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VIRIAL IDENTITIES FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH A CRITICAL COEFFICIENT INVERSE–SQUARE POTENTIAL

TOSHIYUKI SUZUKI

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Abstract. Virial identities for nonlinear Schrödinger equations with some strongly singular potential ($a|x|^{-2}$) are established. Here if $a = a(N) := -(N-2)^2/4$, then $P_{a(N)} := -\Delta + a(N)|x|^{-2}$ is nonnegative selfadjoint in the sense of Friedrichs extension. But the energy class $D((1 + P_{a(N)})^{1/2})$ does not coincide with $H^1(\mathbb{R}^N)$. Thus justification of the virial identities has a lot of difficulties. The identities can be applicable for showing blow-up in finite time and for proving the existence of scattering states.

1. Introduction and main results

In this article we consider the following Cauchy problems for nonlinear Schrödinger equations with inverse-square potentials

$$\begin{cases} i\dfrac{\partial u}{\partial t} = \Big(-\Delta + \dfrac{a}{|x|^2}\Big)u + g(u) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{$\mathbf{(CP)}_a$}$$

where $i = \sqrt{-1}$, $N \geqslant 3$ and

$$a \geqslant a(N) := -\frac{(N-2)^2}{4}. \tag{1.1}$$

Here (1.1) is based on the nonnegative selfadjointness of $P_a := -\Delta + a|x|^{-2}$ in $L^2(\mathbb{R}^N)$ in the sense of Friedrichs extension, which is followed by the usual *Hardy inequality*

$$\big\||x|^{-1}u\big\|_{L^2} \leqslant \frac{2}{N-2}\|\nabla u\|_{L^2} \quad \forall\, f \in H^1(\mathbb{R}^N),\, N \geqslant 3. \tag{1.2}$$

By virtue of $a > a(N)$ we see that $D((1+P_a)^{1/2})$ coincides with $H^1(\mathbb{R}^N)$. Thus a lot of studies of $\mathbf{(CP)}_a$ are available. Specifically for nonlinear problems, Okazawa–Suzuki–Yokota [9] showed the global unique existence of $\mathbf{(CP)}_a$ with power-type nonlinearities under unsatisfactory conditions of a via the contraction methods. The worse

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assumption of a is removed in Okazawa–Suzuki–Yokota [10] by applying the abstract energy methods. For Hartree type nonlinearities (in general non-local nonlinearities with non-convolution) see Suzuki [13]. Moreover, a finite time blow-up for **(CP)**$_a$ is shown in Suzuki [14]. On the other hand, the scattering problems of Hartree equations ($g(u) := u(|x|^{-\gamma} * |u|^2)$, $1 < \gamma < \max\{N, 4\}$) are considered in Suzuki [16] in the weighted energy space $\Sigma^1(\mathbb{R}^N) := H^1(\mathbb{R}^N) \cap D(|x|)$. Whereas Zhang–Zheng [17] studied the scattering problems (especially construction of the wave operators in $H^1(\mathbb{R}^N)$) of power type ($g(u) := |u|^{p-1}u$, $1 + 4/N < p < 1 + 4/(N-2)$) under the unsatisfactory condition of a. Thus we cannot apply their methods to scattering problems for the critical case $a = a(N)$.

On the other hand, we remark if $a = a(N)$, then we see that the energy class $D((1 + P_{a(N)})^{1/2})$ does not coincide with $H^1(\mathbb{R}^N)$. Now we write down the energy space $D((1 + P_{a(N)})^{1/2})$ as $X^1(\mathbb{R}^N)$. The well-posedness for **(CP)**$_a$ in $X^1(\mathbb{R}^N)$ is shown in Suzuki [15] and he analyzed $X^1(\mathbb{R}^N)$ spaces.

To observe the blow-up in finite time and the scattering problem we usually need the *virial identity*. For example, if $g(u) = \lambda|u|^{p-1}u$ ($\lambda \in \mathbb{R}$), then we can calculate formally

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\|P_a^{1/2}u(t)\|_{L^2}^2 + \frac{4N\lambda(p-1)}{p+1}\|u(t)\|_{L^{p+1}}^{p+1}.$$

If $a > a(N)$, then Suzuki [14] justified the virial identity for **(CP)**$_a$. But if $a = a(N)$, then we have never justified owing to the solvability that has not shown in $H^1(\mathbb{R}^N)$. Thus we need to try the case $a = a(N)$. Here we know $H^1(\mathbb{R}^N) \subset X^1(\mathbb{R}^N) \subset H^s(\mathbb{R}^N)$ ($0 < s < 1$). Thus the approximated argument as in Suzuki [14] can be applicable even in $a = a(N)$. Here when we prove the convergence $a \to a(N) + 0$, we need to prepare for the Mellin transform argument as in Suzuki [15] (see Lemma 3.1 and (3.7) for useful results in this article).

This paper is divided into four sections. In Section 2 we give some preliminary results. Notations are prepared in Section 2.1. Sections 2.2 is devoted to the linear operator $-\Delta + a|x|^{-2}$. The virial identities for **(CP)**$_a$ with $a = a(N)$ is justified in Section 3. We give some typical example for the virial identities in Section 3.1. In Section 4 we apply the virial identities to two problems: blow-up in finite time (Section 4.1) and existence of scattering states (Section 4.2).

2. Notations and preliminaries

2.1. Notations

To simplify the notation, we write

$$P_a := -\Delta + a|x|^{-2}, \quad a \geqslant a(N) = -(N-2)^2/4. \tag{2.1}$$

Also we use the notation $A(u) \lesssim B(u)$; an abbreviation of $A(u) \leqslant CB(u)$, where C is independent of u.

$L^p(\mathbb{R}^N)$ is the usual Lebesgue space with norm

$$\|u\|_{L^p} := \Big(\int_{\mathbb{R}^N} |u(x)|^p\,dx\Big)^{1/p}, \quad u \in L^p(\mathbb{R}^N)\ (1 \leqslant p < \infty),$$
$$\|u\|_{L^\infty} := \operatorname{ess\,sup}|u(x)|, \quad u \in L^\infty(\mathbb{R}^N).$$

Let $p \in [1,\infty]$. Then $p' \in [1,\infty]$ denotes the Hölder conjugate $p' := p/(p-1)$. $H^1(\mathbb{R}^N)$ is the usual L^2-type Sobolev space with the norm

$$\|u\|_{H^1} := (\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)^{1/2}, \quad u \in H^1(\mathbb{R}^N).$$

On the other hand, $H^{-1}(\mathbb{R}^N)$ is the dual of $H^1(\mathbb{R}^N)$. Note that we have a usual triplet

$$H^1(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N),$$

where the inclusions are continuous and dense. In particular, we have the Sobolev embeddings

$$H^1(\mathbb{R}^N) \subset L^q(\mathbb{R}^N),\ L^{q'}(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N), \quad 2 \leqslant q \leqslant \frac{2N}{N-2},\ N \geqslant 3.$$

$H^1(\mathbb{R}^N)$ coincides with the energy space $D((1+P_a)^{1/2})$ $(a > a(N))$. Here we denote the energy space $D((1+P_{a(N)})^{1/2})$ as $X^1(\mathbb{R}^N)$. $X^{-1}(\mathbb{R}^N)$ is the dual of $X^1(\mathbb{R}^N)$. As we see

$$H^1(\mathbb{R}^N) \subset X^1(\mathbb{R}^N) \subset H^s(\mathbb{R}^N) \quad (s < 1).$$

In particular, applying fractional Sobolev inequality and [15, Theorem 3.2] we see that for all $u \in X^1(\mathbb{R}^N)$ and $0 \leqslant s < 1$

$$\|u\|_{L^{2N/(N-2s)}} \leqslant C_{N,s}\,\|(-\Delta)^{s/2}u\|_{L^2} \leqslant C_{N,s}C_s\,\|P_{a(N)}^{s/2}u\|_{L^2}.$$

Note that we also obtain for all $u \in X^1(\mathbb{R}^N)$ and $0 \leqslant s < 1$

$$\|u\|_{L^{2N/(N-2s)}} \leqslant C\,\|(1+P_{a(N)})^{1/2}u\|_{L^2}^s\,\|u\|_{L^2}^{1-s}.$$

Define $D(|x|) := \{u \in L^2(\mathbb{R}^N); |x|u \in L^2(\mathbb{R}^N)\}$. Then we denote the weighted energy spaces as

$$\Sigma^1(\mathbb{R}^N) := H^1(\mathbb{R}^N) \cap D(|x|), \quad \Sigma_*^1(\mathbb{R}^N) := X^1(\mathbb{R}^N) \cap D(|x|).$$

Let $I \subset \mathbb{R}$ be an open interval and Y be a Banach space. Then $C(\overline{I};Y)$ is a family of the continuous Y-valued function on $\overline{I}$. On the other hand, the vector-valued Lebesgue space $L^p(I;Y)$ is equipped with norm

$$\|u\|_{L^p(I;Y)} := \big\|\|u(\cdot)\|_Y\big\|_{L^p(I)} < \infty.$$

Moreover the vector-valued Sobolev space $W^{1,p}(I;Y)$ is equipped with norm

$$\|u\|_{W^{1,p}(I;Y)} := \|u\|_{L^p(I;Y)} + \|u'\|_{L^p(I;Y)} < \infty.$$

Here u' denotes the weak derivative of u respect to time variable $t \in I$. Then it is well-known that $W^{1,p}(I;Y) \subset C(\overline{I};Y)$ for $p > 1$.

Strichartz estimates for $\exp(-itP_{a(N)})$ are proved in Suzuki [15, Proposition 4.8].

LEMMA 2.1. *Let $N \geqslant 3$ and (p_j, q_j) be Schrödinger admissible pairs $(j = 0,1,2)$, i.e.,*

$$\frac{2}{p_j} + \frac{N}{q_j} = \frac{N}{2}, \quad p_j > 2,\ q_j \geqslant 2.$$

Then the following inequalities hold for $\varphi \in L^2(\mathbb{R}^N)$ and $\Phi \in L^{p_1'}(\mathbb{R};L^{q_1'}(\mathbb{R}^N))$:

$$\|\exp(-itP_{a(N)})\varphi\|_{L^{p_0}(\mathbb{R};L^{q_0})} \leqslant C\|\varphi\|_{L^2}, \tag{2.2}$$

$$\left\|\int_0^t \exp(-i(t-s)P_{a(N)})\Phi(s,x)\,ds\right\|_{L^{p_2}(\mathbb{R};L^{q_2})} \leqslant C'\|\Phi\|_{L^{p_1'}(\mathbb{R};L^{q_1'})}. \tag{2.3}$$

Here the end point $(\tau,\rho) = (2, 2N/(N-2))$ is open.

2.2. Spherical harmonics decomposition

Next we consider the spherical harmonics decomposition; see [12, Chapter IV] for details. A function $Q : \mathbb{R}^N \to \mathbb{C}$ is said to be *ℓ-th solid harmonic* if Q is harmonic (i.e., $\Delta Q = 0$) and a homogeneous polynomial of degree ℓ (i.e., $Q(x) = |x|^\ell Q(x/|x|)$). Let $\mathcal{Q}_\ell$ be a family of ℓ-th solid harmonic functions. Then $\mathcal{Q}_\ell$ is a finite dimensional vector space. Moreover, $\mathcal{Q}_\ell$ has an orthogonal normal system $\{Y_{\ell,k}\}_k$:

$$\mathcal{Q}_\ell = \operatorname{Span}\{Y_{\ell,k}\}_k, \quad \int_{S^{N-1}} Y_{\ell,k_1}(x')\overline{Y_{\ell,k_2}(x')}\,dS(x') = \begin{cases} 1 & k_1 = k_2, \\ 0 & k_1 \neq k_2. \end{cases}$$

Let $Q \in \mathcal{Q}_\ell$ and $\widetilde{f} \in L^2(0,\infty)$. Then

$$f(x) := |x|^{-\ell-(N-1)/2}\widetilde{f}(|x|)Q(x) \in L^2(\mathbb{R}^N).$$

In fact, we can calculate as follows:

$$\int_{\mathbb{R}^N} |f(x)|^2\,dx = \left(\int_{|x'|=1} |Q(x')|^2\,dS\right)\left(\int_0^\infty |\widetilde{f}(r)|^2\,dr\right) < \infty. \tag{2.4}$$

Thus we define the subspace of $L^2(\mathbb{R}^N)$ as

$$L^2_{=\ell}(\mathbb{R}^N) := \left\{\sum_k |x|^{-\ell-(N-1)/2}\widetilde{f}_k(|x|)Y_{\ell,k}(x);\ \widetilde{f}_k \in L^2(0,\infty),\ Y_{\ell,k} \in \mathcal{Q}_\ell\right\}.$$

In particular, $L^2_{=0}(\mathbb{R}^N) = L^2_{\mathrm{rad}}(\mathbb{R}^N)$, the family of radially symmetric functions. Also we define

$$L^2_{\geqslant d}(\mathbb{R}^N) := \bigoplus_{\ell \geqslant d} L^2_{=\ell}(\mathbb{R}^N).$$

Now we have the following (see e.g. [12, Lemma IV.2.18]).

PROPOSITION 2.2. *Let ℓ, ℓ_1, ℓ_2 be nonnegative integers. Then one has*

(i) *$L^2_{=\ell}(\mathbb{R}^N)$ is a closed subspace of $L^2(\mathbb{R}^N)$;*

(ii) *$L^2_{=\ell_1}(\mathbb{R}^N) \perp L^2_{=\ell_2}(\mathbb{R}^N)$ if $\ell_1 \neq \ell_2$, i.e.,*

$$\int_{\mathbb{R}^N} f_1(x)\overline{f_2(x)}\, dx = 0 \quad \forall\, f_1 \in L^2_{=\ell_1}(\mathbb{R}^N),\ \forall\, f_2 \in L^2_{=\ell_2}(\mathbb{R}^N);$$

(iii) *$L^2(\mathbb{R}^N) = \bigoplus_{\ell=0}^{\infty} L^2_{=\ell}(\mathbb{R}^N)$, i.e., for every $f \in L^2(\mathbb{R}^N)$ there uniquely exists $\{f_\ell\}_\ell \subset L^2(\mathbb{R}^N)$ such that $f_\ell \in L^2_{=\ell}(\mathbb{R}^N)$ for $\ell \in \mathbb{N} \cup \{0\}$ and*

$$f = \sum_{\ell=0}^{\infty} f_\ell \quad \text{(spherical harmonics decomposition).}$$

As seen in Suzuki [15], we have

$$-\Delta f = A_{\mu(\ell)} f \quad \forall\, f \in L^2_{=\ell}(\mathbb{R}^N), \tag{2.5}$$

$$P_a f = A_{\nu(\ell)} f \quad \forall\, f \in L^2_{=\ell}(\mathbb{R}^N), \tag{2.6}$$

where $\lambda = (N-2)/2$ and

$$A_\nu \widetilde{f} := -\partial_r^2 \widetilde{f} - \frac{N-1}{r}\partial_r \widetilde{f} + \frac{\nu^2 - \lambda^2}{r^2}\widetilde{f}, \tag{2.7}$$

$$\mu(\ell) := \mu_\ell = \lambda + \ell. \tag{2.8}$$

$$\nu(\ell) := \nu_\ell = [(\lambda+\ell)^2 + a]^{1/2}. \tag{2.9}$$

Next we introduce the Mellin transform.

DEFINITION 2.1. Let f be a complex-valued measurable function such that $x^{\gamma-1} f(x) \in L^1(0,\infty;\mathbb{C})$ for some $\gamma \in \mathbb{R}$. Then *the Mellin transform* of f is defined as

$$\mathscr{M}[f(r)](z) := \int_0^\infty r^{z-1} f(r)\, dr.$$

In general, let $f \in L^2(\mathbb{R}^N)$ $(N \geqslant 2)$. Then the Mellin transform of f is defined as

$$\mathscr{M}[f(x)](z) := \int_0^\infty r^{z-1} f\Big(\frac{rx}{|x|}\Big)\, dr.$$

Note that $\mathscr{M}[r^{\alpha}f(r)](z)=\mathscr{M}[f(r)](z+\alpha)$ for $\alpha\in\mathbb{R}$. Next we see that

$$\mathscr{M}[\partial_r f](z)=(1-z)\mathscr{M}[f](z-1). \tag{2.10}$$

As in Suzuki [15, (2.20)] (see also [11]) we have for $\nu\geqslant 0$ and $\sigma\geqslant 0$

$$\mathscr{M}[A_{\nu}^{\sigma/2}f](z)=2^{\sigma}\frac{\Gamma\big((z-\lambda+\nu)/2\big)\Gamma\big(1-(z-\sigma-\lambda-\nu)/2\big)}{\Gamma\big((z-\sigma-\lambda+\nu)/2\big)\Gamma\big(1-(z-\lambda-\nu)/2\big)}\mathscr{M}[f](z-\sigma), \tag{2.11}$$

where $\lambda=(N-2)/2$.

Here we have the Plancherel type equality

$$\int_0^{\infty}f(s)\overline{g(s)}s^{N-1}ds=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathscr{M}[f]\Big(\frac{N}{2}+iy\Big)\overline{\mathscr{M}[g]\Big(\frac{N}{2}+iy\Big)}\,dy \tag{2.12}$$

(see Suzuki [15, Lemma 2.5]).

3. Proof of the virial identities

First we consider the key inequalities.

LEMMA 3.1. *Let $a\geqslant a(N)$. Assume that φ is real-valued and radially symmetric. Then*

$$\Big|\mathrm{Im}\int_{\mathbb{R}^N}\overline{x\varphi u}\cdot\nabla u\,dx\Big|\leqslant\|(1+P_a)^{1/2}u\|_{L^2}\,\|x\varphi u\|_{L^2} \tag{3.1}$$

Proof. Let $u=\sum_{\ell,k}u_{\ell,k}(r)Y_{\ell,k}$ be a spherical harmonics decomposition. Then we see that

$$\int_{\mathbb{R}^N}\overline{\varphi u}x\cdot\nabla u\,dx=\sum_{\ell,k}\int_0^{\infty}\overline{\varphi u_{\ell,k}(r)}r(\partial_r u_{\ell,k})(r)\,r^{N-1}\,dr. \tag{3.2}$$

By using (2.12) and (2.10) we obtain that

$$\begin{aligned}
&\int_0^{\infty}\overline{r\varphi u_{\ell,k}(r)}(\partial_r u_{\ell,k})(r)\,r^{N-1}\,dr\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathscr{M}[\partial_r u_{\ell,k}]\Big(\frac{N}{2}+iy\Big)\overline{\mathscr{M}[r\varphi u_{\ell,k}]\Big(\frac{N}{2}+iy\Big)}\,dy\\
&=\frac{N}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathscr{M}[u_{\ell,k}]\Big(\frac{N}{2}-1+iy\Big)\overline{\mathscr{M}[\varphi u_{\ell,k}]\Big(\frac{N}{2}+1+iy\Big)}\,dy\\
&+\frac{i}{2\pi}\int_{-\infty}^{\infty}y\mathscr{M}[u_{\ell,k}]\Big(\frac{N}{2}-1+iy\Big)\overline{\mathscr{M}[\varphi u_{\ell,k}]\Big(\frac{N}{2}+1+iy\Big)}\,dy\\
&=:\frac{N}{2}I_1+I_2.
\end{aligned} \tag{3.3}$$

Here I_1 is calculated

$$I_1=\int_0^{\infty}r^{-1}u_{\ell,k}\overline{r\varphi u_{\ell,k}}\,r^{N-1}\,dr=\int_0^{\infty}\varphi(r)|u_{\ell,k}(r)|^2\,r^{N-1}\,dr\in\mathbb{R}.$$

Thus we have

$$\begin{aligned}
&\operatorname{Im}\int_0^\infty \overline{r\varphi u_{\ell,k}(r)}(\partial_r u_{\ell,k})(r)\,r^{N-1}\,dr = \operatorname{Im} I_2 \\
&= \frac{1}{2\pi}\operatorname{Re}\int_{-\infty}^\infty y\mathscr{M}[u_{\ell,k}]\Big(\frac{N}{2}-1+iy\Big)\overline{\mathscr{M}[\varphi u_{\ell,k}]\Big(\frac{N}{2}+1+iy\Big)}\,dy. \qquad (3.4)
\end{aligned}$$

On the other hand, we see from (2.11) that

$$\begin{aligned}
&\mathscr{M}[A_\nu^{1/2}f]\Big(\frac{N}{2}+iy\Big) \\
&= (\nu-iy)\frac{\Gamma\big((\nu-iy)/2\big)\Gamma\big((\nu+1+iy)/2\big)}{\Gamma\big((\nu+iy)/2\big)\Gamma\big((\nu+1-iy)/2\big)}\mathscr{M}[f]\Big(\frac{N}{2}-1+iy\Big).
\end{aligned}$$

Using $\Gamma(\bar{z}) = \overline{\Gamma(z)}$ we obtain

$$\int_0^\infty |A_\nu^{1/2}f|^2\,r^{N-1}\,dr = \frac{1}{2\pi}\int_{-\infty}^\infty (\nu^2+y^2)\Big|\mathscr{M}[f]\Big(\frac{N}{2}-1+iy\Big)\Big|^2\,dy. \qquad (3.5)$$

Applying (3.5) and (2.12) we calculate

$$\begin{aligned}
&\frac{1}{2\pi}\operatorname{Re}\int_{-\infty}^\infty y\mathscr{M}[u_{\ell,k}]\Big(\frac{N}{2}-1+iy\Big)\overline{\mathscr{M}[\varphi u_{\ell,k}]\Big(\frac{N}{2}+1+iy\Big)}\,dy \\
&\leqslant \frac{1}{2\pi}\Big[\int_{-\infty}^\infty\Big|y\mathscr{M}[u_{\ell,k}]\Big(\frac{N}{2}-1+iy\Big)\Big|^2dy\Big]^{1/2}\Big[\int_{-\infty}^\infty\Big|\mathscr{M}[\varphi u_{\ell,k}]\Big(\frac{N}{2}+1+iy\Big)\Big|^2dy\Big]^{1/2} \\
&= \Big[\int_0^\infty |A_{\nu(\ell)}^{1/2}f|^2\,r^{N-1}\,dr\Big]^{1/2}\Big[\int_0^\infty |r\varphi u_{\ell,k}|^2\,r^{N-1}\,dr\Big]^{1/2}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\Big|\operatorname{Im}\int_0^\infty \overline{r\varphi u_{\ell,k}(r)}(\partial_r u_{\ell,k})(r)\,r^{N-1}\,dr\Big| \\
&\leqslant \Big[\int_0^\infty |A_{\nu(\ell)}^{1/2}f|^2\,r^{N-1}\,dr\Big]^{1/2}\Big[\int_0^\infty |r\varphi u_{\ell,k}|^2\,r^{N-1}\,dr\Big]^{1/2} \\
&= \|P_a^{1/2}(u_{\ell,k}Y_{\ell,k})\|_{L^2}\,\|x\varphi u_{\ell,k}Y_{\ell,k}\|_{L^2}. \qquad (3.6)
\end{aligned}$$

Summing (3.6) over k and ℓ we conclude (3.1) from (3.2). □

REMARK 3.1. Let $\varphi(x) \equiv 1$. In a way similar to Lemma 3.1 we can conclude that for all $u, v \in \Sigma_*^1(\mathbb{R}^N)$

$$\Big|\int_{\mathbb{R}^N} \overline{xu}\cdot\nabla v\,dx\Big| \leqslant \|xu\|_{L^2}\,\|P_{a(N)}^{1/2}v\|_{L^2} + \frac{N}{2}\|u\|_{L^2}\,\|v\|_{L^2}. \qquad (3.7)$$

Now we show the virial identities for **(CP)**$_a$ with $a = a(N)$. To end this we consider approximated problems for **(CP)**$_a$:

$$\begin{cases} i\dfrac{\partial u_{\varepsilon,a}}{\partial t} = \Big(-\Delta+\dfrac{a}{|x|^2}\Big)u_{\varepsilon,a} + g_\varepsilon(u_{\varepsilon,a}) & \text{in } \mathbb{R}\times\mathbb{R}^N, \\ u_{\varepsilon,a}(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \qquad \textbf{(CP)}_{\varepsilon,a}$$

Here g_ε is approximation of g. g and g_ε satisfy weak closedness (see [10, **(G5)**]):

$$\begin{aligned}
&v_m(t) \to v(t)\ (m \to \infty) \text{ weakly in } L^\infty(-T,T;X_S),\\
&g(v_m(t)) \to f(t)\ (m \to \infty) \text{ weakly}^* \text{ in } L^\infty(-T,T;X_S^*)\\
&\Rightarrow\ 0 = \lim_{m\to\infty} \operatorname{Im} \int_{\mathbb{R}^N} g(v_m(t))\overline{v_m(t)}\,dx = \operatorname{Im} \int_{\mathbb{R}^N} f(t)\overline{v(t)}\,dx,
\end{aligned}$$

where $X_S := H^1(\mathbb{R}^N)$ $(a > a(N))$ or $X^1(\mathbb{R}^N)$ $(a = a(N))$. Assume further that $v_m(t) \to v(t)$ in $C([-T,T];L^2(\mathbb{R}^N))$. Then $f(t) = g(v(t))$.

We give three types of nonlinearities g and their approximations as we can consider in this article.

EXAMPLE 3.1. Let $g : \mathbb{C} \to \mathbb{C}$ be power type nonlinearities so that

(N1) $g(0) = 0$ and there exist $p \in [1,(N+2)/(N-2))$ and $K \geqslant 0$ such that

$$|g(z_1) - g(z_2)| \leqslant K(1 + |z_1|^{p-1} + |z_2|^{p-1})|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{C};$$

(N2) $g(x) \in \mathbb{R}$ $(x > 0)$ and $g(e^{i\theta}z) = e^{i\theta}g(z)$ $(z \in \mathbb{C},\ \theta \in \mathbb{R})$.

In such a case we define $g_\varepsilon(u) := \rho_\varepsilon * [g(\rho_\varepsilon * u)]$, where ρ_ε is the Friedrichs mollifier. Moreover, the energy functionals of g and g_ε are

$$G(u) := \int_{\mathbb{R}^N} F(|u(x)|)\,dx, \quad G_\varepsilon(u) := \int_{\mathbb{R}^N} F(|\rho_\varepsilon * u(x)|)\,dx,$$

where F is the primitive integral of g:

$$F(x) := \int_0^x g(s)\,ds \quad \forall\, x > 0.$$

EXAMPLE 3.2. Let

$$g(u) := \lambda\, |x|^{-r}|u|^{p-1}u \quad (\lambda \in \mathbb{R},\ 0 < r < 2,\ 1 \leqslant p < (N+2-2r)/(N-2)).$$

In such a case we define

$$g_\varepsilon(u) := \lambda\, \rho_\varepsilon * [(|x|^2 + \varepsilon)^{-r/2}|\rho_\varepsilon * u|^{p-1}(\rho_\varepsilon * u)].$$

Moreover, the energy functionals of g and g_ε are

$$G(u) := \lambda \int_{\mathbb{R}^N} \frac{|u(x)|^{p+1}}{|x|^r}\,dx, \quad G_\varepsilon(u) := \lambda \int_{\mathbb{R}^N} \frac{|\rho_\varepsilon * u(x)|^{p+1}}{(|x|^2 + \varepsilon)^{r/2}}\,dx.$$

EXAMPLE 3.3. Let $g(u) := u\,K[k](|u|^2)$, where

$$K[k](f) := \int_{\mathbb{R}^N} k(x,y) f(y)\,dy.$$

Here k satisfies three conditions:

(K1) k is a symmetric real-valued function, that is, $k(x,y) = k(y,x) \in \mathbb{R}$ a.a. $x, y \in \mathbb{R}^N$;

(K2) $k \in L_x^\infty(L_y^\infty) + L_x^\beta(L_y^\alpha)$ for some $\alpha, \beta \in [1,\infty]$ such that $\alpha \leqslant \beta$ and $\alpha^{-1} + \beta^{-1} < 4/N$;

(K3) $\widetilde{k}(x,y) := x \cdot \nabla_x k(x,y) + y \cdot \nabla_y k(x,y)$ belongs to $L_x^\infty(L_y^\infty) + L_x^{\tilde{\beta}}(L_y^{\tilde{\alpha}})$ for some $\tilde{\alpha}, \tilde{\beta} \in [1,\infty]$ such that $\tilde{\alpha} \leqslant \tilde{\beta}$ and $\tilde{\alpha}^{-1} + \tilde{\beta}^{-1} < 4/N$.

Note that $L_x^\beta(L_y^\alpha)$ is the family of $k(x,y)$ such that $k(x,\cdot) \in L^\alpha(\mathbb{R}^N)$ a.a. $x \in \mathbb{R}^N$ with $\big\| \|k(x,\cdot)\|_{L^\alpha} \big\|_{L^\beta} < \infty$. In such a case we define $g_\varepsilon(u) := uK[k_\varepsilon](|u|^2)$, where

$$k_\varepsilon(x,y) := \iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho_\varepsilon(x-\xi)\rho_\varepsilon(y-\eta)k(\xi,\eta)\, d\xi\, d\eta.$$

Moreover, the energy functionals of g and g_ε are

$$G(u) := \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} k(x,y)\, |u(x)|^2\, |u(y)|^2\, dx\,dy,$$
$$G_\varepsilon(u) := \frac{1}{4} \iint_{\mathbb{R}^N \times \mathbb{R}^N} k_\varepsilon(x,y)\, |u(x)|^2\, |u(y)|^2\, dx\,dy.$$

REMARK 3.2. Condition **(K2)** implies that we can divide k into $k_R + (k - k_R)$, where

$$k_R(x,y) := \begin{cases} k(x,y) & |k(x,y)| \leqslant R, \\ R & k(x,y) > R, \\ -R & k(x,y) < -R. \end{cases}$$

Here $k_R \in L_x^\infty(L_y^\infty)$ with $\|k_R\|_{L_x^\infty(L_y^\infty)} \leqslant R$ and $\|k - k_R\|_{L_x^\beta(L_y^\alpha)} \to 0$ ($R \to \infty$). Moreover, define

$$\gamma := \left[1 - \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\right]^{-1} \in \left[1, \frac{N}{N-2}\right). \tag{3.8}$$

Then we have (see [13, Lemma 2.4]) if $k \in L_x^\beta(L_y^\alpha)$,

$$\|K[k]f\|_{L^{\gamma'}} \leqslant \|k\|_{L_x^\beta(L_y^\alpha)} \|f\|_{L^\gamma} \quad \forall\, f \in L^\gamma(\mathbb{R}^N). \tag{3.9}$$

By virtue of (3.9), **(K1)** and **(K2)** imply

$$|G(u) - G(v)|, |G_\varepsilon(u) - G_\varepsilon(v)| \leqslant CM^4 \|k - k_R\|_{L_x^\beta(L_y^\alpha)} + RM^3 \|u - v\|_{L^2} \tag{3.10}$$

for every $u, v \in X^1(\mathbb{R}^N)$ with $\|u\|_{X^1} \leqslant M$, $\|v\|_{X^1} \leqslant M$ (see [14, (2.13)]) and

$$G_\varepsilon(\varphi) \to G(\varphi) \quad (\varepsilon \to 0), \quad \varphi \in X^1(\mathbb{R}^N)$$

(see [14, Remark 2.2]).

Here we give a plan to prove the virial identities for **(CP)**$_a$ with $a = a(N)$.

Stage 1. We construct the virial identities for **(CP)**$_{\varepsilon,a}$ with $a > a(N)$. This step has already finished: see Suzuki [14].

Stage 2. To show $u_{\varepsilon,a} \to u_\varepsilon$ $(a \to a(N)+0)$ we need uniform boundedness in $X^1(\mathbb{R}^N)$. Here u_ε is a solution to **(CP)**$_{\varepsilon,a}$ with $a = a(N)$. u_ε is also satisfies a certain virial identity.

Stage 3. Next we let $\varepsilon \to +0$. Applying the Strichartz estimates (2.2) and (2.3) we can show $u_\varepsilon \to u$. Since u satisfies **(CP)**$_a$ with $a = a(N)$, we can prove the desired virial identity.

Before deriving the virial identity for **(CP)**$_a$ with $a = a(N)$ we calculate the first derivative of $\|xu(t)\|_{L^2}^2$.

LEMMA 3.2. *Let $u \in C([-T_1,T_2];X^1(\mathbb{R}^N))$ be a solutions to* **(CP)**$_a$ *with $a = a(N)$ and $u_0 \in \Sigma_*^1(\mathbb{R}^N)$. Then u belongs to $C([-T_1,T_2];\Sigma_*^1(\mathbb{R}^N))$ and satisfies*

$$\frac{d}{dt}\|xu(t)\|_{L^2}^2 = 4\,\mathrm{Im}\int_{\mathbb{R}^N} \nabla u(t)\cdot \overline{xu(t)}\,dx. \tag{3.11}$$

Proof. Using assumption of g we can calculate

$$\frac{d}{dt}\left\|\frac{xu(t)}{\sqrt{1+\delta|x|^2}}\right\|_{L^2}^2 = 4\,\mathrm{Im}\int_{\mathbb{R}^N} \nabla u(t)\cdot \overline{\left[\frac{xu(t)}{(1+\delta|x|^2)^2}\right]}\,dx.$$

Lemma 3.1 with $\varphi(x) = (1+\delta|x|^2)^{-2}$ implies that

$$\left|\frac{d}{dt}\left\|\frac{xu(t)}{\sqrt{1+\delta|x|^2}}\right\|_{L^2}^2\right| \leqslant 4\,\|(1+P_{a(N)})^{1/2}u(t)\|_{L^2}\left\|\frac{xu(t)}{(1+\delta|x|^2)^2}\right\|_{L^2}.$$

Thus we see that

$$\left\|\frac{xu(t)}{\sqrt{1+\delta|x|^2}}\right\|_{L^2} \leqslant \left\|\frac{xu_0}{\sqrt{1+\delta|x|^2}}\right\|_{L^2} + 2\left|\int_0^t \|u(s)\|_{X^1}\,ds\right|.$$

Letting $\delta \to 0$ we conclude that

$$\|xu(t)\|_{L^2} \leqslant \|xu_0\|_{L^2} + 2\left|\int_0^t \|u(s)\|_{X^1}\,ds\right|.$$

In a way similar to the above we also obtain

$$\big|\|xu(T_2)\|_{L^2} - \|xu(T_1)\|_{L^2}\big| \leqslant 2\left|\int_{T_2}^{T_1} \|u(s)\|_{X^1}\,ds\right|;$$

hence u is continuous in $\Sigma_*^1(\mathbb{R}^N)$ and satisfies (3.11). □

Henceforth we prove the virial identities for **(CP)**$_a$ with $a = a(N)$ only the case of nonlinearities as in Example 3.3. Other cases are similar ways.

Summary of Stage 1. Let $u_0 \in H^1(\mathbb{R}^N) \subset X^1(\mathbb{R}^N)$ and $a > a(N)$. Then global weak solution $u_{\varepsilon,a} \in C(\mathbb{R}; H^1(\mathbb{R}^N)) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R}^N))$ to **(CP)**$_{\varepsilon,a}$ with $u_{\varepsilon,a}(0) = u_0$ exists uniquely. Also $u_{\varepsilon,a}$ satisfies the conservation laws

$$\|u_{\varepsilon,a}(t)\|_{L^2} = \|u_0\|_{L^2}, \quad E_{\varepsilon,a}(u_\varepsilon(t)) = E_{\varepsilon,a}(u_0) \quad \forall\, t \in \mathbb{R},$$

where

$$E_{\varepsilon,a}(\varphi) := \frac{1}{2}\|P_a^{1/2}\varphi\|_{L^2}^2 + G_\varepsilon(\varphi), \quad \varphi \in H^1(\mathbb{R}^N).$$

Here the virial identities for **(CP)**$_{\varepsilon,a}$ $(a > a(N))$ is verified in Suzuki [14, Section 3]

$$\begin{aligned}\frac{d^2}{dt^2}\|xu_{\varepsilon,a}(t)\|_{L^2}^2 &= 8\,\|P_a^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 \\ &\qquad - 2\iint_{\mathbb{R}^N\times\mathbb{R}^N} \widetilde{k_\varepsilon}(x,y)|u_{\varepsilon,a}(t,x)|^2\,|u_{\varepsilon,a}(t,y)|^2\,dxdy \quad \forall\, t \in \mathbb{R},\end{aligned}$$

where $\widetilde{k_\varepsilon} \in L_x^\infty(L_y^\infty)$ and

$$\begin{aligned}\widetilde{k_\varepsilon}(x,y) &:= \iint_{\mathbb{R}^N\times\mathbb{R}^N} \rho_\varepsilon(x-\xi)\rho_\varepsilon(y-\eta)\widetilde{k}(\xi,\eta)\,d\xi d\eta \\ &\qquad + \iint_{\mathbb{R}^N\times\mathbb{R}^N} [\widetilde{\rho}_\varepsilon(x-\xi)\rho_\varepsilon(y-\eta) + \rho_\varepsilon(x-\xi)\widetilde{\rho}_\varepsilon(y-\eta)]k(\xi,\eta)\,d\xi d\eta, \\ \widetilde{\rho}_\varepsilon(x) &:= N\rho_\varepsilon(x) + x\cdot\nabla\rho_\varepsilon(x).\end{aligned}$$

Proof of Stage 2. First we show there exists u_ε such that $u_{\varepsilon,a} \to u_\varepsilon$ $(a \to a(N)+0)$ weakly in some sense. Applying the conservation laws we see that

$$\|P_a^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 = \|P_a^{1/2}u_0\|_{L^2}^2 + 2G_\varepsilon(u_0) - 2G_\varepsilon(u_{\varepsilon,a}(t))$$

Since G_ε is continuous in $L^2(\mathbb{R}^N)$, there exists a non-decreasing function $d_\varepsilon : (0,\infty) \to (0,\infty)$ such that $|G_\varepsilon(\varphi)| \leqslant d_\varepsilon(\|\varphi\|_{L^2})$; in this case (Example 3.3) we have $d_\varepsilon(\|\varphi\|_{L^2}) := (1/4)\|k_\varepsilon\|_{L_x^\infty(L_y^\infty)}\|\varphi\|_{L^2}^4$. Thus we conclude that

$$\|P_a^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 \leqslant \|P_a^{1/2}u_0\|_{L^2}^2 + 4d_\varepsilon(\|u_0\|_{L^2}) \quad \forall\, t \in \mathbb{R}.$$

On the other hand,

$$\|P_{a(N)}^{1/2}\varphi\|_{L^2} \leqslant \|P_a^{1/2}\varphi\|_{L^2} \quad \forall\, \varphi \in H^1(\mathbb{R}^N).$$

Combining these we obtain

$$\|P_{a(N)}^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 \leqslant \|P_a^{1/2}u_0\|_{L^2}^2 + 4d_\varepsilon(\|u_0\|_{L^2}) \quad \forall\, t \in \mathbb{R}. \tag{3.12}$$

On the other hand, we can see that

$$\|P_a v\|_{H^{-1}} \leqslant \left(1+\frac{4|a|}{(N-2)^2}\right)^{1/2} \|(1+P_a)^{1/2}v\|_{L^2}.$$

Putting $v := u_{\varepsilon,a}(t)$ we obtain

$$\begin{aligned}\|P_a u_{\varepsilon,a}(t)\|_{H^{-1}} &\leqslant \left(1+\frac{4|a|}{(N-2)^2}\right)^{1/2} \|(1+P_a)^{1/2}u_{\varepsilon,a}(t)\|_{L^2} \\ &\leqslant \left(1+\frac{4|a|}{(N-2)^2}\right)^{1/2} \left[\|(1+P_a)^{1/2}u_0\|_{L^2}^2 + 4d_\varepsilon(\|u_0\|_{L^2})\right]^{1/2}.\end{aligned}$$

Also we can calculate

$$\|g_\varepsilon(u_{\varepsilon,a}(t))\|_{L^2} \leqslant \widetilde{d}_\varepsilon(\|u_{\varepsilon,a}(t)\|_{L^2})\|u_{\varepsilon,a}(t)\|_{L^2} = \widetilde{d}_\varepsilon(\|u_0\|_{L^2})\|u_0\|_{L^2},$$

where $\widetilde{d}_\varepsilon(\|\varphi\|_{L^2}) := \|k_\varepsilon\|_{L^\infty_x(L^\infty_y)}\|\varphi\|_{L^2}^3$. Thus we obtain for all $t \in \mathbb{R}$

$$\|u'_{\varepsilon,a}(t)\|_{H^{-1}} \leqslant [\|(1+P_a)^{1/2}u_0\|_{L^2} + 4d_\varepsilon(\|u_0\|_{L^2})]^{1/2} + \widetilde{d}_\varepsilon(\|u_0\|_{L^2})\|u_0\|_{L^2}. \tag{3.13}$$

Since $X^1(\mathbb{R}^N) \subset H^{-1}(\mathbb{R}^N)$ is continuous, applying the Ascoli-Arzela type lemma (see [2, Proposition 1.1.2]) we conclude that for any $T > 0$ there exist $\{a_j\}_j \subset (a(N),0)$ and $u_\varepsilon(t)$ such that $a_j \to a(N)$ $(j \to \infty)$ and

$$u_{\varepsilon,a_j}(t) \to u_\varepsilon(t)\ (j\to\infty) \quad \forall\, t \in (-T,T) \quad \text{weakly in } X^1(\mathbb{R}^N).$$

Next we show that u_ε satisfies **(CP)**$_a$ with $a = a(N)$ and $g = g_\varepsilon$. Since $g_\varepsilon(u_{\varepsilon,a_j}(t))$ is uniformly bounded in $L^\infty(-T,T;L^2(\mathbb{R}^N))$, we see that $g_\varepsilon(u_{\varepsilon,a_j}(t)) \to f_\varepsilon(t)$ weakly* in $L^\infty(-T,T;L^2(\mathbb{R}^N))$; also in $L^\infty(-T,T;X^{-1}(\mathbb{R}^N))$. By virtue of the weak closedness of g_ε (see **(G5)** in [13, Lemma 3.1]) we have

$$\operatorname{Im}\int_{\mathbb{R}^N} f_\varepsilon(t)\overline{u_\varepsilon(t)}\,dx = 0.$$

Thus we obtain

$$\|u_\varepsilon(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2 = \|u_{\varepsilon,a}(t)\|_{L^2}^2 \quad \forall\, t \in (-T,T),$$

Weak convergence and the convergence of the corresponding norms imply that

$$u_{\varepsilon,a_j}(t) \to u_\varepsilon(t)\ (j\to\infty) \quad \forall\, t \in (-T,T) \quad \text{strongly in } L^2(\mathbb{R}^N).$$

Hence we conclude from the weak closedness of g_ε (in a way similar to the verification of **(G5)** of [13, Lemma 3.1]) that u_ε satisfies $iu'_\varepsilon = P_{a(N)}u_\varepsilon + g_\varepsilon(u_\varepsilon)$ and

$$u_{\varepsilon,a_j}(t) \to u_\varepsilon(t)\ (j\to\infty) \quad \text{in } C([-T,T];L^2(\mathbb{R}^N)).$$

Next we prove

$$u_{\varepsilon,a_j}(t) \to u_\varepsilon(t)\ (j\to\infty) \quad \text{strongly in } X^1(\mathbb{R}^N). \tag{3.14}$$

The conservation of laws imply that

$$\begin{aligned}
&\|(1+P_a)^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 \\
&= 2E_{\varepsilon,a}(u_{\varepsilon,a}(t)) - 2G_\varepsilon(u_{\varepsilon,a}(t)) \\
&= 2E_{\varepsilon,a}(u_0) - 2G_\varepsilon(u_{\varepsilon,a}(t)) \\
&\to 2E_\varepsilon(u_0) - 2G_\varepsilon(u_\varepsilon(t)) \quad (a \to a(N)+0) \\
&= 2E_\varepsilon(u_\varepsilon(t)) - 2G_\varepsilon(u_\varepsilon(t)) = \|(1+P_{a(N)})^{1/2}u_\varepsilon(t)\|_{L^2}^2.
\end{aligned}$$

Moreover, we see that

$$\begin{aligned}
\limsup_{a\to a(N)+0} \|(1+P_{a(N)})^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 &\leqslant \limsup_{a\to a(N)+0} \|(1+P_a)^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 \\
&= \|(1+P_{a(N)})^{1/2}u_\varepsilon(t)\|_{L^2}^2.
\end{aligned}$$

Thus the weak convergence implies (3.14).

Since the strong convergence in $X^1(\mathbb{R}^N)$ and $\|u_{\varepsilon,a_j}(t)\|_{X^1}$ is uniformly bounded in $t \in [-T,T]$, we see from the dominated convergence theorem implies that if $u_0 \in \Sigma^1(\mathbb{R}^N)$, then

$$\|xu_\varepsilon(t)\|_{L^2}^2 = \|xu_0\|_{L^2}^2 + 4t\,\mathrm{Im}\int_{\mathbb{R}^N} \overline{xu_0}\cdot\nabla u_0\,dx + \int_0^t (t-s)V_\varepsilon(u_\varepsilon(s))\,ds,$$

where

$$V_\varepsilon(v) := 8\,\|P_{a(N)}^{1/2}v\|_{L^2}^2 - 2\iint_{\mathbb{R}^N\times\mathbb{R}^N} \widetilde{k_\varepsilon}(x,y)\,|v(x)|^2\,|v(y)|^2\,dxdy.$$

Thus we obtain

$$\frac{d^2}{dt^2}\|xu_\varepsilon(t)\|_{L^2}^2 = V_\varepsilon(u_\varepsilon(t)). \tag{3.15}$$

REMARK 3.3. Let $u_0 \in X^1(\mathbb{R}^N)$ in general. We can apply the continuous dependence of initial values. First set $\{u_{0m}\}_m \subset \Sigma^1(\mathbb{R}^N)$ such that $u_{0m} \to u_0$ strongly in $\Sigma^1_*(\mathbb{R}^N)$. Next let $u_{\varepsilon,m}$ be a unique solution to **(CP)**$_{\varepsilon,a}$ with $u_{\varepsilon,m}(0) = u_{0m}$. Then $u_{\varepsilon,m}$ satisfies

$$\begin{aligned}
&\|xu_{\varepsilon,m}(t)\|_{L^2}^2 \\
&= \|xu_{0m}\|_{L^2}^2 + 4t\,\mathrm{Im}\int_{\mathbb{R}^N} \overline{xu_{0m}}\cdot\nabla u_{0m}\,dx + \int_0^t (t-s)V_\varepsilon(u_{\varepsilon,m}(s))\,ds.
\end{aligned}$$

Since the continuous dependence and the dominated convergence theorem imply that

$$\int_0^t (t-s)V_\varepsilon(u_{\varepsilon,m}(s))\,ds \to \int_0^t (t-s)V_\varepsilon(u_\varepsilon(s))\,ds$$

uniformly in $t \in [-T,T]$. (3.7) yields that

$$\mathrm{Im}\int_{\mathbb{R}^N} \overline{xu_{0m}}\cdot\nabla u_{0m}\,dx \to \mathrm{Im}\int_{\mathbb{R}^N} \overline{xu_0}\cdot\nabla u_0\,dx.$$

Hence we conclude the virial identity (3.15) even when $u_0 \in X^1(\mathbb{R}^N)$.

Proof of Stage 3. First we show the uniform convergence of u_ε $(\varepsilon \to +0)$; see also [14, Lemma 3.2].

PROPOSITION 3.3. *Let* $u \in C([-T_1,T_2];X^1(\mathbb{R}^N))$ *be a local weak solution to* **(CP)**$_a$ *in* $(-T_1,T_2)$. *Then* $u_\varepsilon \to u$ $(\varepsilon \to +0)$ *strongly in* $C([-T_1,T_2];X^1(\mathbb{R}^N))$.

Proof. **Step 1**. First we show the uniform boundedness of u_ε:

$$\|u_\varepsilon(t)\|_{X^1} \leqslant M_0 \quad \forall\, \varepsilon > 0, \forall\, t \in [-T,T]. \tag{3.16}$$

Let $M > \|u_0\|_{X^1}$. Define

$$\tau_\varepsilon := \sup_{T>0}\{\|u_\varepsilon(t)\|_{X^1} \leqslant M,\ t \in [-T,T]\}.$$

If $\tau_\varepsilon = \infty$, then we have proved the uniform boundedness. Thus we assume $\tau_\varepsilon < \infty$. Since $u_\varepsilon \in C(\mathbb{R};X^1(\mathbb{R}^N))$, τ_ε satisfies

$$\|u_\varepsilon(\tau_\varepsilon)\|_{X^1} = M \text{ or } \|u_\varepsilon(-\tau_\varepsilon)\|_{X^1} = M. \tag{3.17}$$

By using the conservation laws we have

$$\begin{aligned}\|u_\varepsilon(t)\|_{X^1}^2 - \|u_0\|_{X^1}^2 &= 2\,[G_\varepsilon(u_0) - G_\varepsilon(u_\varepsilon(t))] \\ &\leqslant 2CM^4\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)} + 2RM^3\,\|u_0 - u_\varepsilon(t)\|_{L^2}.\end{aligned} \tag{3.18}$$

On the other hand, we see from the verification of **(G2)** in [13, Lemma 3.1] (or [13, Lemma 2.5]) that

$$\begin{aligned}\|u_\varepsilon'(t)\|_{X^{-1}} &\leqslant \|P_a u_\varepsilon(t)\|_{X^{-1}} + \|u_\varepsilon(t)\,K[k_\varepsilon](|u_\varepsilon(t)|^2)\|_{X^{-1}} \\ &\leqslant \|u_\varepsilon(t)\|_{X^1} + R_0\,\|u_\varepsilon(t)\|_{L^2}^3 + C\,\|k-k_{R_0}\|_{L_x^\beta(L_y^\alpha)}\|u_\varepsilon(t)\|_{X^1}^3 \\ &\leqslant M + R_0\,\|u_0\|_{L^2}^3 + C\,\|k-k_{R_0}\|_{L_x^\beta(L_y^\alpha)}M^3 =: C(M) \quad \forall\, t \in [-\tau_\varepsilon,\tau_\varepsilon].\end{aligned}$$

Applying [2, Lemma 3.3.6] we obtain

$$\|u_\varepsilon(t) - u_\varepsilon(s)\|_{L^2} \leqslant \sqrt{2}C(M)\,|t-s|^{1/2}, \quad t,s \in [-\tau_\varepsilon,\tau_\varepsilon]. \tag{3.19}$$

Combining (3.19) with setting $s = 0$ into (3.18), we see that

$$\|u_\varepsilon(t)\|_{X^1}^2 - \|u_0\|_{X^1}^2 \leqslant 2CM^4\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)} + 2\sqrt{2}RM^3C(M)|t|^{1/2}.$$

Letting $t = \pm\tau_\varepsilon$ and applying (3.17) we have

$$\tau_\varepsilon^{1/2} \geqslant \frac{3M^2 - 8CM^4\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)}}{8\sqrt{2}RM^3C(M)} > 0;$$

note that $\|k-k_R\|_{L_x^\beta(L_y^\alpha)} \to 0$ $(R\to\infty)$ implies the positivity. Thus we obtain (3.16) by putting

$$T_M := \Big[\frac{3-8CM^2\|k-k_R\|_{L_x^\beta(L_y^\alpha)}}{8\sqrt{2}RMC(M)}\Big]^2 > 0.$$

Step 2. Next we show that $u_\varepsilon \to u$ $(\varepsilon\to 0)$ strongly in $L^\infty(-T_1,T_2;L^2(\mathbb{R}^N))$ and in $L^{r(\gamma)}(-T_1,T_2;L^{2\gamma}(\mathbb{R}^N))$, where γ is defined in (3.8) and $r(\gamma) := 4\gamma/[N(\gamma-1)]$. Note that u and u_ε satisfy the following integral equations:

$$u(t) = \exp(-itP_{a(N)})u_0 - i\int_0^t \exp(-i(t-s)P_{a(N)})\{u(s)\,K[k](|u(s)|^2)\}\,ds,$$
$$u_\varepsilon(t) = \exp(-itP_{a(N)})u_0 - i\int_0^t \exp(-i(t-s)P_{a(N)})\{u_\varepsilon(s)\,K[k_\varepsilon](|u_\varepsilon(s)|^2)\}\,ds.$$

We divide $u(t)-u_\varepsilon(t)$ into $J_1(t;\varepsilon)+J_2(t;\varepsilon)+J_3(t;\varepsilon)$, where

$$J_1(t;\varepsilon) := -i\int_0^t \exp(-i(t-s)P_{a(N)})\{u(s)\big(K[k](|u(s)|^2) - K[k_\varepsilon](|u(s)|^2)\big)\}\,ds,$$
$$J_2(t;\varepsilon) := -i\int_0^t \exp(-i(t-s)P_{a(N)})\{(u(s)-u_\varepsilon(s))K[k_\varepsilon](|u(s)|^2)\}\,ds,$$
$$J_3(t;\varepsilon) := -i\int_0^t \exp(-i(t-s)P_{a(N)})\{u_\varepsilon(s)\,K[k_\varepsilon](|u(s)|^2 - |u_\varepsilon(s)|^2)\}\,ds.$$

For simply we denote $\|f\|_{L_t^\tau(L^\rho)} := \|f\|_{L^\tau(-T,T;L^\rho)}$. Applying the Strichartz estimates (2.3) we have the estimate for J_1

$$\begin{aligned}\|J_1\|_{L_t^\tau(L^\rho)} \leqslant\ & C_{\infty,\tau}\,\|u\,K[k_R-(k_R)_\varepsilon](|u|^2)\|_{L_t^1(L^2)}\\ &+ C_{r(\gamma),\tau}\,\|u\,K[(k-k_R)-(k_\varepsilon-(k_R)_\varepsilon)](|u|^2)\|_{L_t^{r(\gamma)'}(L^{(2\gamma)'})}.\end{aligned}$$

Applying (3.9) and the dominated convergence theorem, we see that

$$\|J_1\|_{L_t^\tau(L^\rho)} \to 0 \quad (\varepsilon\to +0). \tag{3.20}$$

For J_2 applying the Strichartz estimates we have

$$\begin{aligned}&\|J_2\|_{L_t^\tau(L^\rho)}\\ &\leqslant C_{\infty,\tau}\,\|(u-u_\varepsilon)\,K[(k_R)_\varepsilon](|u|^2)\|_{L_t^1(L^2)}\\ &+ C_{r(\gamma),\tau}\,\|(u-u_\varepsilon)\,K[k_\varepsilon-(k_R)_\varepsilon](|u|^2)\|_{L_t^{r(\gamma)'}(L^{(2\gamma)'})}\\ &\leqslant 2C_{\infty,\tau}RT\,\|u\|^2_{L_t^\infty(L^2)}\|u-u_\varepsilon\|_{L_t^\infty(L^2)}\\ &+ C_{r(\gamma),\tau}(2T)^{1-2/r(\gamma)}\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)}\|u\|^2_{L_t^\infty(L^{2\gamma})}\|u-u_\varepsilon\|_{L_t^{r(\gamma)}(L^{2\gamma})}.\end{aligned}$$

In a way similar to J_2, we can evaluate J_3 as follows:

$$\begin{aligned}
&\|J_3\|_{L_t^\tau(L^\rho)} \\
&\leqslant C_{\infty,\tau}\,\|u_\varepsilon K[(k_R)_\varepsilon](|u|^2-|u_\varepsilon|^2)\|_{L_t^1(L^2)} \\
&+C_{r(\gamma),\tau}\,\|u_\varepsilon K[k_\varepsilon-(k_R)_\varepsilon](|u|^2-|u_\varepsilon|^2)\|_{L_t^{r(\gamma)'}(L^{(2\gamma)'})} \\
&\leqslant 2C_{\infty,\tau}RT\,\|u_\varepsilon\|_{L_t^\infty(L^2)}(\|u\|_{L_t^\infty(L^2)}+\|u_\varepsilon\|_{L_t^\infty(L^2)})\|u-u_\varepsilon\|_{L_t^\infty(L^2)} \\
&+C_{r(\gamma),\tau}(2T)^{1-2/r(\gamma)}\,\|k-k_R\|_{L_x^\beta(L_y^\alpha)}\,\|u_\varepsilon\|_{L_t^\infty(L^{2\gamma})} \\
&\times(\|u\|_{L_t^\infty(L^{2\gamma})}+\|u_\varepsilon\|_{L_t^\infty(L^{2\gamma})})\,\|u-u_\varepsilon\|_{L_t^{r(\gamma)}(L^{2\gamma})}.
\end{aligned}$$

Set $(\tau,\rho)=(\infty,2)$ and $(r(\gamma),2\gamma)$. Now we put

$$M:=\max\{\|u_0\|_{L^2},\|u\|_{L^{r(\gamma)}(-T,T;L^{2\gamma})},\sup_{\varepsilon\in(0,1)}\|u_\varepsilon\|_{L^{r(\gamma)}(-T,T;L^{2\gamma})}\}<\infty.$$

Take $T_0\in(0,T)$ such that $6(C_{\infty,\infty}+C_{\infty,r(\gamma)})RM^2T_0\leqslant 1/2$ and $3(C_{r(\gamma),\infty}+C_{r(\gamma),r(\gamma)})\|k-k_R\|_{L_x^\beta(L_y^\alpha)}M^2(2T_0)^{1-2/r(\gamma)}\leqslant 1/2$. Then we obtain

$$\begin{aligned}
&\|u-u_\varepsilon\|_{L^{r(\gamma)}(-T_0,T_0;L^{2\gamma})}+\|u-u_\varepsilon\|_{L^\infty(-T_0,T_0;L^2)} \\
&\leqslant 2\,\|J_1\|_{L^\infty(-T_0,T_0;L^2)}+2\,\|J_1\|_{L^{r(\gamma)}(-T_0,T_0;L^{2\gamma})}.
\end{aligned}\tag{3.21}$$

It follows from (3.20) that

$$\begin{aligned}
u_\varepsilon\to u\quad(\varepsilon\to+0)\ &\text{strongly in } L^\infty(-T_0,T_0;L^2(\mathbb{R}^N)) \\
&\text{and in } L^{r(\gamma)}(-T_0,T_0;L^{2\gamma}(\mathbb{R}^N)).
\end{aligned}\tag{3.22}$$

Extending the interval step by step, we conclude that $u_\varepsilon\to u$ $(\varepsilon\to+0)$ strongly in $L^\infty(-T_1,T_2;L^2(\mathbb{R}^N))$ and in $L^{r(\gamma)}(-T_1,T_2;L^{2\gamma}(\mathbb{R}^N))$.

Step 3. Assume that $u_\varepsilon\not\to u$ $(\varepsilon\to+0)$ in $C([-T_1,T_2];X^1(\mathbb{R}^N))$. Then there exist $\varepsilon_0>0$ and bounded sequences $\{\varepsilon_m\}_m\subset(0,1)$ and $\{t_m\}_m\subset[-T_1,T_2]$ such that

$$\|u_{\varepsilon_m}(t_m)-u(t_m)\|_{X^1}\geqslant\varepsilon_0,\quad m\in\mathbb{N}.$$

We may also assume that $\varepsilon_m\to 0$ and $t_m\to t_0[-T_1,T_2]$ $(m\to\infty)$. Since u belongs to $C([-T_1,T_2];X^1(\mathbb{R}^N))$, we have $\|u(t_m)-u(t_0)\|_{X^1}<\varepsilon_0/2$ for sufficiently large m. Therefore we obtain

$$\|u_{\varepsilon_m}(t_m)-u(t_0)\|_{X^1}>\frac{\varepsilon_0}{2}.$$

On the other hand, it follows from Step 2 that $\|u_{\varepsilon_m}(t_m)-u(t_m)\|_{L^2}\to 0$ $(m\to\infty)$. Since $u\in C(\bar{I};L^2(\mathbb{R}^N))$, we have $\|u(t_m)-u(t_0)\|_{L^2}\to 0$ $(m\to\infty)$. Thus we obtain $\|u_{\varepsilon_m}(t_m)-u(t_0)\|_{L^2}\to 0$ $(m\to\infty)$. This means that $u_{\varepsilon_m}(t_m)\to u(t_0)$ $(m\to\infty)$ strongly in $L^2(\mathbb{R}^N)$ but $u_{\varepsilon_m}(t_m)\not\to u(t_0)$ $(m\to\infty)$ strongly in $X^1(\mathbb{R}^N)$. To derive a contradiction it remains to show that

$$u_{\varepsilon_m}(t_m)\to u(t_0)\ (m\to\infty)\quad\text{strongly in } X^1(\mathbb{R}^N).\tag{3.23}$$

Next we show

$$\|P_{a(N)}^{1/2} u_{\varepsilon_m}(t_m)\|_{L^2}^2 \to \|P_{a(N)}^{1/2} u(t_0)\|_{L^2}^2 \quad (m \to \infty). \tag{3.24}$$

To end this, first we see from the conservation laws that

$$E_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) = E_{\varepsilon_m}(u_0) \to E(u_0) = E(u(t_0)) \quad (m \to \infty). \tag{3.25}$$

Next we prove

$$G_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) \to G(u(t_0)) \quad (m \to \infty). \tag{3.26}$$

Applying (3.10) we calculate

$$\begin{aligned}
&|G_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) - G(u(t_0))| \\
&\leqslant |G_{\varepsilon_m}(u_{\varepsilon_m}(t_m)) - G_{\varepsilon_m}(u(t_0))| + |G_{\varepsilon_m}(u(t_0)) - G(u(t_0))| \\
&\leqslant CM^4 \|k - k_R\|_{L_x^\beta(L_y^\alpha)} + RM^3 \|u_{\varepsilon_m}(t_m) - u(t_0)\|_{L^2} \\
&+ |G_{\varepsilon_m}(u(t_0)) - G(u(t_0))| \\
&\to CM^4 \|k - k_R\|_{L_x^\beta(L_y^\alpha)} \quad (m \to \infty).
\end{aligned}$$

Since $R > 0$ is arbitrary, **(K2)** implies (3.26). Now we can write as for $\varphi \in X^1(\mathbb{R}^N)$

$$E_\varepsilon(\varphi) = \frac{1}{2}\|P_{a(N)}^{1/2}\varphi\|_{L^2}^2 + G_\varepsilon(\varphi), \quad E(\varphi) = \frac{1}{2}\|P_{a(N)}^{1/2}\varphi\|_{L^2}^2 + G(\varphi). \tag{3.27}$$

Combining (3.25) and (3.26) into (3.27) we obtain (3.24).

On the other hand, by the boundedness of $\|u_{\varepsilon_m}(t_m)\|_{X^1}$ there exist $v \in X^1(\mathbb{R}^N)$ and a weak convergent subsequence $\{u_{\varepsilon_{m(j)}}(t_{m(j)})\}_j$ such that $u_{\varepsilon_{m(j)}}(t_{m(j)}) \to v$ $(j \to \infty)$ weakly in $X^1(\mathbb{R}^N)$. Since $u_{\varepsilon_{m(j)}}(t_{m(j)}) \to u(t_0)$ $(j \to \infty)$ strongly in $L^2(\mathbb{R}^N)$, we obtain $u_{\varepsilon_{m(j)}}(t_{m(j)}) \to u(t_0)$ $(j \to \infty)$ weakly in $X^1(\mathbb{R}^N)$. Therefore from the weak convergence in $X^1(\mathbb{R}^N)$ of $\{u_{\varepsilon_{m(j)}}(t_{m(j)})\}_j$ to $u(t_0)$ and the convergence of the corresponding norms we conclude (3.23), a contradiction. □

Now we are the final position to prove the virial identities for **(CP)**$_a$ $(a = a(N))$. Since the strong convergence in $X^1(\mathbb{R}^N)$ and $\|u_\varepsilon(t)\|_{X^1}$ is uniformly bounded in $t \in [-T_1, T_2]$ and ε, we see from the dominated convergence theorem implies that if $u_0 \in \Sigma_*^1(\mathbb{R}^N)$, then

$$\|xu(t)\|_{L^2}^2 = \|xu_0\|_{L^2}^2 + 4t\,\mathrm{Im}\int_{\mathbb{R}^N} \overline{xu_0} \cdot \nabla u_0\,dx + \int_0^t (t-s)V(u_\varepsilon(s))\,ds,$$

where

$$V(v) := 8\|P_{a(N)}^{1/2} v\|_{L^2}^2 - 2\iint_{\mathbb{R}^N \times \mathbb{R}^N} \widetilde{k}(x,y)|v(x)|^2|v(y)|^2\,dxdy$$

(see [14, Remark 2.1]). Thus we obtain

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = V(u(t)).$$

REMARK 3.4. We consider the nonlinearity as in Example 3.1. We obtain

$$\begin{aligned}\frac{d^2}{dt^2}\|xu_{\varepsilon,a}(t)\|_{L^2}^2 = 8\,\|P_a^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 - 8N\int_{\mathbb{R}^N} F(|\rho_\varepsilon * u_{\varepsilon,a}(t)|)\,dx \\ + 4N\int_{\mathbb{R}^N} |\rho_\varepsilon * u_{\varepsilon,a}(t)|\,g(|\rho_\varepsilon * u_{\varepsilon,a}(t)|)\,dx \\ - 8\,\mathrm{Re}\int_{\mathbb{R}^N} \overline{\widetilde{\rho}_\varepsilon * u_{\varepsilon,a}(t)}\,g(\rho_\varepsilon * u_{\varepsilon,a}(t))\,dx\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\|xu_\varepsilon(t)\|_{L^2}^2 = 8\,\|P_a^{1/2}u_\varepsilon(t)\|_{L^2}^2 - 8N\int_{\mathbb{R}^N} F(|\rho_\varepsilon * u_\varepsilon(t)|)\,dx \\ + 4N\int_{\mathbb{R}^N} |\rho_\varepsilon * u_\varepsilon(t)|\,g(|\rho_\varepsilon * u_{\varepsilon,a}(t)|)\,dx \\ - 8\,\mathrm{Re}\int_{\mathbb{R}^N} \overline{\widetilde{\rho}_\varepsilon * u_\varepsilon(t)}\,g(\rho_\varepsilon * u_\varepsilon(t))\,dx.\end{aligned}$$

Thus we see from $\widetilde{\rho}_\varepsilon * f \to 0$ $(\varepsilon \to +0)$ strongly in $L^q(\mathbb{R}^N)$ $(1 \leqslant q < \infty)$ that

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\,\|P_a^{1/2}u(t)\|_{L^2}^2 - 8N\int_{\mathbb{R}^N} F(|u(t)|)\,dx + 4N\int_{\mathbb{R}^N} |u(t)|\,g(|u(t)|)\,dx.$$

REMARK 3.5. We consider the nonlinearity as in Example 3.2. We obtain

$$\begin{aligned}\frac{d^2}{dt^2}\|xu_{\varepsilon,a}(t)\|_{L^2}^2 = 8\,\|P_a^{1/2}u_{\varepsilon,a}(t)\|_{L^2}^2 - 4N\lambda\int_{\mathbb{R}^N} \frac{|\rho_\varepsilon * u_{\varepsilon,a}(t)|^{p+1}}{(|x|^2+\varepsilon)^{r/2}}\,dx \\ + 8\lambda\int_{\mathbb{R}^N} \frac{(N-r)|x|^2+N\varepsilon}{(p+1)(|x|^2+\varepsilon)^{r/2+1}}\,|\rho_\varepsilon * u_{\varepsilon,a}(t)|^{p+1}\,dx \\ - 8\lambda\,\mathrm{Re}\int_{\mathbb{R}^N} \overline{\widetilde{\rho}_\varepsilon * u_{\varepsilon,a}(t)}\,\rho_\varepsilon * u_{\varepsilon,a}(t)\,\frac{|\rho_\varepsilon * u_{\varepsilon,a}(t)|^{p-1}}{(|x|^2+\varepsilon)^{r/2}}\,dx\end{aligned}$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\|xu_\varepsilon(t)\|_{L^2}^2 = 8\,\|P_a^{1/2}u_\varepsilon(t)\|_{L^2}^2 + 4N\lambda\int_{\mathbb{R}^N} \frac{|\rho_\varepsilon * u_\varepsilon(t)|^{p+1}}{(|x|^2+\varepsilon)^{r/2}}\,dx \\ - 8\lambda\int_{\mathbb{R}^N} \frac{(N-r)|x|^2+N\varepsilon}{(p+1)(|x|^2+\varepsilon)^{r/2+1}}\,|\rho_\varepsilon * u_\varepsilon(t)|^{p+1}\,dx \\ - 8\lambda\,\mathrm{Re}\int_{\mathbb{R}^N} \overline{\widetilde{\rho}_\varepsilon * u_\varepsilon(t)}\,\rho_\varepsilon * u_\varepsilon(t)\,\frac{|\rho_\varepsilon * u_\varepsilon(t)|^{p-1}}{(|x|^2+\varepsilon)^{r/2}}\,dx.\end{aligned}$$

Thus we see from $\widetilde{\rho}_\varepsilon * f \to 0$ $(\varepsilon \to +0)$ strongly in $L^q(\mathbb{R}^N)$ $(1 \leqslant q < \infty)$ that

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8\,\|P_a^{1/2}u(t)\|_{L^2}^2 + \frac{4\lambda(Np-N+2r)}{p+1}\int_{\mathbb{R}^N} \frac{|u(t)|^{p+1}}{|x|^r}\,dx.$$

3.1. Typical examples of the virial identity for (CP)$_a$ with $a = a(N)$

(V1) $g(u) := \lambda |u|^{p-1}u$ $(\lambda \in \mathbb{R},\ 1 \leqslant p < (N+2)/(N-2))$

$$\begin{aligned}\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 &= 8\,\|P_{a(N)}^{1/2}u(t)\|_{L^2}^2 + \frac{4\lambda N(p-1)}{p+1}\int_{\mathbb{R}^N}|u(t)|^{p+1}\,dx\\ &= 16E(u(t)) + \frac{4\lambda(Np-N-4)}{p+1}\int_{\mathbb{R}^N}|u(t)|^{p+1}\,dx;\end{aligned}\tag{3.28}$$

(V2) $g(u) := \lambda |x|^{-r}|u|^{p-1}u$ $(\lambda \in \mathbb{R},\ 0 < r < 2,\ 1 \leqslant p < (N+2-2r)/(N-2))$

$$\begin{aligned}\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 &= 8\,\|P_{a(N)}^{1/2}u(t)\|_{L^2}^2 + \frac{4\lambda(Np-N+2r)}{p+1}\int_{\mathbb{R}^N}\frac{|u(t)|^{p+1}}{|x|^r}\,dx\\ &= 16E(u(t)) + \frac{4\lambda(Np-N-4+2r)}{p+1}\int_{\mathbb{R}^N}\frac{|u(t)|^{p+1}}{|x|^r}\,dx;\end{aligned}\tag{3.29}$$

(V3) $g(u) := \lambda(|x|^{-\gamma} * |u|^2)u$ $(\lambda \in \mathbb{R},\ 0 < \gamma < \min\{N,4\})$

$$\begin{aligned}\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 &= 8\,\|P_{a(N)}^{1/2}u(t)\|_{L^2}^2 + 2\lambda\gamma\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(t,x)|^2\,|u(t,y)|^2}{|x-y|^\gamma}\,dxdy\\ &= 16E(u(t)) + 2\lambda(\gamma-2)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(t,x)|^2\,|u(t,y)|^2}{|x-y|^\gamma}\,dxdy;\end{aligned}\tag{3.30}$$

(V4) $g(u) := \lambda u|x|^{-\alpha}[|x|^{-\beta} * (|x|^{-\alpha}|u|^2)]$ $(\lambda \in \mathbb{R},\ \alpha > 0,\ \beta > 0,\ 0 < 2\alpha+\beta < \min\{N,4\})$

$$\begin{aligned}&\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2\\ &= 8\,\|P_{a(N)}^{1/2}u(t)\|_{L^2}^2 + 2\lambda(2\alpha+\beta)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(t,x)|^2\,|u(t,y)|^2}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dxdy\\ &= 16E(u(t)) + 2\lambda(2\alpha+\beta-2)\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(t,x)|^2\,|u(t,y)|^2}{|x|^\alpha\,|x-y|^\beta\,|y|^\alpha}\,dxdy.\end{aligned}\tag{3.31}$$

4. Applications

4.1. Blow-up in finite time

THEOREM 4.1. *Let $a = a(N)$ and $u_0 \in \Sigma_*^1(\mathbb{R}^N)$ with $E(u_0) < 0$. Assume either that*

(B1) *g satisfies* **(N1)**, **(N2)**, *and $2(N+2)F(x) - Nxf(x) \geqslant 0$ for $x > 0$;*

(B2) *$g(u) = \lambda|x|^{-r}|u|^{p-1}u$ with $\lambda < 0$, $0 < r < 2$, and $1 + (4-2r)/N < p < 1 + (4-2r)/(N-2)$;*

(B3) $g(u) = uK[k](|u|^2)$, *where* k *satisfies* **(K1)**, **(K2)**, **(K3)**, *and*

$$k(x,y) + \frac{1}{2}\widetilde{k}(x,y) \geqslant 0.$$

Then there exist $T_1, T_2 > 0$ *such that the unique local weak solution* $u \in C(\bar{I}; X^1(\mathbb{R}^N)) \cap C^1(\bar{I}; X^{-1}(\mathbb{R}^N))$ *to* **(CP)**$_a$ *is exist if* $\bar{I} \subset (-T_1, T_2)$ *and* u *satisfies*

$$\lim_{t \to -T_1+0} \|(1+P_{a(N)})^{1/2} u(t)\|_{L^2} = \infty = \lim_{t \to T_2-0} \|(1+P_{a(N)})^{1/2} u(t)\|_{L^2}, \tag{4.1}$$

that is, the local weak solution blows up in finite time and hence cannot extend globally in time.

Proof. Assume that a local solution $u \in C([-T,T]; X^1(\mathbb{R}^N))$ to **(CP)**$_a$ can be extended globally in time. Then we see that

$$\varphi(t) := \|xu(t)\|_{L^2}^2 \geqslant 0 \quad \forall\, t \in \mathbb{R}.$$

The virial identity for **(CP)**$_a$ and the assumption imply that $\varphi''(t) \leqslant 16E(u_0) < 0$. Thus we obtain

$$\varphi(t) \leqslant \|xu_0\|_{L^2}^2 + \varphi'(0)t + 8E(u_0)t^2 =: \psi(t).$$

Since $E(u_0) < 0$, we can select $T_1, T_2 > 0$ such that $\psi(-T_1) < 0$ and $\psi(T_2) < 0$. Hence $\phi(-T_1) < 0$ and $\phi(T_2) < 0$; this is a contradiction. Note that the last assertion (4.1) follows from [2, Remark 3.1.6 (ii)]. □

4.2. Existence of scattering states

Next we consider the asymptotic completeness for **(CP)**$_a$ with $a = a(N)$. Define

$$Q(u;\tau) := \int_{\mathbb{R}^N} \left[\left| \frac{x}{2}u + i\tau\nabla u \right|^2 - \frac{(N-2)^2}{4|x|^2}\tau^2 |u|^2 \right] dx.$$

We see that

$$Q(u;\tau) = \tau^2 \|P_{a(N)}^{1/2} u\|_{L^2}^2 - \tau \operatorname{Im} \int_{\mathbb{R}^N} x\bar{u} \cdot \nabla u \, dx + \frac{1}{4}\|xu\|_{L^2}^2. \tag{4.2}$$

Applying Lemma 3.1 we obtain

$$\begin{aligned} |Q(u;\tau)| &\leqslant \tau^2 \|P_{a(N)}^{1/2} u\|_{L^2}^2 + |\tau|\, \|P_{a(N)}^{1/2} u\|_{L^2} \|xu\|_{L^2} + \frac{1}{4}\|xu\|_{L^2}^2 \\ &= \left(\frac{1}{2}\|xu\|_{L^2} + |\tau| \|P_{a(N)}^{1/2} u\|_{L^2} \right)^2. \end{aligned}$$

On the other hand, Q is deduced by $\|P_a^{1/2}\cdot\|_{L^2}^2$:

$$\begin{aligned}\left\|P_{a(N)}^{1/2}\left[\exp\left(\frac{|x|^2}{4i\nu}\right)u\right]\right\|_{L^2}^2 &= \int_{\mathbb{R}^N}\left[\left|\nabla\left[\exp\left(\frac{|x|^2}{4i\nu}\right)u\right]\right|^2-\frac{(N-2)^2}{4|x|^2}\left|\exp\left(\frac{|x|^2}{4i\nu}\right)u\right|^2\right]dx\\ &= \int_{\mathbb{R}^N}\left[\left|\exp\left(\frac{|x|^2}{4i\nu}\right)\left[\frac{x}{2i\nu}u+\nabla u\right]\right|^2-\frac{(N-2)^2}{4|x|^2}|u|^2\right]dx\\ &= \frac{1}{\nu^2}\int_{\mathbb{R}^N}\left[\left|\frac{x}{2}u+i\nu\nabla u\right|^2-\frac{(N-2)^2}{4|x|^2}\nu^2|u|^2\right]dx=\frac{1}{\nu^2}Q(u;\nu).\end{aligned}$$

THEOREM 4.2. *Assume either that*

(A1) $g(u)=\lambda|u|^{p-1}u$ *with* $\lambda>0$ *and* $1+2/N<p<1+4/(N-2)$;

(A2) $g(u)=\lambda|x|^{-r}|u|^{p-1}u$ *with* $\lambda>0,\ 0<r<2$, *and* $1+(2-2r)/N<p<1+(4-2r)/(N-2)$;

(A3) $g(u)=\lambda(|x|^{-\gamma}*|u|^2)u$ *with* $\lambda>0$ *and* $1<\gamma<\min\{N,4\}$;

(A4) $g(u)=\lambda u|x|^{-\alpha}[|x|^{-\beta}*(|x|^{-\alpha}|u|^2)]$ *with* $\lambda>0,\ \alpha>0,\ \beta>0$, *and* $1<2\alpha+\beta<\min\{N,4\}$.

Then for every $u_0\in\Sigma_*^1(\mathbb{R}^N)$ *there uniquely exists* $(u_+,u_-)\in L^2(\mathbb{R}^N)^2$ *such that*

$$\begin{cases}\lim\limits_{t\to\infty}\exp(itP_{a(N)})u(t)=u_+ & \textit{strongly in } L^2(\mathbb{R}^N),\\ \lim\limits_{t\to-\infty}\exp(itP_{a(N)})u(t)=u_- & \textit{strongly in } L^2(\mathbb{R}^N).\end{cases}\tag{4.3}$$

Here $u(t)\in C(\mathbb{R};X^1(\mathbb{R}^N))\cap C^1(\mathbb{R};X^{-1}(\mathbb{R}^N))$ *is the unique global solution to* $\textbf{(CP)}_a$ *with* $a=a(N)$ *and* $u(0)=u_0$.

Proof. We divide the proof into three steps.

Step 1. First we consider the pseudo-conformal transform

$$(\mathscr{C}u)(t,x):=(1-t)^{-N/2}u\Big(\frac{t}{1-t},\frac{x}{1-t}\Big)\exp\Big(\frac{-i|x|^2}{4(1-t)}\Big).\tag{4.4}$$

Here $v(t,x):=(\mathscr{C}u)(t,x)$ satisfies $iv'=P_{a(N)}v+(1-t)^{\omega}g(v)$. Note that $G(u)$ satisfies the convexity type inequality:

$$\Big|\int_{\mathbb{R}^N}g(u)\overline{v}\,dx\Big|\leqslant q\,G(u)^{1-1/q}G(v)^{1/q},\quad G(u)\equiv\frac{1}{q}\int_{\mathbb{R}^N}g(u)\overline{u}\,dx.\tag{4.5}$$

Here ω and q are

(A1) $\omega=N(p-1)/2-2,\ q=p+1$;

(A2) $\omega=N(p-1)/2+r-2,\ q=p+1$;

(A3) $\omega=\gamma-2,\ q=4$;

(A4) $\omega=2\alpha+\beta-2,\ q=4$.

Note that it is sufficient to prove that there exists $v_+ \in L^2(\mathbb{R}^N)$ such that

$$v(t) \to v_+ \ (t \to 1-0) \quad \text{strongly in } L^2(\mathbb{R}^N). \tag{4.6}$$

Step 2. We show that for $0 < t < 1$

$$\|P_a^{1/2} v(t)\|_{L^2}^2 \lesssim \begin{cases} (1-t)^\omega & -1 < \omega \leqslant 0, \\ 1 & \omega > 0, \end{cases} \tag{4.7}$$

$$G(v(t)) \lesssim 1. \tag{4.8}$$

To end this we define another energy function.

$$E_1(t) := \frac{1}{2}\|P_{a(N)}^{1/2} v(t)\|_{L^2}^2 + (1-t)^\omega G(v(t)). \tag{4.9}$$

Note that $E_1(t) = E(v(t))$ if $\omega = 0$. Since $v = \mathscr{C}u$ and u satisfies **(CP)**$_a$ with $a = a(N)$, we see from (4.4) and (4.9) that

$$\begin{aligned} E_1(t) &= \frac{1}{8}\|xu(\sigma(t))\|_{L^2}^2 + \frac{(1-t)^{-2}}{2}\|P_{a(N)}^{1/2} u(\sigma(t))\|_{L^2}^2 \\ &\quad - \frac{(1-t)^{-1}}{8}\frac{d}{d\sigma}\|xu(\sigma)\|_{L^2}^2\Big|_{\sigma=\sigma(t)} + (1-t)^{-2}G(u(\sigma(t))) \\ &= (1-t)^{-2}E(u(\sigma(t))) + \frac{1}{8}\|xu(\sigma(t))\|_{L^2}^2 - \frac{(1-t)^{-1}}{4}\frac{d}{d\sigma}\|xu(\sigma)\|_{L^2}^2\Big|_{\sigma=\sigma(t)} \end{aligned}$$

where $\sigma(t) := t/(1-t)$. Hence we calculate the derivative of E_1 by using $\sigma'(t) = (1-t)^{-2}$ and the virial identity (See Section 3.1):

$$\begin{aligned} \frac{d}{dt}E_1(t) &= 2(1-t)^{-3}E(u_0) - \frac{(1-t)^{-3}}{8} \times \frac{d^2}{d\sigma^2}\|xu(\sigma)\|_{L^2}^2\Big|_{\sigma=\sigma(t)} \\ &= 2(1-t)^{-3}E(u_0) - (1-t)^{-3}\Big[\|P_{a(N)}^{1/2} u(\sigma(t))\|_{L^2}^2 + (\omega+2)G(u(\sigma(t)))\Big] \\ &= 2(1-t)^{-3}E(u_0) - 2(1-t)^{-3}E(u(\sigma(t))) - \omega(1-t)^{-3}G(u(\sigma(t))). \end{aligned}$$

It follows from the conservation of laws that

$$\frac{d}{dt}E_1(t) = -\omega(1-t)^{-3}G(u(s(t))) = -\omega(1-t)^{\omega-1}G(v(t)). \tag{4.10}$$

If $\omega \geqslant 0$, then $E_1(t) \leqslant E_1(0)$ for $0 < t < 1$. Hence we see (4.7). (4.7) yields (4.8).

On the other hand, it follows form (4.10) that

$$\frac{d}{dt}[(1-t)^{-\omega}E_1(t)] = -\frac{-\omega}{2}(1-t)^{-\omega-1}\|P_{a(N)}^{1/2} v(t)\|_{L^2}^2.$$

Hence we conclude from $-1 < \omega \leqslant 0$ that $(1-t)^{-\omega}E_1(t) \leqslant E_1(0)$ for $0 < t < 1$. Thus we obtain (4.7) and (4.8).

Step 3. If $\omega \geqslant 0$, then we see that

$$v \in L^\infty(0,1;X^1(\mathbb{R}^N)) \cap W^{1,\infty}(0,1;X^{-1}(\mathbb{R}^N)) \subset C([0,1];L^2(\mathbb{R}^N))$$

and hence we simply obtain (4.6). Thus we assume $-1 < \omega \leqslant 0$ and prove (4.6).

First we verify that

$$v(t) \to v_+ \ (t \to 1-0) \quad \text{strongly in } X^{-1}(\mathbb{R}^N). \tag{4.11}$$

We see from

$$\begin{aligned}
\|g(v)\|_{X^{-1}} &= \sup\Big\{\mathrm{Re}\int_{\mathbb{R}^N} g(v)\overline{\psi}\,dx; \|\psi\|_{X^1} \leqslant 1\Big\} \\
&\leqslant \sup\Big\{(p+1)G(v)^{1-1/q}G(\psi)^{1/q}; \|\psi\|_{X^1} \leqslant 1\Big\} \\
&\lesssim G(v)^{1-1/q} \sup\big\{G(\psi)^{1/q}; \|\psi\|_{X^1} \leqslant 1\big\} \\
&\lesssim G(v)^{1-1/q}.
\end{aligned} \tag{4.12}$$

On the other hand, we have

$$\begin{aligned}
&\|P_{a(N)}v\|_{X^{-1}} \\
&= \sup\Big\{\mathrm{Re}\,\langle P_{a(N)}v, \psi\rangle_{X^{-1},X^1}; \|\psi\|_{X^1} \leqslant 1\Big\} \\
&= \sup\Big\{\mathrm{Re}\,\langle (1+P_{a(N)})^{1/2}v, (1+P_{a(N)})^{1/2}\psi\rangle_{L^2} - \mathrm{Re}\,\langle v, \psi\rangle_{L^2}; \|\psi\|_{X^1} \leqslant 1\Big\} \\
&\leqslant \sup\Big\{\|v\|_{X^1}\,\|\psi\|_{X^1} + \|v\|_{L^2}\,\|\psi\|_{L^2}; \|\psi\|_{X^1} \leqslant 1\Big\} \\
&\leqslant 2\,\|v\|_{X^1}.
\end{aligned} \tag{4.13}$$

Combining (4.13) into (4.7) and (4.12) into (4.8) we obtain

$$\begin{aligned}
\|v'(t)\|_{X^{-1}} &\leqslant \|P_{a(N)}v(t)\|_{X^{-1}} + (1-t)^\omega \|g(v)\|_{X^{-1}} \\
&\lesssim \|v(t)\|_{X^1} + (1-t)^\omega G(v(t))^{p/(p+1)} \\
&\lesssim (1-t)^{\omega/2} + (1-t)^\omega \lesssim (1-t)^\omega.
\end{aligned} \tag{4.14}$$

Thus (4.7) and (4.14) yield that $v \in W^{1,1/(\delta-\omega)}(0,1;X^{-1}(\mathbb{R}^N)) \subset C([0,1];X^{-1}(\mathbb{R}^N))$ for sufficiently small $\delta > 0$. Thus we see (4.11).

Next we confirm that

$$v(t) \to v_+ \ (t \to 1-0) \quad \text{weakly in } L^2(\mathbb{R}^N). \tag{4.15}$$

Since $\|v(t)\|_{L^2} = \|v_0\|_{L^2}$ for $t \in [0,1)$, for any sequence $\{t_j\}_j \subset (0,1)$ there exists $v_{+1} \in L^2(\mathbb{R}^N)$ and a subsequence $\{j(k)\}_k \subset \{j\}$ such that

$$v(t_{j(k)}) \to v_{+1} \ (k \to \infty) \quad \text{weakly in } L^2(\mathbb{R}^N) \subset X^{-1}(\mathbb{R}^N).$$

Here (4.11) implies that $v_{+1} = v_+$. Thus (4.15) is verified. Hence it follows from (4.15) that

$$\langle v_+ - v(t), v_+ \rangle_{L^2} \to 0 \quad t \to 1-0. \tag{4.16}$$

Next we show that

$$\langle v_+ - v(t), v(t) \rangle_{L^2} \to 0 \quad t \to 1-0. \tag{4.17}$$

Let $0 < t < \tau < 1$. We calculate

$$\begin{aligned} |\langle v(\tau) - v(t), v(t) \rangle_{L^2}| &\leqslant \int_t^\tau |\langle v'(s), v(t) \rangle_{X^{-1}, X^1}| \, ds \\ &\leqslant 2 \int_t^\tau \|P_{a(N)}^{1/2} v(s)\|_{L^2} \|P_{a(N)}^{1/2} v(t)\|_{L^2} \, ds \\ &\quad + \int_t^\tau (1-s)^\omega \left| \int_{\mathbb{R}^N} g(v(s)) \, v(t) \, dx \right| ds. \end{aligned}$$

Here the convexity (4.5) implies that

$$\left| \int_{\mathbb{R}^N} g(v(s)) \, v(t) \, dx \right| \leqslant q \, G(v(s))^{1-1/q} \, G(v(t))^{1/q}.$$

Applying (4.7) and (4.8) and letting $\tau \to 1-0$ we ensure (4.17):

$$|\langle v_+ - v(t), v(t) \rangle_{L^2}| \lesssim \int_t^1 (1-t)^{\omega/2} (1-s)^{\omega/2} \, ds + \int_t^1 (1-s)^\omega \, ds \lesssim (1-t)^{\omega+1}.$$

Finally we conclude from (4.16) and (4.17) that

$$\|v_+ - v(t)\|_{L^2}^2 = \langle v_+ - v(t), v_+ \rangle_{L^2} - \langle v_+ - v(t), v(t) \rangle_{L^2} \to 0 \quad t \to 1-0.$$

This is nothing but (4.6). $\square$

REMARK 4.1. Under the conditions in Theorem 4.2 let u be a solution to **(CP)**$_a$ with $u(0) = u_0 \in \Sigma_*^1(\mathbb{R}^N)$. Then $w(t) := \overline{u(-t)}$ satisfies **(CP)**$_a$ with $w(0) = \overline{u_0}$. Hence we conclude that the scattering states

$$\begin{aligned} \lim_{t \to \infty} \exp(itP_{a(N)}) u(t) &= u_+ = \Omega_+ u_0 \quad \text{strongly in } L^2(\mathbb{R}^N), \\ \lim_{t \to -\infty} \exp(itP_{a(N)}) u(t) &= u_- = \Omega_- u_0 \quad \text{strongly in } L^2(\mathbb{R}^N) \end{aligned}$$

satisfy $\Omega_- \varphi = \overline{\Omega_+ \overline{\varphi}}$ for $\varphi \in \Sigma_*^1(\mathbb{R}^N)$.

REMARK 4.2. Assume either that

(A1a) $g(u) = \lambda |u|^{p-1} u$ with $\lambda > 0$ and $1 \leqslant p \leqslant 1 + 2/N$;
(A2a) $g(u) = \lambda |x|^{-r} |u|^{p-1} u$ with $\lambda > 0$, $0 < r \leqslant 1$ and $1 \leqslant p \leqslant 1 + (2-2r)/N$;
(A3a) $g(u) = \lambda (|x|^{-\gamma} * |u|^2) u$ with $\lambda > 0$ and $0 < \gamma \leqslant 1$;

(A4a) $g(u) = \lambda u|x|^{-\alpha}[|x|^{-\beta} * (|x|^{-\alpha}|u|^2)]$ with $\lambda > 0$, $\alpha > 0$, $\beta > 0$ and $0 < 2\alpha + \beta \leqslant 1$.

Then if $u_0 \in \Sigma_*^1(\mathbb{R}^N)$ satisfies

$$\lim_{t\to\infty} \exp(itP_{a(N)})u(t) = u_+, \quad \text{strongly in } L^2(\mathbb{R}^N),$$
$$\lim_{t\to-\infty} \exp(itP_{a(N)})u(t) = u_- \quad \text{strongly in } L^2(\mathbb{R}^N)$$

for some $u_+, u_- \in L^2(\mathbb{R}^N)$, then $u_+ = 0 = u_-$ and hence $u_0 = 0$. Therefore **(CP)**$_a$ with $a = a(N)$ has no free scattering state; see e.g. [7, Theorem 3.1 (2)].

REMARK 4.3. Assume either that

(A1b) $g(u) = \lambda |u|^{4/N}u$ with $\lambda > 0$;
(A2b) $g(u) = \lambda |x|^{-r}|u|^{(4-2r)/N}u$ with $\lambda > 0$ and $0 < r < 2$;
(A3b) $g(u) = \lambda(|x|^{-2} * |u|^2)u$ with $\lambda > 0$;
(A4b) $g(u) = \lambda u|x|^{-\alpha}[|x|^{-\beta} * (|x|^{-\alpha}|u|^2)]$ with $\lambda > 0$, $\alpha > 0$, $\beta > 0$ and $2\alpha + \beta = 2$.

If u is a solution to $iu' = P_{a(N)}u + g(u)$, then $v(t,x) = (\mathscr{C}u)(t,x)$ satisfies also $iv' = P_{a(N)}v + g(v)$. This problem has global unique solution v belongs to $C(\mathbb{R};\Sigma_*^1(\mathbb{R}^N)) \cap C^1(\mathbb{R};X^{-1}(\mathbb{R}^N))$ if $v(T_0) \in \Sigma_*^1(\mathbb{R}^N)$. Especially, we can put $T_0 = 1$. Thus we can construct wave operators $W_+ : u_+ \mapsto u(0)$ and $W_- : u_- \mapsto u(0)$ in $\Sigma_*^1(\mathbb{R}^N)$ such that

$$\lim_{t\to\infty} \exp(itP_{a(N)})u(t) = W_+u_+ \quad \text{strongly in } \Sigma_*^1(\mathbb{R}^N),$$
$$\lim_{t\to-\infty} \exp(itP_{a(N)})u(t) = W_-u_- \quad \text{strongly in } \Sigma_*^1(\mathbb{R}^N).$$

Note that $W_-\varphi = \overline{W_+\overline{\varphi}}$. Other nonlinearities seem to be applied another method. Partial constructions are available in Suzuki [16] in $\Sigma^1(\mathbb{R}^N)$ for $a > -(N-2)^2/4$ with **(A3)** $(1 < \gamma < 2)$ applying the contraction method. The case $a = a(N)$ is open.

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Toshiyuki Suzuki
Department of Mathematics
Faculty of Engineering, Kanagawa University
Kanagawa, Japan
e-mail: t21.suzuki@gmail.com