, x) \in \mathbb{R}$ a.a. $x, y \in \mathbb{R}^N$;

(K2) $k \in L_x^\infty(L_y^\alpha) + L_x^\beta(L_y^\alpha)$ for some $\alpha, \beta \in [1, \infty]$ such that $\alpha \leq \beta$ and $\alpha^{-1} + \beta^{-1} < 4/N$;

(K3) $\tilde{k}(x, y) := x \cdot \nabla_x k(x, y) + y \cdot \nabla_y k(x, y)$ belongs to $L_x^\infty(L_y^{\tilde{\alpha}}) + L_x^{\tilde{\beta}}(L_y^{\tilde{\alpha}})$ for some $\tilde{\alpha}, \tilde{\beta} \in [1, \infty]$ such that $\tilde{\alpha} \leq \tilde{\beta}$ and $\tilde{\alpha}^{-1} + \tilde{\beta}^{-1} < 4/N$.

Note that $L_x^\beta(L_y^\alpha)$ is the family of $k(x, y)$ such that $k(x, \cdot) \in L^\alpha(\mathbb{R}^N)$ a.a. $x \in \mathbb{R}^N$ with $\| \|k(x, \cdot)\|_{L^\alpha} \|_{L^\beta} < \infty$. In such a case we define $g_\varepsilon(u) := uK[k_\varepsilon](|u|^2)$, where

$$k_\varepsilon(x, y) := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho_\varepsilon(x - \xi) \rho_\varepsilon(y - \eta) k(\xi, \eta) d\xi d\eta.$$

Moreover, the energy functionals of g and g_ε are

$$G(u) := \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} k(x, y) |u(x)|^2 |u(y)|^2 dx dy,$$

$$G_\varepsilon(u) := \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} k_\varepsilon(x, y) |u(x)|^2 |u(y)|^2 dx dy.$$

REMARK 3.2. Condition **(K2)** implies that we can divide k into $k_R + (k - k_R)$, where

$$k_R(x, y) := \begin{cases} k(x, y) & |k(x, y)| \leq R, \\ R & k(x, y) > R, \\ -R & k(x, y) < -R. \end{cases}$$

Here $k_R \in L_x^\infty(L_y^\infty)$ with $\|k_R\|_{L_x^\infty(L_y^\infty)} \leq R$ and $\|k - k_R\|_{L_x^\beta(L_y^\alpha)} \rightarrow 0$ ($R \rightarrow \infty$). Moreover, define

$$\gamma := \left[1 - \frac{1}{2} \left(\frac{1}{\alpha} + \frac{1}{\beta} \right) \right]^{-1} \in \left[1, \frac{N}{N-2} \right). \tag{3.8}$$

Then we have (see [13, Lemma 2.4]) if $k \in L_x^\beta(L_y^\alpha)$,

$$\|K[k]f\|_{L^\gamma} \leq \|k\|_{L_x^\beta(L_y^\alpha)} \|f\|_{L^\gamma} \quad \forall f \in L^\gamma(\mathbb{R}^N). \tag{3.9}$$

By virtue of (3.9), **(K1)** and **(K2)** imply

$$|G(u) - G(v)|, |G_\varepsilon(u) - G_\varepsilon(v)| \leq CM^4 \|k - k_R\|_{L_x^\beta(L_y^\alpha)} + RM^3 \|u - v\|_{L^2} \tag{3.10}$$

for every $u, v \in X^1(\mathbb{R}^N)$ with $\|u\|_{X^1} \leq M, \|v\|_{X^1} \leq M$ (see [14, (2.13)]) and

$$G_\varepsilon(\varphi) \rightarrow G(\varphi) \quad (\varepsilon \rightarrow 0), \quad \varphi \in X^1(\mathbb{R}^N)$$

(see [14, Remark 2.2]).

Here we give a plan to prove the virial identities for $(\mathbf{CP})_a$ with $a = a(N)$.

Stage 1. We construct the virial identities for $(\mathbf{CP})_{\varepsilon,a}$ with $a > a(N)$. This step has already finished: see Suzuki [14].

Stage 2. To show $u_{\varepsilon,a} \rightarrow u_\varepsilon$ ($a \rightarrow a(N) + 0$) we need uniform boundedness in $X^1(\mathbb{R}^N)$. Here u_ε is a solution to $(\mathbf{CP})_{\varepsilon,a}$ with $a = a(N)$. u_ε is also satisfies a certain virial identity.

Stage 3. Next we let $\varepsilon \rightarrow +0$. Applying the Strichartz estimates (2.2) and (2.3) we can show $u_\varepsilon \rightarrow u$. Since u satisfies $(\mathbf{CP})_a$ with $a = a(N)$, we can prove the desired virial identity.

Before deriving the virial identity for $(\mathbf{CP})_a$ with $a = a(N)$ we calculate the first derivative of $\|xu(t)\|_{L^2}^2$.

LEMMA 3.2. *Let $u \in C([-T_1, T_2]; X^1(\mathbb{R}^N))$ be a solutions to $(\mathbf{CP})_a$ with $a = a(N)$ and $u_0 \in \Sigma_*^1(\mathbb{R}^N)$. Then u belongs to $C([-T_1, T_2]; \Sigma_*^1(\mathbb{R}^N))$ and satisfies*

$$\frac{d}{dt} \|xu(t)\|_{L^2}^2 = 4 \operatorname{Im} \int_{\mathbb{R}^N} \nabla u(t) \cdot \overline{xu(t)} dx. \tag{3.11}$$

Proof. Using assumption of g we can calculate

$$\frac{d}{dt} \left\| \frac{xu(t)}{\sqrt{1 + \delta|x|^2}} \right\|_{L^2}^2 = 4 \operatorname{Im} \int_{\mathbb{R}^N} \nabla u(t) \cdot \overline{\left[\frac{xu(t)}{(1 + \delta|x|^2)^2} \right]} dx.$$

Lemma 3.1 with $\varphi(x) = (1 + \delta|x|^2)^{-2}$ implies that

$$\left| \frac{d}{dt} \left\| \frac{xu(t)}{\sqrt{1 + \delta|x|^2}} \right\|_{L^2}^2 \right| \leq 4 \|(1 + P_{a(N)})^{1/2} u(t)\|_{L^2} \left\| \frac{xu(t)}{(1 + \delta|x|^2)^2} \right\|_{L^2}.$$

Thus we see that

$$\left\| \frac{xu(t)}{\sqrt{1 + \delta|x|^2}} \right\|_{L^2} \leq \left\| \frac{xu_0}{\sqrt{1 + \delta|x|^2}} \right\|_{L^2} + 2 \left| \int_0^t \|u(s)\|_{X^1} ds \right|.$$

Letting $\delta \rightarrow 0$ we conclude that

$$\|xu(t)\|_{L^2} \leq \|xu_0\|_{L^2} + 2 \left| \int_0^t \|u(s)\|_{X^1} ds \right|.$$

In a way similar to the above we also obtain

$$\left| \|xu(T_2)\|_{L^2} - \|xu(T_1)\|_{L^2} \right| \leq 2 \left| \int_{T_2}^{T_1} \|u(s)\|_{X^1} ds \right|;$$

hence u is continuous in $\Sigma_*^1(\mathbb{R}^N)$ and satisfies (3.11). \square

Henceforth we prove the virial identities for $(\mathbf{CP})_a$ with $a = a(N)$ only the case of nonlinearities as in Example 3.3. Other cases are similar ways.

Summary of Stage 1. Let $u_0 \in H^1(\mathbb{R}^N) \subset X^1(\mathbb{R}^N)$ and $a > a(N)$. Then global weak solution $u_{\varepsilon,a} \in C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ to $(\mathbf{CP})_{\varepsilon,a}$ with $u_{\varepsilon,a}(0) = u_0$ exists uniquely. Also $u_{\varepsilon,a}$ satisfies the conservation laws

$$\|u_{\varepsilon,a}(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E_{\varepsilon,a}(u_{\varepsilon}(t)) = E_{\varepsilon,a}(u_0) \quad \forall t \in \mathbb{R},$$

where

$$E_{\varepsilon,a}(\varphi) := \frac{1}{2} \|P_a^{1/2} \varphi\|_{L^2}^2 + G_{\varepsilon}(\varphi), \quad \varphi \in H^1(\mathbb{R}^N).$$

Here the virial identities for $(\mathbf{CP})_{\varepsilon,a}$ ($a > a(N)$) is verified in Suzuki [14, Section 3]

$$\begin{aligned} \frac{d^2}{dt^2} \|xu_{\varepsilon,a}(t)\|_{L^2}^2 &= 8 \|P_a^{1/2} u_{\varepsilon,a}(t)\|_{L^2}^2 \\ &\quad - 2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \tilde{k}_{\varepsilon}(x,y) |u_{\varepsilon,a}(t,x)|^2 |u_{\varepsilon,a}(t,y)|^2 dx dy \quad \forall t \in \mathbb{R}, \end{aligned}$$

where $\tilde{k}_{\varepsilon} \in L_x^{\infty}(L_y^{\infty})$ and

$$\begin{aligned} \tilde{k}_{\varepsilon}(x,y) &:= \iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho_{\varepsilon}(x-\xi) \rho_{\varepsilon}(y-\eta) \tilde{k}(\xi,\eta) d\xi d\eta \\ &\quad + \iint_{\mathbb{R}^N \times \mathbb{R}^N} [\tilde{\rho}_{\varepsilon}(x-\xi) \rho_{\varepsilon}(y-\eta) + \rho_{\varepsilon}(x-\xi) \tilde{\rho}_{\varepsilon}(y-\eta)] k(\xi,\eta) d\xi d\eta, \\ \tilde{\rho}_{\varepsilon}(x) &:= N \rho_{\varepsilon}(x) + x \cdot \nabla \rho_{\varepsilon}(x). \end{aligned}$$

Proof of Stage 2. First we show there exists u_{ε} such that $u_{\varepsilon,a} \rightarrow u_{\varepsilon}$ ($a \rightarrow a(N) + 0$) weakly in some sense. Applying the conservation laws we see that

$$\|P_a^{1/2} u_{\varepsilon,a}(t)\|_{L^2}^2 = \|P_a^{1/2} u_0\|_{L^2}^2 + 2G_{\varepsilon}(u_0) - 2G_{\varepsilon}(u_{\varepsilon,a}(t))$$

Since G_{ε} is continuous in $L^2(\mathbb{R}^N)$, there exists a non-decreasing function $d_{\varepsilon} : (0, \infty) \rightarrow (0, \infty)$ such that $|G_{\varepsilon}(\varphi)| \leq d_{\varepsilon}(\|\varphi\|_{L^2})$; in this case (Example 3.3) we have $d_{\varepsilon}(\|\varphi\|_{L^2}) := (1/4) \|k_{\varepsilon}\|_{L_x^{\infty}(L_y^{\infty})} \|\varphi\|_{L^2}^4$. Thus we conclude that

$$\|P_a^{1/2} u_{\varepsilon,a}(t)\|_{L^2}^2 \leq \|P_a^{1/2} u_0\|_{L^2}^2 + 4d_{\varepsilon}(\|u_0\|_{L^2}) \quad \forall t \in \mathbb{R}.$$

On the other hand,

$$\|P_{a(N)}^{1/2} \varphi\|_{L^2} \leq \|P_a^{1/2} \varphi\|_{L^2} \quad \forall \varphi \in H^1(\mathbb{R}^N).$$

Combining these we obtain

$$\|P_{a(N)}^{1/2} u_{\varepsilon,a}(t)\|_{L^2}^2 \leq \|P_a^{1/2} u_0\|_{L^2}^2 + 4d_{\varepsilon}(\|u_0\|_{L^2}) \quad \forall t \in \mathbb{R}. \tag{3.12}$$

On the other hand, we can see that

$$\|P_a v\|_{H^{-1}} \leq \left(1 + \frac{4|a|}{(N-2)^2}\right)^{1/2} \|(1 + P_a)^{1/2} v\|_{L^2}.$$

Putting $v := u_{\varepsilon,a}(t)$ we obtain

$$\begin{aligned} \|P_a u_{\varepsilon,a}(t)\|_{H^{-1}} &\leq \left(1 + \frac{4|a|}{(N-2)^2}\right)^{1/2} \|(1 + P_a)^{1/2} u_{\varepsilon,a}(t)\|_{L^2} \\ &\leq \left(1 + \frac{4|a|}{(N-2)^2}\right)^{1/2} [\|(1 + P_a)^{1/2} u_0\|_{L^2}^2 + 4d_\varepsilon(\|u_0\|_{L^2})]^{1/2}. \end{aligned}$$

Also we can calculate

$$\|g_\varepsilon(u_{\varepsilon,a}(t))\|_{L^2} \leq \tilde{d}_\varepsilon(\|u_{\varepsilon,a}(t)\|_{L^2}) \|u_{\varepsilon,a}(t)\|_{L^2} = \tilde{d}_\varepsilon(\|u_0\|_{L^2}) \|u_0\|_{L^2},$$

where $\tilde{d}_\varepsilon(\|\varphi\|_{L^2}) := \|k_\varepsilon\|_{L_x^\infty(L_y^\infty)} \|\varphi\|_{L^2}^3$. Thus we obtain for all $t \in \mathbb{R}$

$$\|u'_{\varepsilon,a}(t)\|_{H^{-1}} \leq [\|(1 + P_a)^{1/2} u_0\|_{L^2} + 4d_\varepsilon(\|u_0\|_{L^2})]^{1/2} + \tilde{d}_\varepsilon(\|u_0\|_{L^2}) \|u_0\|_{L^2}. \quad (3.13)$$

Since $X^1(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N)$ is continuous, applying the Ascoli-Arzelà type lemma (see [2, Proposition 1.1.2]) we conclude that for any $T > 0$ there exist $\{a_j\}_j \subset (a(N), 0)$ and $u_\varepsilon(t)$ such that $a_j \rightarrow a(N)$ ($j \rightarrow \infty$) and

$$u_{\varepsilon,a_j}(t) \rightarrow u_\varepsilon(t) \quad (j \rightarrow \infty) \quad \forall t \in (-T, T) \quad \text{weakly in } X^1(\mathbb{R}^N).$$

Next we show that u_ε satisfies **(CP)_a** with $a = a(N)$ and $g = g_\varepsilon$. Since $g_\varepsilon(u_{\varepsilon,a_j}(t))$ is uniformly bounded in $L^\infty(-T, T; L^2(\mathbb{R}^N))$, we see that $g_\varepsilon(u_{\varepsilon,a_j}(t)) \rightarrow f_\varepsilon(t)$ weakly* in $L^\infty(-T, T; L^2(\mathbb{R}^N))$; also in $L^\infty(-T, T; X^{-1}(\mathbb{R}^N))$. By virtue of the weak closedness of g_ε (see **(G5)** in [13, Lemma 3.1]) we have

$$\text{Im} \int_{\mathbb{R}^N} f_\varepsilon(t) \overline{u_\varepsilon(t)} dx = 0.$$

Thus we obtain

$$\|u_\varepsilon(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 = \|u_{\varepsilon,a}(t)\|_{L^2}^2 \quad \forall t \in (-T, T),$$

Weak convergence and the convergence of the corresponding norms imply that

$$u_{\varepsilon,a_j}(t) \rightarrow u_\varepsilon(t) \quad (j \rightarrow \infty) \quad \forall t \in (-T, T) \quad \text{strongly in } L^2(\mathbb{R}^N).$$

Hence we conclude from the weak closedness of g_ε (in a way similar to the verification of **(G5)** of [13, Lemma 3.1]) that u_ε satisfies $iu'_\varepsilon = P_{a(N)} u_\varepsilon + g_\varepsilon(u_\varepsilon)$ and

$$u_{\varepsilon,a_j}(t) \rightarrow u_\varepsilon(t) \quad (j \rightarrow \infty) \quad \text{in } C([-T, T]; L^2(\mathbb{R}^N)).$$

Next we prove

$$u_{\varepsilon,a_j}(t) \rightarrow u_\varepsilon(t) \quad (j \rightarrow \infty) \quad \text{strongly in } X^1(\mathbb{R}^N). \quad (3.14)$$

The conservation of laws imply that

$$\begin{aligned} & \| (1 + P_a)^{1/2} u_{\varepsilon,a}(t) \|_{L^2}^2 \\ &= 2E_{\varepsilon,a}(u_{\varepsilon,a}(t)) - 2G_{\varepsilon}(u_{\varepsilon,a}(t)) \\ &= 2E_{\varepsilon,a}(u_0) - 2G_{\varepsilon}(u_{\varepsilon,a}(t)) \\ &\rightarrow 2E_{\varepsilon}(u_0) - 2G_{\varepsilon}(u_{\varepsilon}(t)) \quad (a \rightarrow a(N) + 0) \\ &= 2E_{\varepsilon}(u_{\varepsilon}(t)) - 2G_{\varepsilon}(u_{\varepsilon}(t)) = \| (1 + P_{a(N)})^{1/2} u_{\varepsilon}(t) \|_{L^2}^2. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} \limsup_{a \rightarrow a(N)+0} \| (1 + P_{a(N)})^{1/2} u_{\varepsilon,a}(t) \|_{L^2}^2 &\leq \limsup_{a \rightarrow a(N)+0} \| (1 + P_a)^{1/2} u_{\varepsilon,a}(t) \|_{L^2}^2 \\ &= \| (1 + P_{a(N)})^{1/2} u_{\varepsilon}(t) \|_{L^2}^2. \end{aligned}$$

Thus the weak convergence implies (3.14).

Since the strong convergence in $X^1(\mathbb{R}^N)$ and $\|u_{\varepsilon,a_j}(t)\|_{X^1}$ is uniformly bounded in $t \in [-T, T]$, we see from the dominated convergence theorem implies that if $u_0 \in \Sigma^1(\mathbb{R}^N)$, then

$$\|xu_{\varepsilon}(t)\|_{L^2}^2 = \|xu_0\|_{L^2}^2 + 4t \operatorname{Im} \int_{\mathbb{R}^N} \overline{xu_0} \cdot \nabla u_0 dx + \int_0^t (t-s) V_{\varepsilon}(u_{\varepsilon}(s)) ds,$$

where

$$V_{\varepsilon}(v) := 8 \|P_{a(N)}^{1/2} v\|_{L^2}^2 - 2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \widetilde{k}_{\varepsilon}(x, y) |v(x)|^2 |v(y)|^2 dx dy.$$

Thus we obtain

$$\frac{d^2}{dt^2} \|xu_{\varepsilon}(t)\|_{L^2}^2 = V_{\varepsilon}(u_{\varepsilon}(t)). \tag{3.15}$$

REMARK 3.3. Let $u_0 \in X^1(\mathbb{R}^N)$ in general. We can apply the continuous dependence of initial values. First set $\{u_{0m}\}_m \subset \Sigma^1(\mathbb{R}^N)$ such that $u_{0m} \rightarrow u_0$ strongly in $\Sigma^1_*(\mathbb{R}^N)$. Next let $u_{\varepsilon,m}$ be a unique solution to **(CP)** $_{\varepsilon,a}$ with $u_{\varepsilon,m}(0) = u_{0m}$. Then $u_{\varepsilon,m}$ satisfies

$$\begin{aligned} & \|xu_{\varepsilon,m}(t)\|_{L^2}^2 \\ &= \|xu_{0m}\|_{L^2}^2 + 4t \operatorname{Im} \int_{\mathbb{R}^N} \overline{xu_{0m}} \cdot \nabla u_{0m} dx + \int_0^t (t-s) V_{\varepsilon}(u_{\varepsilon,m}(s)) ds. \end{aligned}$$

Since the continuous dependence and the dominated convergence theorem imply that

$$\int_0^t (t-s) V_{\varepsilon}(u_{\varepsilon,m}(s)) ds \rightarrow \int_0^t (t-s) V_{\varepsilon}(u_{\varepsilon}(s)) ds$$

uniformly in $t \in [-T, T]$. (3.7) yields that

$$\operatorname{Im} \int_{\mathbb{R}^N} \overline{xu_{0m}} \cdot \nabla u_{0m} dx \rightarrow \operatorname{Im} \int_{\mathbb{R}^N} \overline{xu_0} \cdot \nabla u_0 dx.$$

Hence we conclude the virial identity (3.15) even when $u_0 \in X^1(\mathbb{R}^N)$.

Proof of Stage 3. First we show the uniform convergence of u_ε ($\varepsilon \rightarrow +0$); see also [14, Lemma 3.2].

PROPOSITION 3.3. *Let $u \in C([-T_1, T_2]; X^1(\mathbb{R}^N))$ be a local weak solution to $(\mathbf{CP})_a$ in $(-T_1, T_2)$. Then $u_\varepsilon \rightarrow u$ ($\varepsilon \rightarrow +0$) strongly in $C([-T_1, T_2]; X^1(\mathbb{R}^N))$.*

Proof. Step 1. First we show the uniform boundedness of u_ε :

$$\|u_\varepsilon(t)\|_{X^1} \leq M_0 \quad \forall \varepsilon > 0, \forall t \in [-T, T]. \tag{3.16}$$

Let $M > \|u_0\|_{X^1}$. Define

$$\tau_\varepsilon := \sup_{T>0} \{ \|u_\varepsilon(t)\|_{X^1} \leq M, t \in [-T, T] \}.$$

If $\tau_\varepsilon = \infty$, then we have proved the uniform boundedness. Thus we assume $\tau_\varepsilon < \infty$. Since $u_\varepsilon \in C(\mathbb{R}; X^1(\mathbb{R}^N))$, τ_ε satisfies

$$\|u_\varepsilon(\tau_\varepsilon)\|_{X^1} = M \text{ or } \|u_\varepsilon(-\tau_\varepsilon)\|_{X^1} = M. \tag{3.17}$$

By using the conservation laws we have

$$\begin{aligned} \|u_\varepsilon(t)\|_{X^1}^2 - \|u_0\|_{X^1}^2 &= 2[G_\varepsilon(u_0) - G_\varepsilon(u_\varepsilon(t))] \\ &\leq 2CM^4 \|k - k_R\|_{L_x^\beta(L_y^\alpha)} + 2RM^3 \|u_0 - u_\varepsilon(t)\|_{L^2}. \end{aligned} \tag{3.18}$$

On the other hand, we see from the verification of $(\mathbf{G2})$ in [13, Lemma 3.1] (or [13, Lemma 2.5]) that

$$\begin{aligned} \|u'_\varepsilon(t)\|_{X^{-1}} &\leq \|P_a u_\varepsilon(t)\|_{X^{-1}} + \|u_\varepsilon(t)K[k_\varepsilon](|u_\varepsilon(t)|^2)\|_{X^{-1}} \\ &\leq \|u_\varepsilon(t)\|_{X^1} + R_0 \|u_\varepsilon(t)\|_{L^2}^3 + C \|k - k_{R_0}\|_{L_x^\beta(L_y^\alpha)} \|u_\varepsilon(t)\|_{X^1}^3 \\ &\leq M + R_0 \|u_0\|_{L^2}^3 + C \|k - k_{R_0}\|_{L_x^\beta(L_y^\alpha)} M^3 =: C(M) \quad \forall t \in [-\tau_\varepsilon, \tau_\varepsilon]. \end{aligned}$$

Applying [2, Lemma 3.3.6] we obtain

$$\|u_\varepsilon(t) - u_\varepsilon(s)\|_{L^2} \leq \sqrt{2}C(M) |t - s|^{1/2}, \quad t, s \in [-\tau_\varepsilon, \tau_\varepsilon]. \tag{3.19}$$

Combining (3.19) with setting $s = 0$ into (3.18), we see that

$$\|u_\varepsilon(t)\|_{X^1}^2 - \|u_0\|_{X^1}^2 \leq 2CM^4 \|k - k_R\|_{L_x^\beta(L_y^\alpha)} + 2\sqrt{2}RM^3 C(M) |t|^{1/2}.$$

Letting $t = \pm \tau_\varepsilon$ and applying (3.17) we have

$$\tau_\varepsilon^{1/2} \geq \frac{3M^2 - 8CM^4 \|k - k_R\|_{L_x^\beta(L_y^\alpha)}}{8\sqrt{2}RM^3 C(M)} > 0;$$

note that $\|k - k_R\|_{L_x^\beta(L_y^\alpha)} \rightarrow 0$ ($R \rightarrow \infty$) implies the positivity. Thus we obtain (3.16) by putting

$$T_M := \left[\frac{3 - 8CM^2 \|k - k_R\|_{L_x^\beta(L_y^\alpha)}}{8\sqrt{2}RMC(M)} \right]^2 > 0.$$

Step 2. Next we show that $u_\varepsilon \rightarrow u$ ($\varepsilon \rightarrow 0$) strongly in $L^\infty(-T_1, T_2; L^2(\mathbb{R}^N))$ and in $L^{r(\gamma)}(-T_1, T_2; L^{2\gamma}(\mathbb{R}^N))$, where γ is defined in (3.8) and $r(\gamma) := 4\gamma/[N(\gamma - 1)]$. Note that u and u_ε satisfy the following integral equations:

$$\begin{aligned} u(t) &= \exp(-itP_{a(N)})u_0 - i \int_0^t \exp(-i(t-s)P_{a(N)})\{u(s)K[k](|u(s)|^2)\} ds, \\ u_\varepsilon(t) &= \exp(-itP_{a(N)})u_0 - i \int_0^t \exp(-i(t-s)P_{a(N)})\{u_\varepsilon(s)K[k_\varepsilon](|u_\varepsilon(s)|^2)\} ds. \end{aligned}$$

We divide $u(t) - u_\varepsilon(t)$ into $J_1(t; \varepsilon) + J_2(t; \varepsilon) + J_3(t; \varepsilon)$, where

$$\begin{aligned} J_1(t; \varepsilon) &:= -i \int_0^t \exp(-i(t-s)P_{a(N)})\{u(s)(K[k](|u(s)|^2) - K[k_\varepsilon](|u(s)|^2))\} ds, \\ J_2(t; \varepsilon) &:= -i \int_0^t \exp(-i(t-s)P_{a(N)})\{(u(s) - u_\varepsilon(s))K[k_\varepsilon](|u(s)|^2)\} ds, \\ J_3(t; \varepsilon) &:= -i \int_0^t \exp(-i(t-s)P_{a(N)})\{u_\varepsilon(s)K[k_\varepsilon](|u(s)|^2 - |u_\varepsilon(s)|^2)\} ds. \end{aligned}$$

For simply we denote $\|f\|_{L_t^\tau(L^p)} := \|f\|_{L^\tau(-T, T; L^p)}$. Applying the Strichartz estimates (2.3) we have the estimate for J_1

$$\begin{aligned} \|J_1\|_{L_t^\tau(L^p)} &\leq C_{\infty, \tau} \|uK[k_R - (k_R)_\varepsilon](|u|^2)\|_{L_t^1(L^2)} \\ &\quad + C_{r(\gamma), \tau} \|uK[(k - k_R) - (k_\varepsilon - (k_R)_\varepsilon)](|u|^2)\|_{L_t^{r(\gamma)'}(L^{2\gamma}')} \end{aligned}$$

Applying (3.9) and the dominated convergence theorem, we see that

$$\|J_1\|_{L_t^\tau(L^p)} \rightarrow 0 \quad (\varepsilon \rightarrow +0). \tag{3.20}$$

For J_2 applying the Strichartz estimates we have

$$\begin{aligned} &\|J_2\|_{L_t^\tau(L^p)} \\ &\leq C_{\infty, \tau} \|(u - u_\varepsilon)K[(k_R)_\varepsilon](|u|^2)\|_{L_t^1(L^2)} \\ &\quad + C_{r(\gamma), \tau} \|(u - u_\varepsilon)K[k_\varepsilon - (k_R)_\varepsilon](|u|^2)\|_{L_t^{r(\gamma)'}(L^{2\gamma}')} \\ &\leq 2C_{\infty, \tau}RT \|u\|_{L_t^\infty(L^2)}^2 \|u - u_\varepsilon\|_{L_t^\infty(L^2)} \\ &\quad + C_{r(\gamma), \tau} (2T)^{1-2/r(\gamma)} \|k - k_R\|_{L_x^\beta(L_y^\alpha)} \|u\|_{L_t^\infty(L^{2\gamma})}^2 \|u - u_\varepsilon\|_{L_t^{r(\gamma)'}(L^{2\gamma}')} \end{aligned}$$

In a way similar to J_2 , we can evaluate J_3 as follows:

$$\begin{aligned} & \|J_3\|_{L_t^\tau(L^\rho)} \\ & \leq C_{\infty,\tau} \|u_\varepsilon K[(k_R)_\varepsilon] (|u|^2 - |u_\varepsilon|^2)\|_{L_t^1(L^2)} \\ & + C_{r(\gamma),\tau} \|u_\varepsilon K[k_\varepsilon - (k_R)_\varepsilon] (|u|^2 - |u_\varepsilon|^2)\|_{L_t^{r(\gamma)'}(L^{2\gamma'})} \\ & \leq 2C_{\infty,\tau} RT \|u_\varepsilon\|_{L_t^\infty(L^2)} (\|u\|_{L_t^\infty(L^2)} + \|u_\varepsilon\|_{L_t^\infty(L^2)}) \|u - u_\varepsilon\|_{L_t^\infty(L^2)} \\ & + C_{r(\gamma),\tau} (2T)^{1-2/r(\gamma)} \|k - k_R\|_{L_x^\beta(L_x^\alpha)} \|u_\varepsilon\|_{L_t^\infty(L^{2\gamma})} \\ & \times (\|u\|_{L_t^\infty(L^{2\gamma})} + \|u_\varepsilon\|_{L_t^\infty(L^{2\gamma})}) \|u - u_\varepsilon\|_{L_t^{r(\gamma)}(L^{2\gamma})}. \end{aligned}$$

Set $(\tau, \rho) = (\infty, 2)$ and $(r(\gamma), 2\gamma)$. Now we put

$$M := \max\{\|u_0\|_{L^2}, \|u\|_{L^{r(\gamma)}(-T, T; L^{2\gamma})}, \sup_{\varepsilon \in (0, 1)} \|u_\varepsilon\|_{L^{r(\gamma)}(-T, T; L^{2\gamma})}\} < \infty.$$

Take $T_0 \in (0, T)$ such that $6(C_{\infty,\infty} + C_{\infty,r(\gamma)})RM^2T_0 \leq 1/2$ and $3(C_{r(\gamma),\infty} + C_{r(\gamma),r(\gamma)})\|k - k_R\|_{L_x^\beta(L_x^\alpha)} M^2(2T_0)^{1-2/r(\gamma)} \leq 1/2$. Then we obtain

$$\begin{aligned} & \|u - u_\varepsilon\|_{L^{r(\gamma)}(-T_0, T_0; L^{2\gamma})} + \|u - u_\varepsilon\|_{L^\infty(-T_0, T_0; L^2)} \\ & \leq 2\|J_1\|_{L^\infty(-T_0, T_0; L^2)} + 2\|J_1\|_{L^{r(\gamma)}(-T_0, T_0; L^{2\gamma})}. \end{aligned} \tag{3.21}$$

It follows from (3.20) that

$$\begin{aligned} u_\varepsilon & \rightarrow u \quad (\varepsilon \rightarrow +0) \text{ strongly in } L^\infty(-T_0, T_0; L^2(\mathbb{R}^N)) \\ & \text{and in } L^{r(\gamma)}(-T_0, T_0; L^{2\gamma}(\mathbb{R}^N)). \end{aligned} \tag{3.22}$$

Extending the interval step by step, we conclude that $u_\varepsilon \rightarrow u$ ($\varepsilon \rightarrow +0$) strongly in $L^\infty(-T_1, T_2; L^2(\mathbb{R}^N))$ and in $L^{r(\gamma)}(-T_1, T_2; L^{2\gamma}(\mathbb{R}^N))$.

Step 3. Assume that $u_\varepsilon \not\rightarrow u$ ($\varepsilon \rightarrow +0$) in $C([-T_1, T_2]; X^1(\mathbb{R}^N))$. Then there exist $\varepsilon_0 > 0$ and bounded sequences $\{\varepsilon_m\}_m \subset (0, 1)$ and $\{t_m\}_m \subset [-T_1, T_2]$ such that

$$\|u_{\varepsilon_m}(t_m) - u(t_m)\|_{X^1} \geq \varepsilon_0, \quad m \in \mathbb{N}.$$

We may also assume that $\varepsilon_m \rightarrow 0$ and $t_m \rightarrow t_0[-T_1, T_2]$ ($m \rightarrow \infty$). Since u belongs to $C([-T_1, T_2]; X^1(\mathbb{R}^N))$, we have $\|u(t_m) - u(t_0)\|_{X^1} < \varepsilon_0/2$ for sufficiently large m . Therefore we obtain

$$\|u_{\varepsilon_m}(t_m) - u(t_0)\|_{X^1} > \frac{\varepsilon_0}{2}.$$

On the other hand, it follows from Step 2 that $\|u_{\varepsilon_m}(t_m) - u(t_m)\|_{L^2} \rightarrow 0$ ($m \rightarrow \infty$). Since $u \in C(\bar{I}; L^2(\mathbb{R}^N))$, we have $\|u(t_m) - u(t_0)\|_{L^2} \rightarrow 0$ ($m \rightarrow \infty$). Thus we obtain $\|u_{\varepsilon_m}(t_m) - u(t_0)\|_{L^2} \rightarrow 0$ ($m \rightarrow \infty$). This means that $u_{\varepsilon_m}(t_m) \rightarrow u(t_0)$ ($m \rightarrow \infty$) strongly in $L^2(\mathbb{R}^N)$ but $u_{\varepsilon_m}(t_m) \not\rightarrow u(t_0)$ ($m \rightarrow \infty$) strongly in $X^1(\mathbb{R}^N)$. To derive a contradiction it remains to show that

$$u_{\varepsilon_m}(t_m) \rightarrow u(t_0) \quad (m \rightarrow \infty) \quad \text{strongly in } X^1(\mathbb{R}^N). \tag{3.23}$$

Next we show

$$\|P_{a(N)}^{1/2} u_{\varepsilon_m}(t_m)\|_{L^2}^2 \rightarrow \|P_{a(N)}^{1/2} u(t_0)\|_{L^2}^2 \quad (m \rightarrow \infty). \tag{3.24}$$

To end this, first we see from the conservation laws that

$$E_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) = E_{\varepsilon_m}(u_0) \rightarrow E(u_0) = E(u(t_0)) \quad (m \rightarrow \infty). \tag{3.25}$$

Next we prove

$$G_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) \rightarrow G(u(t_0)) \quad (m \rightarrow \infty). \tag{3.26}$$

Applying (3.10) we calculate

$$\begin{aligned} & |G_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) - G(u(t_0))| \\ & \leq |G_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) - G_{\varepsilon_m}(u(t_0))| + |G_{\varepsilon_m}(u(t_0)) - G(u(t_0))| \\ & \leq CM^4 \|k - k_R\|_{L_x^\beta(L_y^\alpha)} + RM^3 \|u_{\varepsilon_m}(t_m) - u(t_0)\|_{L^2} \\ & + |G_{\varepsilon_m}(u(t_0)) - G(u(t_0))| \\ & \rightarrow CM^4 \|k - k_R\|_{L_x^\beta(L_y^\alpha)} \quad (m \rightarrow \infty). \end{aligned}$$

Since $R > 0$ is arbitrary, **(K2)** implies (3.26). Now we can write as for $\varphi \in X^1(\mathbb{R}^N)$

$$E_\varepsilon(\varphi) = \frac{1}{2} \|P_{a(N)}^{1/2} \varphi\|_{L^2}^2 + G_\varepsilon(\varphi), \quad E(\varphi) = \frac{1}{2} \|P_{a(N)}^{1/2} \varphi\|_{L^2}^2 + G(\varphi). \tag{3.27}$$

Combining (3.25) and (3.26) into (3.27) we obtain (3.24).

On the other hand, by the boundedness of $\|u_{\varepsilon_m}(t_m)\|_{X^1}$ there exist $v \in X^1(\mathbb{R}^N)$ and a weak convergent subsequence $\{u_{\varepsilon_{m(j)}}(t_{m(j)})\}_j$ such that $u_{\varepsilon_{m(j)}}(t_{m(j)}) \rightarrow v$ ($j \rightarrow \infty$) weakly in $X^1(\mathbb{R}^N)$. Since $u_{\varepsilon_{m(j)}}(t_{m(j)}) \rightarrow u(t_0)$ ($j \rightarrow \infty$) strongly in $L^2(\mathbb{R}^N)$, we obtain $u_{\varepsilon_{m(j)}}(t_{m(j)}) \rightarrow u(t_0)$ ($j \rightarrow \infty$) weakly in $X^1(\mathbb{R}^N)$. Therefore from the weak convergence in $X^1(\mathbb{R}^N)$ of $\{u_{\varepsilon_{m(j)}}(t_{m(j)})\}_j$ to $u(t_0)$ and the convergence of the corresponding norms we conclude (3.23), a contradiction. \square

Now we are the final position to prove the virial identities for **(CP)_a** ($a = a(N)$). Since the strong convergence in $X^1(\mathbb{R}^N)$ and $\|u_\varepsilon(t)\|_{X^1}$ is uniformly bounded in $t \in [-T_1, T_2]$ and ε , we see from the dominated convergence theorem implies that if $u_0 \in \Sigma_*^1(\mathbb{R}^N)$, then

$$\|xu(t)\|_{L^2}^2 = \|xu_0\|_{L^2}^2 + 4t \operatorname{Im} \int_{\mathbb{R}^N} \overline{xu_0} \cdot \nabla u_0 \, dx + \int_0^t (t-s)V(u_\varepsilon(s)) \, ds,$$

where

$$V(v) := 8 \|P_{a(N)}^{1/2} v\|_{L^2}^2 - 2 \iint_{\mathbb{R}^N \times \mathbb{R}^N} \tilde{k}(x, y) |v(x)|^2 |v(y)|^2 \, dx dy$$

(see [14, Remark 2.1]). Thus we obtain

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = V(u(t)).$$

REMARK 3.4. We consider the nonlinearity as in Example 3.1. We obtain

$$\begin{aligned} \frac{d^2}{dt^2} \|xu_{\varepsilon,a}(t)\|_{L^2}^2 &= 8 \|P_a^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 - 8N \int_{\mathbb{R}^N} F(|\rho_\varepsilon * u_{\varepsilon,a}(t)|) dx \\ &\quad + 4N \int_{\mathbb{R}^N} |\rho_\varepsilon * u_{\varepsilon,a}(t)| g(|\rho_\varepsilon * u_{\varepsilon,a}(t)|) dx \\ &\quad - 8 \operatorname{Re} \int_{\mathbb{R}^N} \overline{\tilde{\rho}_\varepsilon * u_{\varepsilon,a}(t)} g(\rho_\varepsilon * u_{\varepsilon,a}(t)) dx \end{aligned}$$

and

$$\begin{aligned} \frac{d^2}{dt^2} \|xu_\varepsilon(t)\|_{L^2}^2 &= 8 \|P_a^{1/2}u_\varepsilon(t)\|_{L^2}^2 - 8N \int_{\mathbb{R}^N} F(|\rho_\varepsilon * u_\varepsilon(t)|) dx \\ &\quad + 4N \int_{\mathbb{R}^N} |\rho_\varepsilon * u_\varepsilon(t)| g(|\rho_\varepsilon * u_\varepsilon(t)|) dx \\ &\quad - 8 \operatorname{Re} \int_{\mathbb{R}^N} \overline{\tilde{\rho}_\varepsilon * u_\varepsilon(t)} g(\rho_\varepsilon * u_\varepsilon(t)) dx. \end{aligned}$$

Thus we see from $\tilde{\rho}_\varepsilon * f \rightarrow 0$ ($\varepsilon \rightarrow +0$) strongly in $L^q(\mathbb{R}^N)$ ($1 \leq q < \infty$) that

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8 \|P_a^{1/2}u(t)\|_{L^2}^2 - 8N \int_{\mathbb{R}^N} F(|u(t)|) dx + 4N \int_{\mathbb{R}^N} |u(t)| g(|u(t)|) dx.$$

REMARK 3.5. We consider the nonlinearity as in Example 3.2. We obtain

$$\begin{aligned} \frac{d^2}{dt^2} \|xu_{\varepsilon,a}(t)\|_{L^2}^2 &= 8 \|P_a^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 - 4N\lambda \int_{\mathbb{R}^N} \frac{|\rho_\varepsilon * u_{\varepsilon,a}(t)|^{p+1}}{(|x|^2 + \varepsilon)^{r/2}} dx \\ &\quad + 8\lambda \int_{\mathbb{R}^N} \frac{(N-r)|x|^2 + N\varepsilon}{(p+1)(|x|^2 + \varepsilon)^{r/2+1}} |\rho_\varepsilon * u_{\varepsilon,a}(t)|^{p+1} dx \\ &\quad - 8\lambda \operatorname{Re} \int_{\mathbb{R}^N} \overline{\tilde{\rho}_\varepsilon * u_{\varepsilon,a}(t)} \rho_\varepsilon * u_{\varepsilon,a}(t) \frac{|\rho_\varepsilon * u_{\varepsilon,a}(t)|^{p-1}}{(|x|^2 + \varepsilon)^{r/2}} dx \end{aligned}$$

and

$$\begin{aligned} \frac{d^2}{dt^2} \|xu_\varepsilon(t)\|_{L^2}^2 &= 8 \|P_a^{1/2}u_\varepsilon(t)\|_{L^2}^2 + 4N\lambda \int_{\mathbb{R}^N} \frac{|\rho_\varepsilon * u_\varepsilon(t)|^{p+1}}{(|x|^2 + \varepsilon)^{r/2}} dx \\ &\quad - 8\lambda \int_{\mathbb{R}^N} \frac{(N-r)|x|^2 + N\varepsilon}{(p+1)(|x|^2 + \varepsilon)^{r/2+1}} |\rho_\varepsilon * u_\varepsilon(t)|^{p+1} dx \\ &\quad - 8\lambda \operatorname{Re} \int_{\mathbb{R}^N} \overline{\tilde{\rho}_\varepsilon * u_\varepsilon(t)} \rho_\varepsilon * u_\varepsilon(t) \frac{|\rho_\varepsilon * u_\varepsilon(t)|^{p-1}}{(|x|^2 + \varepsilon)^{r/2}} dx. \end{aligned}$$

Thus we see from $\tilde{\rho}_\varepsilon * f \rightarrow 0$ ($\varepsilon \rightarrow +0$) strongly in $L^q(\mathbb{R}^N)$ ($1 \leq q < \infty$) that

$$\frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 = 8 \|P_a^{1/2}u(t)\|_{L^2}^2 + \frac{4\lambda(Np - N + 2r)}{p+1} \int_{\mathbb{R}^N} \frac{|u(t)|^{p+1}}{|x|^r} dx.$$

3.1. Typical examples of the virial identity for (CP)_a with $a = a(N)$

(V1) $g(u) := \lambda |u|^{p-1}u$ ($\lambda \in \mathbb{R}$, $1 \leq p < (N + 2)/(N - 2)$)

$$\begin{aligned} \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 &= 8 \|P_{a(N)}^{1/2}u(t)\|_{L^2}^2 + \frac{4\lambda N(p-1)}{p+1} \int_{\mathbb{R}^N} |u(t)|^{p+1} dx \\ &= 16E(u(t)) + \frac{4\lambda(Np-N-4)}{p+1} \int_{\mathbb{R}^N} |u(t)|^{p+1} dx; \end{aligned} \tag{3.28}$$

(V2) $g(u) := \lambda |x|^{-r}|u|^{p-1}u$ ($\lambda \in \mathbb{R}$, $0 < r < 2$, $1 \leq p < (N + 2 - 2r)/(N - 2)$)

$$\begin{aligned} \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 &= 8 \|P_{a(N)}^{1/2}u(t)\|_{L^2}^2 + \frac{4\lambda(Np-N+2r)}{p+1} \int_{\mathbb{R}^N} \frac{|u(t)|^{p+1}}{|x|^r} dx \\ &= 16E(u(t)) + \frac{4\lambda(Np-N-4+2r)}{p+1} \int_{\mathbb{R}^N} \frac{|u(t)|^{p+1}}{|x|^r} dx; \end{aligned} \tag{3.29}$$

(V3) $g(u) := \lambda(|x|^{-\gamma} * |u|^2)u$ ($\lambda \in \mathbb{R}$, $0 < \gamma < \min\{N, 4\}$)

$$\begin{aligned} \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 &= 8 \|P_{a(N)}^{1/2}u(t)\|_{L^2}^2 + 2\lambda\gamma \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^\gamma} dx dy \\ &= 16E(u(t)) + 2\lambda(\gamma-2) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x-y|^\gamma} dx dy; \end{aligned} \tag{3.30}$$

(V4) $g(u) := \lambda u |x|^{-\alpha} [|x|^{-\beta} * (|x|^{-\alpha}|u|^2)]$ ($\lambda \in \mathbb{R}$, $\alpha > 0$, $\beta > 0$, $0 < 2\alpha + \beta < \min\{N, 4\}$)

$$\begin{aligned} \frac{d^2}{dt^2} \|xu(t)\|_{L^2}^2 &= 8 \|P_{a(N)}^{1/2}u(t)\|_{L^2}^2 + 2\lambda(2\alpha + \beta) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x|^\alpha |x-y|^\beta |y|^\alpha} dx dy \\ &= 16E(u(t)) + 2\lambda(2\alpha + \beta - 2) \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(t,x)|^2 |u(t,y)|^2}{|x|^\alpha |x-y|^\beta |y|^\alpha} dx dy. \end{aligned} \tag{3.31}$$

4. Applications

4.1. Blow-up in finite time

THEOREM 4.1. *Let $a = a(N)$ and $u_0 \in \Sigma_*^1(\mathbb{R}^N)$ with $E(u_0) < 0$. Assume either that*

(B1) *g satisfies **(N1)**, **(N2)**, and $2(N + 2)F(x) - Nxf(x) \geq 0$ for $x > 0$;*

(B2) *$g(u) = \lambda |x|^{-r}|u|^{p-1}u$ with $\lambda < 0$, $0 < r < 2$, and $1 + (4 - 2r)/N < p < 1 + (4 - 2r)/(N - 2)$;*

(B3) $g(u) = uK[k](|u|^2)$, where k satisfies (K1), (K2), (K3), and

$$k(x, y) + \frac{1}{2}\tilde{k}(x, y) \geq 0.$$

Then there exist $T_1, T_2 > 0$ such that the unique local weak solution $u \in C(\bar{I}; X^1(\mathbb{R}^N)) \cap C^1(\bar{I}; X^{-1}(\mathbb{R}^N))$ to (CP)_a is exist if $\bar{I} \subset (-T_1, T_2)$ and u satisfies

$$\lim_{t \rightarrow -T_1+0} \|(1 + P_{a(N)})^{1/2}u(t)\|_{L^2} = \infty = \lim_{t \rightarrow T_2-0} \|(1 + P_{a(N)})^{1/2}u(t)\|_{L^2}, \tag{4.1}$$

that is, the local weak solution blows up in finite time and hence cannot extend globally in time.

Proof. Assume that a local solution $u \in C([-T, T]; X^1(\mathbb{R}^N))$ to (CP)_a can be extended globally in time. Then we see that

$$\varphi(t) := \|xu(t)\|_{L^2}^2 \geq 0 \quad \forall t \in \mathbb{R}.$$

The virial identity for (CP)_a and the assumption imply that $\varphi''(t) \leq 16E(u_0) < 0$. Thus we obtain

$$\varphi(t) \leq \|xu_0\|_{L^2}^2 + \varphi'(0)t + 8E(u_0)t^2 =: \psi(t).$$

Since $E(u_0) < 0$, we can select $T_1, T_2 > 0$ such that $\psi(-T_1) < 0$ and $\psi(T_2) < 0$. Hence $\phi(-T_1) < 0$ and $\phi(T_2) < 0$; this is a contradiction. Note that the last assertion (4.1) follows from [2, Remark 3.1.6 (ii)]. \square

4.2. Existence of scattering states

Next we consider the asymptotic completeness for (CP)_a with $a = a(N)$. Define

$$Q(u; \tau) := \int_{\mathbb{R}^N} \left[\left| \frac{x}{2}u + i\tau \nabla u \right|^2 - \frac{(N-2)^2}{4|x|^2} \tau^2 |u|^2 \right] dx.$$

We see that

$$Q(u; \tau) = \tau^2 \|P_{a(N)}^{1/2}u\|_{L^2}^2 - \tau \operatorname{Im} \int_{\mathbb{R}^N} x\bar{u} \cdot \nabla u \, dx + \frac{1}{4} \|xu\|_{L^2}^2. \tag{4.2}$$

Applying Lemma 3.1 we obtain

$$\begin{aligned} |Q(u; \tau)| &\leq \tau^2 \|P_{a(N)}^{1/2}u\|_{L^2}^2 + |\tau| \|P_{a(N)}^{1/2}u\|_{L^2} \|xu\|_{L^2} + \frac{1}{4} \|xu\|_{L^2}^2 \\ &= \left(\frac{1}{2} \|xu\|_{L^2} + |\tau| \|P_{a(N)}^{1/2}u\|_{L^2} \right)^2. \end{aligned}$$

On the other hand, Q is deduced by $\|P_a^{1/2} \cdot\|_{L^2}^2$:

$$\begin{aligned} \left\| P_{a(N)}^{1/2} \left[\exp\left(\frac{|x|^2}{4iv}\right) u \right] \right\|_{L^2}^2 &= \int_{\mathbb{R}^N} \left[\left| \nabla \left[\exp\left(\frac{|x|^2}{4iv}\right) u \right] \right|^2 - \frac{(N-2)^2}{4|x|^2} \left| \exp\left(\frac{|x|^2}{4iv}\right) u \right|^2 \right] dx \\ &= \int_{\mathbb{R}^N} \left[\left| \exp\left(\frac{|x|^2}{4iv}\right) \left[\frac{x}{2iv} u + \nabla u \right] \right|^2 - \frac{(N-2)^2}{4|x|^2} |u|^2 \right] dx \\ &= \frac{1}{v^2} \int_{\mathbb{R}^N} \left[\left| \frac{x}{2} u + iv \nabla u \right|^2 - \frac{(N-2)^2}{4|x|^2} v^2 |u|^2 \right] dx = \frac{1}{v^2} Q(u; v). \end{aligned}$$

THEOREM 4.2. Assume either that

- (A1) $g(u) = \lambda |u|^{p-1} u$ with $\lambda > 0$ and $1 + 2/N < p < 1 + 4/(N-2)$;
- (A2) $g(u) = \lambda |x|^{-r} |u|^{p-1} u$ with $\lambda > 0$, $0 < r < 2$, and $1 + (2-2r)/N < p < 1 + (4-2r)/(N-2)$;
- (A3) $g(u) = \lambda (|x|^{-\gamma} * |u|^2) u$ with $\lambda > 0$ and $1 < \gamma < \min\{N, 4\}$;
- (A4) $g(u) = \lambda u |x|^{-\alpha} [|x|^{-\beta} * (|x|^{-\alpha} |u|^2)]$ with $\lambda > 0$, $\alpha > 0$, $\beta > 0$, and $1 < 2\alpha + \beta < \min\{N, 4\}$.

Then for every $u_0 \in \Sigma_*^1(\mathbb{R}^N)$ there uniquely exists $(u_+, u_-) \in L^2(\mathbb{R}^N)^2$ such that

$$\begin{cases} \lim_{t \rightarrow \infty} \exp(itP_{a(N)})u(t) = u_+ & \text{strongly in } L^2(\mathbb{R}^N), \\ \lim_{t \rightarrow -\infty} \exp(itP_{a(N)})u(t) = u_- & \text{strongly in } L^2(\mathbb{R}^N). \end{cases} \tag{4.3}$$

Here $u(t) \in C(\mathbb{R}; X^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; X^{-1}(\mathbb{R}^N))$ is the unique global solution to **(CP)_a** with $a = a(N)$ and $u(0) = u_0$.

Proof. We divide the proof into three steps.

Step 1. First we consider the pseudo-conformal transform

$$(\mathcal{C}u)(t, x) := (1-t)^{-N/2} u\left(\frac{t}{1-t}, \frac{x}{1-t}\right) \exp\left(\frac{-i|x|^2}{4(1-t)}\right). \tag{4.4}$$

Here $v(t, x) := (\mathcal{C}u)(t, x)$ satisfies $iv' = P_{a(N)}v + (1-t)^\omega g(v)$. Note that $G(u)$ satisfies the convexity type inequality:

$$\left| \int_{\mathbb{R}^N} g(u) \bar{\nabla} dx \right| \leq q G(u)^{1-1/q} G(v)^{1/q}, \quad G(u) \equiv \frac{1}{q} \int_{\mathbb{R}^N} g(u) \bar{u} dx. \tag{4.5}$$

Here ω and q are

- (A1) $\omega = N(p-1)/2 - 2$, $q = p + 1$;
- (A2) $\omega = N(p-1)/2 + r - 2$, $q = p + 1$;
- (A3) $\omega = \gamma - 2$, $q = 4$;
- (A4) $\omega = 2\alpha + \beta - 2$, $q = 4$.

Note that it is sufficient to prove that there exists $v_+ \in L^2(\mathbb{R}^N)$ such that

$$v(t) \rightarrow v_+ \quad (t \rightarrow 1 - 0) \quad \text{strongly in } L^2(\mathbb{R}^N). \tag{4.6}$$

Step 2. We show that for $0 < t < 1$

$$\|P_a^{1/2}v(t)\|_{L^2}^2 \lesssim \begin{cases} (1-t)^\omega & -1 < \omega \leq 0, \\ 1 & \omega > 0, \end{cases} \tag{4.7}$$

$$G(v(t)) \lesssim 1. \tag{4.8}$$

To end this we define another energy function.

$$E_1(t) := \frac{1}{2} \|P_{a(N)}^{1/2}v(t)\|_{L^2}^2 + (1-t)^\omega G(v(t)). \tag{4.9}$$

Note that $E_1(t) = E(v(t))$ if $\omega = 0$. Since $v = \mathcal{C}u$ and u satisfies **(CP)_a** with $a = a(N)$, we see from (4.4) and (4.9) that

$$\begin{aligned} E_1(t) &= \frac{1}{8} \|xu(\sigma(t))\|_{L^2}^2 + \frac{(1-t)^{-2}}{2} \|P_{a(N)}^{1/2}u(\sigma(t))\|_{L^2}^2 \\ &\quad - \frac{(1-t)^{-1}}{8} \frac{d}{d\sigma} \|xu(\sigma)\|_{L^2}^2 \Big|_{\sigma=\sigma(t)} + (1-t)^{-2} G(u(\sigma(t))) \\ &= (1-t)^{-2} E(u(\sigma(t))) + \frac{1}{8} \|xu(\sigma(t))\|_{L^2}^2 - \frac{(1-t)^{-1}}{4} \frac{d}{d\sigma} \|xu(\sigma)\|_{L^2}^2 \Big|_{\sigma=\sigma(t)} \end{aligned}$$

where $\sigma(t) := t/(1-t)$. Hence we calculate the derivative of E_1 by using $\sigma'(t) = (1-t)^{-2}$ and the virial identity (See Section 3.1):

$$\begin{aligned} \frac{d}{dt} E_1(t) &= 2(1-t)^{-3} E(u_0) - \frac{(1-t)^{-3}}{8} \times \frac{d^2}{d\sigma^2} \|xu(\sigma)\|_{L^2}^2 \Big|_{\sigma=\sigma(t)} \\ &= 2(1-t)^{-3} E(u_0) - (1-t)^{-3} \left[\|P_{a(N)}^{1/2}u(\sigma(t))\|_{L^2}^2 + (\omega + 2)G(u(\sigma(t))) \right] \\ &= 2(1-t)^{-3} E(u_0) - 2(1-t)^{-3} E(u(\sigma(t))) - \omega(1-t)^{-3} G(u(\sigma(t))). \end{aligned}$$

It follows from the conservation of laws that

$$\frac{d}{dt} E_1(t) = -\omega(1-t)^{-3} G(u(s(t))) = -\omega(1-t)^{\omega-1} G(v(t)). \tag{4.10}$$

If $\omega \geq 0$, then $E_1(t) \leq E_1(0)$ for $0 < t < 1$. Hence we see (4.7). (4.7) yields (4.8).

On the other hand, it follows from (4.10) that

$$\frac{d}{dt} [(1-t)^{-\omega} E_1(t)] = -\frac{\omega}{2} (1-t)^{-\omega-1} \|P_{a(N)}^{1/2}v(t)\|_{L^2}^2.$$

Hence we conclude from $-1 < \omega \leq 0$ that $(1-t)^{-\omega} E_1(t) \leq E_1(0)$ for $0 < t < 1$. Thus we obtain (4.7) and (4.8).

Step 3. If $\omega \geq 0$, then we see that

$$v \in L^\infty(0, 1; X^1(\mathbb{R}^N)) \cap W^{1,\infty}(0, 1; X^{-1}(\mathbb{R}^N)) \subset C([0, 1]; L^2(\mathbb{R}^N))$$

and hence we simply obtain (4.6). Thus we assume $-1 < \omega \leq 0$ and prove (4.6).

First we verify that

$$v(t) \rightarrow v_+ (t \rightarrow 1 - 0) \quad \text{strongly in } X^{-1}(\mathbb{R}^N). \tag{4.11}$$

We see from

$$\begin{aligned} \|g(v)\|_{X^{-1}} &= \sup \left\{ \operatorname{Re} \int_{\mathbb{R}^N} g(v) \overline{\psi} dx; \|\psi\|_{X^1} \leq 1 \right\} \\ &\leq \sup \left\{ (p+1)G(v)^{1-1/q} G(\psi)^{1/q}; \|\psi\|_{X^1} \leq 1 \right\} \\ &\lesssim G(v)^{1-1/q} \sup \left\{ G(\psi)^{1/q}; \|\psi\|_{X^1} \leq 1 \right\} \\ &\lesssim G(v)^{1-1/q}. \end{aligned} \tag{4.12}$$

On the other hand, we have

$$\begin{aligned} &\|P_{a(N)}v\|_{X^{-1}} \\ &= \sup \left\{ \operatorname{Re} \langle P_{a(N)}v, \psi \rangle_{X^{-1}, X^1}; \|\psi\|_{X^1} \leq 1 \right\} \\ &= \sup \left\{ \operatorname{Re} \langle (1 + P_{a(N)})^{1/2}v, (1 + P_{a(N)})^{1/2}\psi \rangle_{L^2} - \operatorname{Re} \langle v, \psi \rangle_{L^2}; \|\psi\|_{X^1} \leq 1 \right\} \\ &\leq \sup \left\{ \|v\|_{X^1} \|\psi\|_{X^1} + \|v\|_{L^2} \|\psi\|_{L^2}; \|\psi\|_{X^1} \leq 1 \right\} \\ &\leq 2 \|v\|_{X^1}. \end{aligned} \tag{4.13}$$

Combining (4.13) into (4.7) and (4.12) into (4.8) we obtain

$$\begin{aligned} \|v'(t)\|_{X^{-1}} &\leq \|P_{a(N)}v(t)\|_{X^{-1}} + (1-t)^\omega \|g(v)\|_{X^{-1}} \\ &\lesssim \|v(t)\|_{X^1} + (1-t)^\omega G(v(t))^{p/(p+1)} \\ &\lesssim (1-t)^{\omega/2} + (1-t)^\omega \lesssim (1-t)^\omega. \end{aligned} \tag{4.14}$$

Thus (4.7) and (4.14) yield that $v \in W^{1,1/(\delta-\omega)}(0, 1; X^{-1}(\mathbb{R}^N)) \subset C([0, 1]; X^{-1}(\mathbb{R}^N))$ for sufficiently small $\delta > 0$. Thus we see (4.11).

Next we confirm that

$$v(t) \rightarrow v_+ (t \rightarrow 1 - 0) \quad \text{weakly in } L^2(\mathbb{R}^N). \tag{4.15}$$

Since $\|v(t)\|_{L^2} = \|v_0\|_{L^2}$ for $t \in [0, 1)$, for any sequence $\{t_j\}_j \subset (0, 1)$ there exists $v_{+1} \in L^2(\mathbb{R}^N)$ and a subsequence $\{j(k)\}_k \subset \{j\}$ such that

$$v(t_{j(k)}) \rightarrow v_{+1} (k \rightarrow \infty) \quad \text{weakly in } L^2(\mathbb{R}^N) \subset X^{-1}(\mathbb{R}^N).$$

Here (4.11) implies that $v_{+1} = v_+$. Thus (4.15) is verified. Hence it follows from (4.15) that

$$\langle v_+ - v(t), v_+ \rangle_{L^2} \rightarrow 0 \quad t \rightarrow 1 - 0. \tag{4.16}$$

Next we show that

$$\langle v_+ - v(t), v(t) \rangle_{L^2} \rightarrow 0 \quad t \rightarrow 1 - 0. \tag{4.17}$$

Let $0 < t < \tau < 1$. We calculate

$$\begin{aligned} |\langle v(\tau) - v(t), v(t) \rangle_{L^2}| &\leq \int_t^\tau |\langle v'(s), v(t) \rangle_{X^{-1}, X^1}| ds \\ &\leq 2 \int_t^\tau \|P_{a(N)}^{1/2} v(s)\|_{L^2} \|P_{a(N)}^{1/2} v(t)\|_{L^2} ds \\ &\quad + \int_t^\tau (1-s)^\omega \left| \int_{\mathbb{R}^N} g(v(s)) v(t) dx \right| ds. \end{aligned}$$

Here the convexity (4.5) implies that

$$\left| \int_{\mathbb{R}^N} g(v(s)) v(t) dx \right| \leq q G(v(s))^{1-1/q} G(v(t))^{1/q}.$$

Applying (4.7) and (4.8) and letting $\tau \rightarrow 1 - 0$ we ensure (4.17):

$$|\langle v_+ - v(t), v(t) \rangle_{L^2}| \lesssim \int_t^1 (1-t)^{\omega/2} (1-s)^{\omega/2} ds + \int_t^1 (1-s)^\omega ds \lesssim (1-t)^{\omega+1}.$$

Finally we conclude from (4.16) and (4.17) that

$$\|v_+ - v(t)\|_{L^2}^2 = \langle v_+ - v(t), v_+ \rangle_{L^2} - \langle v_+ - v(t), v(t) \rangle_{L^2} \rightarrow 0 \quad t \rightarrow 1 - 0.$$

This is nothing but (4.6). \square

REMARK 4.1. Under the conditions in Theorem 4.2 let u be a solution to $(\mathbf{CP})_a$ with $u(0) = u_0 \in \Sigma_*^1(\mathbb{R}^N)$. Then $w(t) := \overline{u(-t)}$ satisfies $(\mathbf{CP})_a$ with $w(0) = \overline{u_0}$. Hence we conclude that the scattering states

$$\begin{aligned} \lim_{t \rightarrow \infty} \exp(itP_{a(N)})u(t) &= u_+ = \Omega_+ u_0 \quad \text{strongly in } L^2(\mathbb{R}^N), \\ \lim_{t \rightarrow -\infty} \exp(itP_{a(N)})u(t) &= u_- = \Omega_- u_0 \quad \text{strongly in } L^2(\mathbb{R}^N) \end{aligned}$$

satisfy $\Omega_- \varphi = \overline{\Omega_+ \overline{\varphi}}$ for $\varphi \in \Sigma_*^1(\mathbb{R}^N)$.

REMARK 4.2. Assume either that

(A1a) $g(u) = \lambda |u|^{p-1} u$ with $\lambda > 0$ and $1 \leq p \leq 1 + 2/N$;

(A2a) $g(u) = \lambda |x|^{-r} |u|^{p-1} u$ with $\lambda > 0$, $0 < r \leq 1$ and $1 \leq p \leq 1 + (2 - 2r)/N$;

(A3a) $g(u) = \lambda (|x|^{-\gamma} * |u|^2) u$ with $\lambda > 0$ and $0 < \gamma \leq 1$;

(A4a) $g(u) = \lambda u |x|^{-\alpha} [|x|^{-\beta} * (|x|^{-\alpha} |u|^2)]$ with $\lambda > 0$, $\alpha > 0$, $\beta > 0$ and $0 < 2\alpha + \beta \leq 1$.

Then if $u_0 \in \Sigma_*^1(\mathbb{R}^N)$ satisfies

$$\begin{aligned} \lim_{t \rightarrow \infty} \exp(itP_{a(N)})u(t) &= u_+, \quad \text{strongly in } L^2(\mathbb{R}^N), \\ \lim_{t \rightarrow -\infty} \exp(itP_{a(N)})u(t) &= u_- \quad \text{strongly in } L^2(\mathbb{R}^N) \end{aligned}$$

for some $u_+, u_- \in L^2(\mathbb{R}^N)$, then $u_+ = 0 = u_-$ and hence $u_0 = 0$. Therefore **(CP)_a** with $a = a(N)$ has no free scattering state; see e.g. [7, Theorem 3.1 (2)].

REMARK 4.3. Assume either that

(A1b) $g(u) = \lambda |u|^{4/N} u$ with $\lambda > 0$;

(A2b) $g(u) = \lambda |x|^{-r} |u|^{(4-2r)/N} u$ with $\lambda > 0$ and $0 < r < 2$;

(A3b) $g(u) = \lambda (|x|^{-2} * |u|^2) u$ with $\lambda > 0$;

(A4b) $g(u) = \lambda u |x|^{-\alpha} [|x|^{-\beta} * (|x|^{-\alpha} |u|^2)]$ with $\lambda > 0$, $\alpha > 0$, $\beta > 0$ and $2\alpha + \beta = 2$.

If u is a solution to $iu' = P_{a(N)}u + g(u)$, then $v(t, x) = (\mathcal{C}u)(t, x)$ satisfies also $iv' = P_{a(N)}v + g(v)$. This problem has global unique solution v belongs to $C(\mathbb{R}; \Sigma_*^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; X^{-1}(\mathbb{R}^N))$ if $v(T_0) \in \Sigma_*^1(\mathbb{R}^N)$. Especially, we can put $T_0 = 1$. Thus we can construct wave operators $W_+ : u_+ \mapsto u(0)$ and $W_- : u_- \mapsto u(0)$ in $\Sigma_*^1(\mathbb{R}^N)$ such that

$$\begin{aligned} \lim_{t \rightarrow \infty} \exp(itP_{a(N)})u(t) &= W_+ u_+ \quad \text{strongly in } \Sigma_*^1(\mathbb{R}^N), \\ \lim_{t \rightarrow -\infty} \exp(itP_{a(N)})u(t) &= W_- u_- \quad \text{strongly in } \Sigma_*^1(\mathbb{R}^N). \end{aligned}$$

Note that $W_- \varphi = \overline{W_+ \varphi}$. Other nonlinearities seem to be applied another method. Partial constructions are available in Suzuki [16] in $\Sigma^1(\mathbb{R}^N)$ for $a > -(N - 2)^2/4$ with **(A3)** ($1 < \gamma < 2$) applying the contraction method. The case $a = a(N)$ is open.

Acknowledgements. The author would like to thank the referees for reading the manuscript carefully.

REFERENCES

- [1] N. BURQ, F. PLANCHON, J. STALKER, A. S. TAHVILDAR-ZADEH, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*, J. Funct. Anal. **203** (2003), 519–549.
- [2] T. CAZENAVE, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, 10. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, 2003.
- [3] T. CAZENAVE, F. B. WEISSLER, *Rapidly decaying solutions of the nonlinear Schrödinger equation*, Comm. Math. Phys. **147** (1992), 75–100.
- [4] J. GINIBRE, G. VELO, *On a class of nonlinear Schrödinger equations with nonlocal interaction*, Math. Z. **170** (1980), 109–136.
- [5] R. T. GLASSEY, *On the blowing up of solutions to the Cauchy problem for nonlinear Schrödinger equations*, J. Math. Phys. **18** (1977), 1794–1797.

- [6] N. HAYASHI, T. OZAWA, *Scattering theory in the weighted $L^2(\mathbb{R}^n)$ spaces for some Schrödinger equations*, Ann. Inst. Henri Poincaré **48** (1988), 17–37.
- [7] N. HAYASHI, Y. TSUTSUMI, *Scattering theory for Hartree type equations*, Ann. Inst. Henri Poincaré **46** (1987), 187–213.
- [8] E. H. LIEB, B. SIMON, *The Hartree-Fock theory for Coulomb systems*, Comm. Math. Phys. **53** (1977), 185–194.
- [9] N. OKAZAWA, T. SUZUKI, T. YOKOTA, *Cauchy problem for nonlinear Schrödinger equations with inverse-square potentials*, Appl. Anal. **91** (2012), 1605–1629.
- [10] N. OKAZAWA, T. SUZUKI, T. YOKOTA, *Energy methods for abstract nonlinear Schrödinger equations*, Evol. Equ. Control Theory, **1** (2012), 337–354.
- [11] F. PLANCHON, J. STALKER, A. S. TAHVILDAR-ZADEH, *L^p estimates for the wave equation with the inverse-square potential*, Discrete Continuous Dyn. Systems **9** (2003), 427–442.
- [12] E. M. STEIN, G. WEISS, “Introduction to Fourier Analysis on Euclidean Spaces,” Princeton University Press, Princeton, NJ, 1971.
- [13] T. SUZUKI, *Energy methods for Hartree type equation with inverse-square potentials*, Evol. Equ. Control Theory, **2** (2013), 531–542.
- [14] T. SUZUKI, *Blowup of nonlinear Schrödinger equations with inverse-square potentials*, Differ. Equ. Appl., **6** (2014), 309–333.
- [15] T. SUZUKI, *Solvability of nonlinear Schrödinger equations with some critical singular potential via generalized Hardy-Rellich inequalities*, Funkcial. Ekvac., **59** (2016), 1–34.
- [16] T. SUZUKI, *Scattering theory for Hartree equations with inverse-square potentials*, Appl. Anal., **96** (2017), 2032–2043.
- [17] J. ZHANG, J. ZHENG, *Scattering theory for nonlinear Schrödinger equations with inverse-square potential*, J. Funct. Anal. **267** (2014), 2907–2932.

(Received August 4, 2016)

Toshiyuki Suzuki
Department of Mathematics
Faculty of Engineering, Kanagawa University
Kanagawa, Japan
e-mail: t21.suzuki@gmail.com