

## FRACTIONAL LYAPUNOV INEQUALITIES ON SPHERICAL SHELLS

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*Abstract.* This paper, deals with Lyapunov inequalities of conformable fractional boundary value problems on an  $N$ -dimensional spherical shell. Applicability of these Lyapunov inequalities will be examined by establishing the disconjugacy as a nonexistence criterion for nontrivial solutions, lower bound estimation for eigenvalues of the corresponding fractional eigenvalue problem, upper bound estimation for maximum number of zeros of the nontrivial solutions and distance between consecutive zeros of an oscillatory solution.

### 1. Introduction

The theory of fractional calculus that acts on the arbitrary order differentiation and integration, tries to generalize the ordinary calculus. But, unfortunately expected generalization has not occurred by now. As we know, fractional calculus essentially was constructed based on the Riemann-Liouville fractional operators, see [16],[18],[19],[22]. On the other hand, V E. Tarasov in references [25],[26] proves that all of Riemann-Liouville based differentiation operators do not satisfy in the Leibniz and chain rules as in the ordinary calculus. Overcomming this inconvenience, more recently R. Khalil et al, in [15], introduced a new definition for fractional order differentiation operators that generalizes the limit approach of the classic differentiation. They called these operators conformable fractional differentiation operators that will be presented in the next section.

But about the Lyapunov inequalities, this is well known that the concept of the Lyapunov inequality turns to the deep studies of the Russian mathematician A. M. Lyapunov on stability of motion, in the late XIX century. Maybe the best interpretation of the Lyapunov inequalities can be stated as follows:

**THEOREM 1.1.** (cf. [5]-[9],[13]) *If the boundary value problem*

$$\begin{cases} y''(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = 0 = y(b), \end{cases} \quad (1.1)$$

*has a nontrivial solution, where  $q$  is a real and continuous function, then*

$$\int_a^b |q(s)| ds > \frac{4}{b-a}. \quad (1.2)$$

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Since then, the literature has been developed by many generalizations and refinements of the Lyapunov inequality (1.2) by now. Since we are interested in study of the fractional approach of the Lyapunov inequalities, one may suggest for instance the basic papers [4]-[12],[20],[23] for detailed consultation. One of the greatest advantages of the Lyapunov inequalities in comparison with other ones turns to their ability in establishing collection of qualitative behavior of related differential or difference equations such as stability, disconjugacy, solvability and spectral properties. Further more, applying these inequalities one may estimate maximum number of zeros for nontrivial solutions of considered problems and distance between consecutive zeros of the oscillatory solutions.

So, now it seems that why we interested in the study of these inequalities for fractional order differential and difference equations. To the best of our knowledge, investigation about Lyapunov inequalities for differential and difference equations of fractional order introduced for first time in literature by Portuguese mathematician R. A. C. Ferreira. We briefly state the works of the Ferreira as follows.

The author in [5], in the late 2013, studied the following two-point fractional boundary value problem

$$\begin{cases} ({}_aD^\alpha y)(t) + q(t)y(t) = 0, & a < t < b, 1 < \alpha \leq 2, \\ y(a) = 0, \quad y(b) = 0, \end{cases} \quad (1.3)$$

where  ${}_aD^\alpha$  stands for the left sided Riemann-Liouville fractional derivative of order  $\alpha$  and  $q : [a, b] \rightarrow \mathbb{R}$  is a continuous function. The author using properties of the corresponding Green function to the (1.3) obtained a Lyapunov inequality of the form

$$\int_a^b |q(s)| ds > \Gamma(\alpha) \left( \frac{4}{b-a} \right)^{\alpha-1}. \quad (1.4)$$

The Lyapunov inequality (1.4) generalizes the classic Lyapunov inequality (1.2), (excepted  $\alpha = 2$ ). He used then the Lyapunov inequality (1.4) to prove that a Mittag-Leffler function of the fractional eigenvalue problem corresponding to the (1.3) has no real zeros on a determined interval.

The author in [8], (2014), proved that if  $u(t)$  is a nontrivial solution of Caputo fractional boundary value problem

$$\begin{cases} ({}^cD_{0+}^\alpha u)(t) + q(t)u(t) = 0, & a < t < b, 1 < \alpha \leq 2, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \quad (1.5)$$

then the Lyapunov inequality

$$\int_a^b |q(s)| ds > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}, \quad (1.6)$$

holds, provided that  $q(t)$  is a real continuous function.

Also, the author in [9], (2015), considered the following two-point fractional  $\Delta$ -difference boundary value problem

$$\begin{cases} (\Delta^\alpha y)(t) + q(t + \alpha - 1)y(t + \alpha - 1) = 0, & t \in \mathbb{N}_0^{b+1}, \\ y(\alpha - 2) = 0 = y(\alpha + b + 1), \text{ or } y(\alpha - 2) = 0 = \Delta y(\alpha + b), \end{cases} \quad (1.7)$$

and using Green function technique, achieved to the following discrete fractional Lyapunov inequality corresponding to the first pair of boundary conditions:

$$\sum_{s=0}^{b+1} |q(s + \alpha - 1)| > 4\Gamma(\alpha) \frac{\Gamma(b + \alpha + 2)\Gamma^2\left(\frac{b}{2} + 2\right)}{(b + 2\alpha)(b + 2)\Gamma^2\left(\frac{b}{2} + \alpha\right)\Gamma(b + 3)}, \quad b : \text{ even},$$

$$\sum_{s=0}^{b+1} |q(s + \alpha - 1)| > \Gamma(\alpha) \frac{\Gamma(b + \alpha + 2)\Gamma^2\left(\frac{b+3}{2}\right)}{\Gamma^2\left(\frac{b+1}{2} + \alpha\right)\Gamma(b + 3)}, \quad b : \text{ odd}.$$
(1.8)

In the second step, the author obtained for the second pair of boundary conditions, the following Lyapunov inequality:

$$\sum_{s=0}^{b+1} |q(s + \alpha - 1)| > \frac{1}{(b + 2)\Gamma(\alpha - 1)}.$$
(1.9)

### 2. Conformable Fractional Calculus

This section includes a brief overview on the conformable fractional calculus that makes this paper self contained in view point of the fractional calculus community.

DEFINITION 2.1. [15] The left and right sided conformable fractional derivatives of order  $n - 1 < \alpha \leq n$ ,  $n \in \mathbb{N}$ , for  $n$ -differentiable function  $f(t)$  on  $t \in (a, b)$  are defined by:

$$T_{a+}^{\alpha} f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(t + \varepsilon(t - a)^{[\alpha]-\alpha}) - f^{([\alpha]-1)}(t)}{\varepsilon},$$
(2.1)

$$T_{b-}^{\alpha} f(t) = (-1)^n \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(t + \varepsilon(b - t)^{[\alpha]-\alpha}) - f^{([\alpha]-1)}(t)}{\varepsilon},$$
(2.2)

where  $[\alpha]$  is the smallest integer greater than or equal to  $\alpha$ .

Taking  $h = \varepsilon(t - a)^{[\alpha]-\alpha}$  in (2.1) and  $h = \varepsilon(b - t)^{[\alpha]-\alpha}$  in (2.2), a direct calculation leads us to the following golden identities of the conformable fractional calculus:

$$T_{a+}^{\alpha} f(t) = (t - a)^{[\alpha]-\alpha} f^{(n)}(t),$$
(2.3)

$$T_{b-}^{\alpha} f(t) = (-1)^n (b - t)^{[\alpha]-\alpha} f^{(n)}(t).$$
(2.4)

REMARK 2.2. Taking  $\alpha = n$  in golden identities (2.3) and (2.4), one has

$$T_{a+}^n f(t) = f^{(n)}(t),$$

$$T_{b-}^n f(t) = (-1)^n f^{(n)}(t).$$

Also taking  $\alpha = 0$ , then  $n = [\alpha] = \alpha = 0$ . We define

$$T_{a+}^0 f(t) = T_{b-}^0 f(t) = f(t).$$

We notice that, apparently there is happening a serious misunderstanding about nature of the conformable fractional calculus. As nobody can represent the zeroth-order ordinary differentiation via limit approach, so we must not expect that for finding the zeroth-order conformable fractional derivative, we shall take  $\alpha = 0$  in definitions (2.1) or (2.2). Instead, following conformability of the new fractional operators we must use the golden identities (2.3) and (2.4) to define the zeroth-order conformable fractional derivatives as we do it in the ordinary calculus. See for details the references [14] and [21].

Since we are going to study Lyapunov inequalities on an N-dimensional spherical shell, so we have to define conformable fractional partial derivatives.

DEFINITION 2.3. Assume  $n - 1 < \alpha \leq n$ ,  $n \in \mathbb{N}$  and  $t = (t_1, t_2, \dots, t_N) \in [a, b]^N$ . If  $f : [a, b]^N \rightarrow \mathbb{R}$ , then left and right sided conformable fractional partial derivatives of order  $\alpha$  are defined as follows, respectively,

$$\left(\frac{\partial_{a^+}^\alpha}{\partial t_k} f\right)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(t_1, t_2, \dots, t_{k-1}, t_k + \varepsilon(t-a)^{[\alpha]-\alpha}, t_{k+1}, \dots, t_N) - f^{([\alpha]-1)}(t_1, t_2, \dots, t_N)}{\varepsilon}, \quad (2.5)$$

$$\left(\frac{\partial_{b^-}^\alpha}{\partial t_k} f\right)(t) = (-1)^n \lim_{\varepsilon \rightarrow 0} \frac{f^{([\alpha]-1)}(t_1, t_2, \dots, t_{k-1}, t_k + \varepsilon(b-t)^{[\alpha]-\alpha}, t_{k+1}, \dots, t_N) - f^{([\alpha]-1)}(t_1, t_2, \dots, t_N)}{\varepsilon}. \quad (2.6)$$

Making use of the golden identities (2.3), (2.4) for special order  $0 < \alpha \leq 1$ , one may derive the following lemma.

LEMMA 2.4. Assume  $0 < \alpha \leq 1$ . Then,

1.  $T_{a^+}^\alpha \left( \frac{(t-a)^\alpha}{\alpha} \right) = 1;$
2.  $T_{a^+}^\alpha \left( e^{\frac{(t-a)^\alpha}{\alpha}} \right) = e^{\frac{(t-a)^\alpha}{\alpha}};$
3.  $T_{a^+}^\alpha \left( \sin \left( \frac{(t-a)^\alpha}{\alpha} \right) \right) = \cos \left( \frac{(t-a)^\alpha}{\alpha} \right);$
4.  $T_{a^+}^\alpha \left( \cos \left( \frac{(t-a)^\alpha}{\alpha} \right) \right) = -\sin \left( \frac{(t-a)^\alpha}{\alpha} \right).$

In the following technical lemma, one can observe that why we call the fractional differentiation operators (2.1) and (2.2), *conformable*.

LEMMA 2.5. Assume  $0 < \alpha \leq 1$ . Suppose that  $f$  and  $g$  are two real-valued  $\alpha$ -differentiable functions on an interval  $(a, b)$  in the sense of (2.1) and for each  $t > a$ , we have  $g(t) \neq 0$ . Then the fractional Leibniz and chain rules are as follows, respectively,

- (i)  $(T_{a^+}^\alpha f g)(t) = (T_{a^+}^\alpha f)(t) \cdot g(t) + f(t) \cdot (T_{a^+}^\alpha g)(t);$
- (ii)  $(T_{a^+}^\alpha f \circ g)(t) = (T_{a^+}^\alpha f)(t) \cdot (T_{a^+}^\alpha g)(g(t)) \cdot (g(t) - g(a))^{\alpha-1}.$

*Proof.* Making use of the golden identity (2.3), yields the following

$$\begin{aligned} (T_{a^+}^\alpha fg)(t) &= (t-a)^{1-\alpha}(fg)'(t) \\ &= (t-a)^{1-\alpha} \left\{ f'(t)g(t) + f(t).g'(t) \right\} \\ &= (t-a)^{1-\alpha} f'(t).g(t) + f(t).(t-a)^{1-\alpha} g'(t) \\ &= (T_{a^+}^\alpha f)(t).g(t) + f(t).(T_{a^+}^\alpha g)(t). \end{aligned}$$

To prove the assertion (ii), note that

$$\begin{aligned} (T_{a^+}^\alpha f \circ g)(t) &= (t-a)^{1-\alpha}(f \circ g)'(t) \\ &= (t-a)^{1-\alpha} \left\{ g'(t).f'(g(t)) \right\} \\ &= (T_{a^+}^\alpha g)(t).(g(t)-g(a))^{1-\alpha}.f'(g(t)).(g(t)-g(a))^{\alpha-1} \\ &= (T_{a^+}^\alpha f \circ g)(t) = (T_{a^+}^\alpha f)(g(t)).(T_{a^+}^\alpha g)(t).(g(t)-g(a))^{\alpha-1}. \end{aligned}$$

This completes the proof.  $\square$

DEFINITION 2.6. (cf. [1]) Assume that  $n-1 < \alpha \leq n, n \in \mathbb{N}$ . If  $f \in L([a, b]; \mathbb{R})$ , then the left and right sided conformable fractional integrals are defined as follows:

$$I_{a^+}^\alpha f(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} (s-a)^{\alpha-n} f(s) ds, \tag{2.7}$$

$$I_{b^-}^\alpha f(t) = \frac{1}{(n-1)!} \int_t^b (s-t)^{n-1} (b-s)^{\alpha-n} f(s) ds. \tag{2.8}$$

The composition rules due to the conformable fractional operators can be represented as below. These rules will play a fundamental role in the next section.

LEMMA 2.7. (cf. [1]) Let  $n-1 < \alpha \leq n, n \in \mathbb{N}$  and  $f : [a, b] \rightarrow \mathbb{R}$  be an  $n$ -differentiable function on  $(a, b)$ . Then for each  $t \in (a, b)$ ,

$$(I_{a^+}^\alpha T_{a^+}^\alpha f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^k}{k!}, \tag{2.9}$$

$$(I_{b^-}^\alpha T_{b^-}^\alpha f)(t) = f(t) - \sum_{k=0}^{n-1} (-1)^k \frac{f^{(k)}(b)(b-t)^k}{k!}. \tag{2.10}$$

### 3. Fractional Lyapunov Inequalities

In this position, we consider the main problem of the paper as follows:

$$(T_{a^+}^\alpha u)(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha < 2, \tag{3.1}$$

subject to the boundary conditions

$$u(a) = 0, \quad u(b) = 0, \tag{3.2}$$

where  $q \in C([a, b]; \mathbb{R})$ . At this moment, we draw a multi step way that leads us to the Lyapunov inequality of the conformable fractional boundary value problem (3.1)-(3.2). The first step can be stated as follows.

LEMMA 3.1. *Let  $h \in C([a, b]; \mathbb{R})$ . Then the conformable fractional boundary value problem*

$$\begin{cases} (T_{a^+}^\alpha u)(t) + h(t) = 0, & a < t < b, \quad 1 < \alpha < 2, \\ u(a) = 0, \quad u(b) = 0, \end{cases} \quad (3.3)$$

in a unique manner can be transformed into the integral equation

$$u(t) = \int_a^b \mathcal{G}(t, s)h(s)ds, \quad (3.4)$$

in which

$$\mathcal{G}(t, s) = \frac{1}{b-a} \begin{cases} [(t-a)(b-s) - (b-a)(t-s)](s-a)^{\alpha-2}; & a < s \leq t < b, \\ (t-a)(b-s)(s-a)^{\alpha-2} & ; \quad a < t \leq s < b. \end{cases} \quad (3.5)$$

is called the Green function of the boundary value problem (3.3).

*Proof.* Concentrating on the governing equation

$$(T_{a^+}^\alpha u)(t) + h(t) = 0,$$

and then taking conformable fractional integral  $I_{a^+}^\alpha$  on both sides of it, in the light of the composition rule (2.9) gives us

$$u(t) = c_0 + c_1(t-a) - \int_a^t (t-s)(s-a)^{\alpha-2}h(s)ds. \quad (3.6)$$

Now, making use of the first boundary condition  $u(a) = 0$  in (3.6), one has  $c_0 = 0$ . Next, imposing the second boundary condition  $u(b) = 0$  into the (3.6), it follows that

$$c_1 = \frac{1}{b-a} \int_a^b (b-s)(s-a)^{\alpha-2}h(s)ds. \quad (3.7)$$

Consequently, substituting the uniquely determined coefficients  $c_0$  and  $c_1$  into the (3.6), gives us the following

$$\begin{aligned} u(t) &= \frac{t-a}{b-a} \int_a^b (b-s)(s-a)^{\alpha-2}h(s)ds - \int_a^t (t-s)(s-a)^{\alpha-2}h(s)ds \\ &= \frac{1}{b-a} \int_a^t [(t-a)(b-s) - (b-a)(t-s)](s-a)^{\alpha-2}h(s)ds \\ &\quad + \frac{1}{b-a} \int_t^b (t-a)(b-s)(s-a)^{\alpha-2}h(s)ds \\ &= \int_a^b \mathcal{G}(t, s)h(s)ds. \end{aligned}$$

This completes the proof.  $\square$

The second step, includes an analytic assessment of the Green function  $\mathcal{G}(t, s)$  as follows.

LEMMA 3.2. *The Green function  $\mathcal{G}(t, s)$  defined by (3.5), satisfies the following analytic components:*

(A<sub>1</sub>)  $\mathcal{G}(t, s)$  is continuous on  $(a, b) \times (a, b)$ ;

$$(A_1) \quad \sup_{t, s \in (a, b)} \mathcal{G}(t, s) = \mathcal{G}_2 \left( \frac{a + (\alpha - 1)b}{\alpha}, \frac{a + (\alpha - 1)b}{\alpha} \right) = \frac{1}{\alpha} \left( \frac{(\alpha - 1)(b - a)}{\alpha} \right)^{\alpha - 1}.$$

*Proof.* The assertion (A<sub>1</sub>) is immediate. So, we focus on the property (A<sub>2</sub>). To prove it, we recall once again the Green function  $\mathcal{G}(t, s)$  as consequences

$$\mathcal{G}(t, s) = \begin{cases} \mathcal{G}_1(t, s); & a < s \leq t < b, \\ \mathcal{G}_2(t, s); & a < t \leq s < b, \end{cases}$$

where

$$\mathcal{G}_1(t, s) = \frac{[(t - a)(b - s) - (b - a)(t - s)](s - a)^{\alpha - 2}}{b - a}, \tag{3.8}$$

$$\mathcal{G}_2(t, s) = \frac{(t - a)(b - s)(s - a)^{\alpha - 2}}{b - a}. \tag{3.9}$$

As can be observed,

$$\sup_{t, s \in (a, b)} \mathcal{G}(t, s) = \sup_{t, s \in (a, b)} \mathcal{G}_2(t, s).$$

In one hand

$$\frac{\partial}{\partial t} \mathcal{G}_2(t, s) = \frac{(b - s)(s - a)^{\alpha - 2}}{b - a} > 0, \tag{3.10}$$

that yields

$$\sup_{t \in (a, b)} \mathcal{G}_2(t, s) = \mathcal{G}_2(s, s) = \frac{(b - s)(s - a)^{\alpha - 1}}{b - a}, \tag{3.11}$$

and on the other hand,

$$\frac{d}{ds} \mathcal{G}_2(s, s) = \frac{(s - a)^{\alpha - 2}}{b - a} \{a + (\alpha - 1)b - \alpha s\}.$$

Thus, it follows that

$$\frac{d}{ds} \mathcal{G}_2(s, s) \begin{cases} > 0; & s < \frac{a + (\alpha - 1)b}{\alpha}, \\ < 0; & s > \frac{a + (\alpha - 1)b}{\alpha}. \end{cases}$$

Thereby, we have the following

$$\sup_{s \in (a,b)} \mathcal{G}_2(s,s) = \mathcal{G}_2 \left( \frac{a + (\alpha - 1)b}{\alpha}, \frac{a + (\alpha - 1)b}{\alpha} \right). \quad (3.12)$$

As a result, making use of the (3.10) and (3.12) we conclude that

$$\sup_{t,s \in (a,b)} \mathcal{G}(t,s) = \mathcal{G}_2 \left( \frac{a + (\alpha - 1)b}{\alpha}, \frac{a + (\alpha - 1)b}{\alpha} \right) = \frac{1}{\alpha} \left( \frac{(\alpha - 1)(b - a)}{\alpha} \right)^{\alpha - 1}.$$

So, the proof is completed now.  $\square$

Throughout this paper, everywhere we discuss on the real line  $\mathbb{R}$ , we are concerned with the Banach space  $(\mathcal{B}; \|\cdot\|)$  such that

$$\mathcal{B} := C^n([a, b]), \quad \|u\| = \sup_{t \in (a,b)} \{|u(t)| \mid u \in \mathcal{B}\}.$$

In this position, we are ready to the final step, that is extracting Lyapunov inequality of the conformable fractional boundary value problem (3.1)-(3.2). This step directly relies on the previous two steps and is presented as below.

**THEOREM 3.3.** *Let  $u(t)$  is a nontrivial solution of the conformable fractional boundary value problem (3.1)-(3.2). Then the Lyapunov inequality*

$$\int_a^b |q(s)| ds > \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha - 1}} \frac{1}{(b - a)^{\alpha - 1}}, \quad (3.13)$$

holds.

*Proof.* In the light of Lemma 3.1, we can transform the boundary value problem (3.1)-(3.2) into the integral equation

$$u(t) = \int_a^b \mathcal{G}(t,s) q(s) u(s) ds,$$

in which  $\mathcal{G}(t,s)$  denotes the Green function (3.5). Since  $u(t)$  is a nontrivial solution of the boundary value problem (3.1)-(3.2), making use of the property  $(A_2)$  in Lemma 3.2, yields

$$\|u\| < \|u\| \frac{1}{\alpha} \left( \frac{(\alpha - 1)(b - a)}{\alpha} \right)^{\alpha - 1} \int_a^b |q(s)| ds,$$

that is,

$$\int_a^b |q(s)| ds > \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha - 1}} \frac{1}{(b - a)^{\alpha - 1}}.$$

This completes the proof.  $\square$



It is time to migration from the real line  $\mathbb{R}$  into the  $N$ -dimensional space  $\mathbb{R}^N$  for  $N \in \mathbb{N}$ . Based on the references [2] and [3], the elements of this space are introduced as follows:

$$A := B(0, b) - \overline{B(0, a)}, \quad 0 < a < b, \quad (\text{spherical shell}); \tag{3.14}$$

$$B(0, r) := \{x \in \mathbb{R}^N \mid |x| < r, r > 0\}, \quad (\text{N-dimensional ball centered by 0 with radius } r); \tag{3.15}$$

$$S^{N-1} := \{x \in \mathbb{R}^N \mid |x| = 1\}, \quad (\text{unit sphere on the } \mathbb{R}^N); \tag{3.16}$$

$$\omega_N := \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}, \quad (\text{surface area of } S^{N-1}); \tag{3.17}$$

$$|\cdot|, \quad (\text{Euclidean norm on the } \mathbb{R}^N). \tag{3.18}$$

REMARK 3.4. (cf. [2],[3] and [24], pp. 149-150)

(R<sub>1</sub>) Each  $x \in \mathbb{R}^N - \{0\}$ , can be uniquely represented as  $x = r\omega$  where  $r \in \mathbb{R}^+$  and  $\omega \in S^{N-1}$ . Here  $r = |x| > 0$  and  $\omega = \frac{x}{r} \in S^{N-1}$ .

(R<sub>2</sub>) For  $u \in C(\overline{A})$ , one has

$$\int_A F(x)dx = \int_{S^{N-1}} \left( \int_a^b F(r\omega)r^{N-1}dr \right) d\omega. \tag{3.19}$$

In particular,  $\overline{A} = [a, b] \times S^{N-1}$ .

(R<sub>3</sub>) As a special case of (3.19), one may take  $F(x) \equiv 1$ . So, we have

$$\int_A dx = \int_{S^{N-1}} \left( \int_a^b r^{N-1}dr \right) d\omega = \frac{b^N - a^N}{N} \cdot \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}. \tag{3.20}$$

At this moment, the recent supplementary data related to the spherical shell  $\overline{A}$ , enables us to direct transfer the conformable fractional boundary value problem (3.1)-(3.2) and its Lyapunov inequality (3.13) over the spherical shell  $\overline{A}$ .

THEOREM 3.5. Let  $q \in C(\overline{A})$  and  $u(x)$  be a nontrivial solution of the conformable fractional partial differential equation

$$\left( \frac{\partial_{a^+}^\alpha}{\partial r} u \right) (x) + q(x)u(x) = 0, \quad x \in \overline{A}, \tag{3.21}$$

subject to the boundary conditions

$$u(\partial B(0, a)) = 0, \quad u(\partial B(0, b)) = 0, \tag{3.22}$$

where  $u(x) \neq 0$  for each  $x \in A$ . Then, the multivariate Lyapunov inequality

$$\int_A |q(x)|dx > \frac{2a^{N-1}\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}}, \tag{3.23}$$

holds.

*Proof.* Paying attention on  $(R_1)$  in Remark 3.4, the governing equation (3.21), one may rewrite it as follows:

$$\left(\frac{\partial_{a^+}^\alpha}{\partial r}\right)(r\omega) + q(r\omega)u(r\omega) = 0, \quad (r, \omega) \in [a, b] \times S^{N-1}, \tag{3.24}$$

where  $q(\cdot\omega) \in C([a, b])$ , for each  $\omega \in S^{N-1}$ . Note that in this case

$$u(a\omega) = 0, \quad u(b\omega) = 0, \quad w \in S^{N-1}, \tag{3.25}$$

in which,  $u(r\omega) \neq 0$  for each  $r \in (a, b)$  and  $\omega \in S^{N-1}$ . Thereby, for a fixed  $\omega \in S^{N-1}$ , by the Lyapunov inequality (3.13) we conclude that

$$\begin{aligned} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}} &< \int_a^b |q(r\omega)| dr = \int_a^b |q(r\omega)| r^{N-1} r^{1-N} dr \\ &\leq \left(\int_a^b |q(r\omega)| r^{N-1} dr\right) a^{1-N}. \end{aligned}$$

This leads us to the inequality

$$\int_a^b |q(r\omega)| r^{N-1} dr > a^{N-1} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}}, \quad \omega \in S^{N-1}. \tag{3.26}$$

Consequently, making use of the (3.17) and  $(R_2)$  in Remark 3.4, it follows that

$$\int_{S^{N-1}} \left(\int_a^b |q(r\omega)| r^{N-1} dr\right) d\omega > \left(a^{N-1} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}}\right) \left(\frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}\right).$$

Equivalently,

$$\int_A |q(x)| dx > \frac{2a^{N-1}\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}}.$$

This completes the proof.  $\square$

REMARK 3.6. Considering the nonnegative part of  $q(x)$  as

$$q^+(x) = \max\{0, q(x)\} = \frac{q(x) + |q(x)|}{2},$$

the following multivariate Lyapunov inequality is immediate

$$\int_A q^+(x) dx > \frac{2a^{N-1}\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}}.$$

Now, having the multivariate Lyapunov inequality (3.23) in hand, we are ready to examine its applicability to establish the qualitative dynamic of the conformable fractional boundary value problem (3.21)-(3.22) over the spherical shell  $\bar{A}$ .

### 4. Applications

In this section, we are going to represent the applied aspect of the multivariate Lyapunov inequality (3.23). To this end, we establish qualitative dynamic of the conformable fractional boundary value problem (3.21)-(3.22), consisting of its disconjugacy as a non-existence criterion for nontrivial solutions, Lower bound estimation for eigenvalues in corresponding eigenvalue problem, distance between consecutive zeros of the nontrivial solution and zero count estimation for nontrivial solutions. So, we begin with the following definition.

DEFINITION 4.1. Suppose  $q \in C(\bar{A})$  and  $n - 1 < \alpha < n, n \in \mathbb{N}$ . The conformable fractional differential equation

$$\left(\frac{\partial_{a^+}^\alpha u}{\partial r}\right)(x) + q(x)u(x) = 0, \quad x \in \bar{A},$$

is said to be disconjugate on spherical shell  $\bar{A}$ , if and only if each nontrivial solution  $u(x)$  has less than  $[\alpha] + 1$  zeros on  $\bar{A}$

In the light of this definition, we can now introduce a non-existence criterion for nontrivial solutions as follows.

THEOREM 4.2. Assume that  $1 < \alpha < 2$ . If

$$\int_A |q(x)|dx \leq \frac{2a^{N-1}\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}}, \tag{4.1}$$

then, the conformable fractional boundary value problem (3.21)-(3.22) is disconjugate on  $\bar{A}$ .

*Proof.* Suppose on contrary that the conformable fractional boundary value problem (3.21)-(3.22) is not disconjugate on  $\bar{A}$ . Therefore, there exists at least one nontrivial solution  $u(x), x \in \bar{A}$  such that  $u(x)$  has at least two zeros on  $\bar{A}$ . This means that according to the essential Theorem 3.5, the multivariate Lyapunov inequality

$$\int_A |q(x)|dx > \frac{2a^{N-1}\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}},$$

is satisfied. Resulted contradiction with the assumption (4.1) proves the desired assertion.  $\square$

LEMMA 4.3. Assume that the assumptions of Theorem 4.2 are satisfied. Then, the conformable fractional boundary value problem (3.21)-(3.22) has no nontrivial solution on  $\bar{A}$ .

*Proof.* Since the proof is the same as of Theorem 4.2, we omit it here.  $\square$

Second application is devoted to a lower bound estimation for the eigenvalues of the conformable fractional eigenvalue problem corresponding to the conformable fractional boundary value problem (3.21)-(3.22).

**THEOREM 4.4.** *Assume  $1 < \alpha < 2$  and  $u(x)$  is a nontrivial solution of the conformable fractional eigenvalue problem*

$$\begin{cases} \left(\frac{\partial_{a^+}^\alpha}{\partial r}\right)u(x) + \lambda u(x) = 0, & x \in \bar{A}, \\ u(\partial B(0, a)) = 0, & u(\partial B(0, b)) = 0, \end{cases} \tag{4.2}$$

where  $u(x) \neq 0$  for each  $x \in A$ . Then

$$|\lambda| > N \frac{a^{N-1}}{b^N - a^N} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}}. \tag{4.3}$$

*Proof.* Replacing  $\lambda$  with  $q(x) \in C(\bar{A})$  in the conformable fractional boundary value problem (3.21)-(3.22), and taking into account (3.20), in Remark 3.4 the following is immediate

$$\begin{aligned} \int_A |\lambda| dx &= \int_{S^{N-1}} \left( \int_a^b |\lambda| r^{N-1} dr \right) d\omega \\ &= |\lambda| \int_{S^{N-1}} \int_a^b r^{N-1} dr d\omega \\ &= |\lambda| \frac{b^N - a^N}{N} \cdot \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}. \end{aligned}$$

Now, taking a look at the multivariate Lyapunov inequality (3.23), it follows that

$$|\lambda| \frac{b^N - a^N}{N} \cdot \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} > \frac{2a^{N-1}\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}}.$$

Equivalently, we conclude that

$$|\lambda| > N \frac{a^{N-1}}{b^N - a^N} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(b - a)^{\alpha-1}}.$$

This completes the proof.  $\square$

In this position, we are going to establish maximum number of zeros of a nontrivial solution of the conformable fractional boundary value problem (3.21)-(3.22), making use of the its multivariate Lyapunov inequality (3.23).

**THEOREM 4.5.** *Assume that  $u(x)$  is a nontrivial solution of the conformable fractional boundary value problem (3.21)-(3.22). Suppose  $\{r_k \omega\}_{k=1}^{2M+1}$  be an increasing*

sequence of zeros of the  $u(x)$  on  $\bar{A}$ , such that  $r_k \geq 1, k = 1, \dots, 2M + 1$ , vary on a compact interval  $I$  having length  $l(I)$ . Then,

$$M < \left\{ \frac{\Gamma(\frac{N}{2}) (l(I)(\alpha - 1))^{\alpha-1}}{2\pi^{\frac{N}{2}} \alpha^\alpha} \int_{S^{N-1}} \sum_{k=1}^M \left( \int_{r_{2k-1}}^{r_{2k+1}} |q(r\omega)| r^{N-1} dr \right) d\omega \right\} \frac{1}{\alpha} \tag{4.4}$$

*Proof.* Since  $r_k \omega$  for each  $k = 1, 2, \dots, 2M + 1$  are roots of the nontrivial solution  $u(x)$  on  $\bar{A}$ , so we can apply Theorem 3.5 for each subinterval  $[r_{2k+1}, r_{2k-1}] \subset I, k = 1, 2, \dots, M$ . Hence, in the light of the assumption  $r_k \geq 1, k = 1, \dots, 2M + 1$ , it follows that

$$\begin{aligned} \int_{S^{N-1}} \left( \int_{r_{2k-1}}^{r_{2k+1}} |q(r\omega)| r^{N-1} dr \right) d\omega &> \left( r_{2k-1}^{N-1} \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \frac{1}{(r_{2k+1} - r_{2k-1})^{\alpha-1}} \right) \left( \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \right) \\ &\geq \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \cdot \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \cdot (r_{2k+1} - r_{2k-1})^{-\alpha+1}. \end{aligned}$$

Now, summing over both sides of the recent inequality from  $k = 1$  to  $M$ , one has

$$\int_{S^{N-1}} \sum_{k=1}^M \left( \int_{r_{2k-1}}^{r_{2k+1}} |q(r\omega)| r^{N-1} dr \right) d\omega > \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \cdot \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \sum_{k=1}^M (r_{2k+1} - r_{2k-1})^{-\alpha+1}. \tag{4.5}$$

Let us define concave up function  $\psi(x) := x^{-\alpha+1}, \alpha > 1$  on positive half-line  $(0, \infty)$ . So, we have

$$\frac{1}{M} \sum_{k=1}^M \psi(a_k) \geq \psi \left( \frac{1}{M} \sum_{k=1}^M a_k \right), \quad (\text{see [6]}),$$

where  $a_k := r_{2k+1} - r_{2k-1} > 0$ . Therefore, it follows that

$$\begin{aligned} \int_{S^{N-1}} \sum_{k=1}^M \left( \int_{r_{2k-1}}^{r_{2k+1}} |q(r\omega)| r^{N-1} dr \right) d\omega &> \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \cdot \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \sum_{k=1}^M (r_{2k+1} - r_{2k-1})^{-\alpha+1} \\ &= \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \cdot \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} M \left( \frac{1}{M} \sum_{k=1}^M (r_{2k+1} - r_{2k-1}) \right)^{-\alpha+1} \\ &\geq \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \cdot \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} M^\alpha (l(I))^{-\alpha+1}. \end{aligned}$$

After simplification, we conclude that

$$M < \left\{ \frac{\Gamma(\frac{N}{2}) (l(I)(\alpha - 1))^{\alpha-1}}{2\pi^{\frac{N}{2}} \alpha^\alpha} \int_{S^{N-1}} \sum_{k=1}^M \left( \int_{r_{2k-1}}^{r_{2k+1}} |q(r\omega)| r^{N-1} dr \right) d\omega \right\} \frac{1}{\alpha},$$

which coincides with our desired result.  $\square$

The last application due to the multivariate Lyapunov inequality (3.23), is dealt with oscillatory solutions of the conformable fractional boundary value problem (3.21)-(3.22). In this stage, we demonstrate that distance between consecutive zeros of each oscillatory solution of (3.21)-(3.22), when their indices tend to infinity can not be finite.

**THEOREM 4.6.** *Assume that  $u(x)$  is an oscillatory solution of the conformable fractional boundary value problem (3.21)-(3.22). Suppose  $\{r_n\omega\}_{n=1}^\infty$ ,  $r_n \geq 1$  be an increasing sequence of zeros of  $u(x)$  in positive half-line  $[0, \infty)$ . If for any positive constant  $M$ ,*

$$\int_r^{r+M} |q(s\omega)|s^{N-1} ds \rightarrow 0, \quad \text{as } s \rightarrow \infty, \tag{4.6}$$

then,  $r_{n+1} - r_n \rightarrow \infty$ , as  $n \rightarrow \infty$ .

*Proof.* Relying on this fact that  $\{r_n\omega\}_{n=1}^\infty$  is a sequence of zeros of the oscillatory solution  $u(x)$  of the conformable fractional boundary value problem (3.21)-(3.22), we can apply Theorem 3.5 for each consecutive zeros  $r_n\omega$  and  $r_{n+1}\omega$ . In this case we reach to the following multivariate Lyapunov inequality

$$\int_{S^{N-1}} \left( \int_{r_n}^{r_{n+1}} |q(r\omega)|r^{N-1} dr \right) d\omega > \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \left( \frac{1}{r_{n+1} - r_n} \right)^{\alpha-1} \left( \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \right). \tag{4.7}$$

Now, suppose on contrary that for a large enough index  $n$ , there exists a subsequence  $\{r_{n_k}\omega\}_{k=1}^\infty$  of the sequence  $\{r_n\omega\}_{n=1}^\infty$  and there exists a positive constant  $M$  such hat  $r_{n_k+1} - r_{n_k} \leq M$ , for large enough  $k$ . So, we have

$$\int_{S^{N-1}} \left( \int_{r_{n_k}}^{r_{n_k+1}} |q(r\omega)|r^{N-1} dr \right) d\omega > \frac{\alpha^\alpha}{(\alpha - 1)^{\alpha-1}} \left( \frac{1}{r_{n_k+1} - r_{n_k}} \right)^{\alpha-1} \left( \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})} \right). \tag{4.8}$$

Equivalently, we conclude that

$$1 < \frac{\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}} \frac{(M(\alpha - 1))^{\alpha-1}}{\alpha^\alpha} \int_{S^{N-1}} \underbrace{\left( \int_{r_{n_k}}^{r_{n_k+1}} |q(r\omega)|r^{N-1} dr \right)}_{\rightarrow 0} d\omega \rightarrow 0,$$

as  $k \rightarrow \infty$ . The resulting algebraic contradiction, completes the proof.  $\square$

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