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A NEW OSCILLATORY CRITERION FOR THE GENERALIZED HILL'S EQUATION

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Abstract. In this note we use an oscillatory theorem for the second order linear ordinary differential equation in order to establish an oscillatory criterion for the generalized Hill's equation. We formulate a hypothesis about representation of the sum of periodic functions with rational dependent periods by a sum of periodic functions with rational independent periods.

1. Introduction

Let $q(t)$ be a real valued continuous function on $[t_0;+\infty)$. Consider the equation

$$\phi''(t)+q(t)\phi(t)=0, \quad t \geqslant t_0. \tag{1}$$

Hereafter we will consider only the real solutions of this and other equations.

DEFINITION 1. Eq. (1) is said to be oscillatory if its each solution has arbitrarily large zeroes.

The study of the oscillatory behavior of Eq. (1) is an important problem of the qualitative theory of differential equations and many works are devoted to it (see [1] and cited works therein, [2 -12]).

In the case of periodic function $q(t)=q_T(t)$, where $T>0$ is the minimal period of $q_T(t)$, Eq. (1) was first studied by G. W. Hill (the Hill's equation) in connection with motion of the moon in a periodic gravitation field (see [13]):

$$\phi''(t)+q_T(t)\phi(t)=0, \quad t \geqslant t_0. \tag{$\mathscr{H}$}$$

In particular for $q_T(t)=a+b\cos t$, $t \geqslant t_0$, where a and $b \neq 0$ are some real constants, Eq. $(\mathscr{H})$ was firstly studied by M. Emile Mathieu in 1886 (the Mathieu's equation), which has very important applications (see for example [14]). One of the generalizations of the Hill's equation is Eq. (1) with $q(t)=q_{T_1}(t)+q_{T_2}(t)$, $t \geqslant t_0$, where $q_{T_1}(t)$

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and $q_{T_2}(t)$ are periodic functions with the rational independent periods T_1 and T_2 respectively:

$$\phi''(t) + [q_{T_1}(t) + q_{T_2}(t)]\phi(t) = 0, \quad t \geqslant t_0. \qquad (\widetilde{\mathscr{H}})$$

It is well known that (see for example [15]), that Eq. $(\mathscr{H})$ is oscillatory if

$$\int\limits_{t_0}^{t_0+T} q_T(\tau)d\tau \geqslant 0.$$

This result was generalized in [12], where it was shown that Eq. $(\widetilde{\mathscr{H}})$ is oscillatory if

$$\frac{1}{T_1}\int\limits_{t_0}^{t_0+T_1} q_{T_1}(\tau)d\tau + \frac{1}{T_2}\int\limits_{t_0}^{t_0+T_2} q_{T_2}(\tau)d\tau \geqslant 0.$$

In this note we use an oscillatory theorem, proven in [12], for establishing an oscillatory criterion for a new generalization of Eq. $(\widetilde{\mathscr{H}})$.Our result is a generalization of the above mentioned result for Eq. $(\widetilde{\mathscr{H}})$.

2. Auxiliary propositions

In this section we formulate the oscillation theorem from [12] and prove an important lemma. They will be used in the section 3 for establishing an oscillatory criterion for the generalized Hill's equation.

Denote by Ω the set of positive and continuously differentiable functions on $[t_0;+\infty)$. For any $f \in \Omega$ denote

$$I_{q,f} \equiv \int\limits_{t_0}^{+\infty} \exp\left\{\int\limits_{t_0}^{t} \frac{d\tau}{f(\tau)} \int\limits_{t_0}^{\tau} \left[2f(s)q(s) - \frac{1}{2}\frac{f'(s)^2}{f(s)}\right] ds\right\} dt.$$

Denote $A^{\pm}_{q,\lambda} \equiv \left\{t \geqslant t_0 : \pm\left(\lambda + \int\limits_{t_0}^{t} q(\tau)d\tau\right) \geqslant 0\right\}$, $\lambda \in R$.

THEOREM 1. *Let for some $f \in \Omega$ the following conditions be satisfied:*
1). $I_{q,f} = +\infty$;
2). there exists an infinitely large sequence $\{\theta_n\}_{n=1}^{+\infty}$ such, that

$$\sup_{n\geqslant 1}\left\{\frac{1}{f(\theta_n)}\int\limits_{t_0}^{\theta_n}\left[4f(\tau)q(\tau) - \frac{f'(\tau)^2}{f(\tau)}\right]d\tau - 4\int\limits_{t_0}^{\theta_n} q(\tau)d\tau\right\} < +\infty,$$

and let for some $\lambda \in R$
3). $\int\limits_{A^{+}_{q,\lambda}} d\tau = +\infty$; 4). $\int\limits_{A^{-}_{q,\lambda}} \left(\lambda + \int\limits_{t_0}^{\tau} q(s)ds\right)^2 d\tau = +\infty$;
Then Eq. (1) is oscillatory.

See proof in [12].

DEFINITION 2. The numbers $a_1,...,a_n$ are said to be rational independent (or linearly independent over the field of rational numbers), if for the arbitrary not all zero integers $j_1,...,j_n$, the inequality $\sum\limits_{k=1}^{n} j_k a_k \neq 0$ holds.

REMARK 1. The rational independence of the numbers a_1 and a_2 means that the relation a_1/a_2 is irrational.

LEMMA 1. *Let the positive numbers $T_1,...,T_n$ be rational independent. Then for each $\varepsilon > 0$ and for the arbitrary real numbers $\xi_1,...,\xi_n$ there exist infinitely large sequences of positive integers $\{m_{kj}\}_{j=1}^{+\infty}$, $k=\overline{1,n}$ such, that*

$$|m_{1j}T_1 - m_{kj}T_k - \xi_k| \leqslant \varepsilon,\ k=\overline{2,n},\ j=1,2,.... \tag{2}$$

Proof. For an arbitrary real number x denote by $\{x\}$ its fractional part, and by $[x]$ - its integer part ($x=[x]+\{x\},\ \{x\}\in[0;1)$). Since $T_1,...,T_n$ are rational independent (see [16], p 59, Theorem 6.3 and Example 6.1) the set of points $\left(\left\{m\frac{T_1}{T_2}\right\},...,\left\{m\frac{T_1}{T_n}\right\}\right)$, $m=1,2,...$ is everywhere dense in $[0;1]^{n-1}$. Therefore there exists an infinitely large sequence of positive integers $m_{1j},\ j=1,2,...$ such that

$$\left|\left\{m_{1j}\frac{T_1}{T_k}\right\} - \left\{\frac{\xi_k}{T_k}\right\}\right| < \frac{\varepsilon}{T_k},\ k=\overline{2,n},\ j=1,2,.... \tag{3}$$

Denote: $m_{kj} \equiv \left[m_{1j}\frac{T_1}{T_k}\right] - \left[\frac{\xi_k}{T_k}\right]$, $k=\overline{2,n},\ j=1,2,....$ We will assume $m_{1j},\ j=1,2,....$ so large that $m_{kj}>0,\ k=\overline{2,n},\ j=1,2,...$. It is evident that the sequences $\{m_{kj}\}_{j=1}^{+\infty}$, $k=\overline{2,n}$ are infinitely large. From (3) we have: $\left|m_{1j}\frac{T_1}{T_k} - m_{kj} - \frac{\xi_k}{T_k}\right| < \frac{\varepsilon}{T_k}$, $k=\overline{2,n}$, $j=1,2,....$ From here it follows (2). The lemma is proved. □

3. Oscillation criterion for the generalized Hill's equation

Let $H_0(t),\ H_1(t),...,H_n(t)$ be real valued continuous functions on $[t_0;+\infty)$ and let $H(t)\equiv\sum\limits_{k=0}^{n} H_k(t),\ t\geqslant t_0$. Consider the equation

$$\phi''(t) + H(t)\phi(t) = 0, \quad t \geqslant t_0. \tag{4}$$

THEOREM 2. *Let the integral $\int\limits_{t_0}^{+\infty} H_0(\tau)d\tau$ be convergent (conditionally), and let $H_1(t),...,H_n(t)$ be periodic functions with the rational independent periods $T_1,...,T_n$ respectively such that $H_1(t)\not\equiv 0$, and*

$$\sum_{k=1}^{n} \frac{1}{T_k} \int\limits_{t_0}^{t_0+T_k} H_k(\tau)d\tau \geqslant 0. \tag{5}$$

Then Eq. (4) is oscillatory.

Proof. Let us prove the theorem only for the case

$$\int_{t_0}^{t_0+T_k} H_k(\tau)d\tau = 0, \quad k = \overline{1,n}. \tag{6}$$

The proof in the general case can be derived from the realized proof by using the Sturm comparison criterion (see [17], p. 332). Let $h_k(t) \equiv \int_{t_0}^{t} H_k(\tau)d\tau,\ t \geqslant t_0,\ k = \overline{1,n}$. It is easy to derive from (6) that $h_k(t)$ is a periodic function with period T_k $(k = \overline{1,n})$. Denote $\overline{h_k} \equiv \frac{1}{T_k}\int_{t_0}^{t_0+T_k} h_k(\tau)d\tau,\ k = \overline{1,n}$. Then

$$h_k(t) = \overline{h_k} + h_k^0(t),\ t \geqslant t_0,\ k = \overline{1,n}, \tag{7}$$

where

$$\int_{t_0}^{t_0+T_k} h_k^0(\tau)d\tau = 0,\ k = \overline{1,n}. \tag{8}$$

By virtue of mean value theorem the equality $\overline{h_k} = h_k(\xi_k)$ holds for some $\xi_k \in [t_0; t_0 + T_k],\ (k = \overline{1,n})$. Then since $h_k(t) = h_k(\xi_k) + \int_{\xi_k}^{t} H_k(\tau)d\tau,\ t \geqslant t_0,\ k = \overline{1,n}$, from (6) and (7) it follows, that

$$h_k^0(t) = \int_{\xi_k + mT_k}^{t} H_k(\tau)d\tau,\ t \geqslant t_0,\ k = \overline{1,n}, \tag{9}$$

for each $m = 0, 1, \ldots$. Denote $M \equiv \min\{M_1, M_2\}$, where $M_1 \equiv \Big|\min\limits_{t\in[t_0;t_0+T_1]} h_1^0(t)\Big|$, $M_2 \equiv \max\limits_{t\in[t_0;t_0+T_1]} h_1^0(t)$. From (8) and $H_1(t) \not\equiv 0$ it follows that $M > 0$. Since by virtue of (6) and (9), we have $h_k^0(\xi_k) = 0$, and therefore, for enough small value of $\delta > 0$ the following inequality holds

$$|h_k^0(t)| < \frac{M}{8},\ t \in [\xi_k - \delta; \xi_k + \delta],\ k = \overline{2,n}. \tag{10}$$

Let ξ_0 be a minimum point of the function $h_1^0(t)$. We have:

$$\int_{\xi_0}^{t} H_1(\tau)d\tau = -h_1(\xi_0) + h_1(t) = -h_1^0(\xi_0) + h_1^0(t),\ t \geqslant t_0. \tag{11}$$

It is evident that

$$-h_1^0(\xi_0) \geqslant M. \tag{12}$$

Since the integral $\int\limits_{t_0}^{+\infty} H_0(\tau)d\tau$ is convergent we chose $T \geqslant t_0$ so large that

$$\left|\int\limits_{t}^{+\infty} H_0(\tau)d\tau\right| < \frac{M}{8}, \quad t \geqslant T. \tag{13}$$

Since $T_1,...,T_n$ are rational independent on the basis of Lemma 1 we chose the positive integers n_0 and m_k, $k=\overline{2,n}$ such that

$$|\xi_k + m_k T_k - \xi_0 - n_0 T_1| < \delta, \quad k=\overline{2,n}; \tag{14}$$

$$\xi_0 + n_0 T_1 \geqslant T, \tag{15}$$

and take $t_1 \equiv \xi_0 + n_0 T_1$. Denote: $g_k(t) \equiv \int\limits_{t_1}^{t} H_k(\tau)d\tau$, $t \geqslant t_1$, $k=\overline{0,n}$. By (13) and (15) we have:

$$|g_0(t)| \leqslant \frac{M}{4}, \quad t \geqslant t_1. \tag{16}$$

By virtue of (6) and (11) the following equality takes place

$$g_1(t) = -h_1^0(\xi_0) + h_1^0(t), \quad t \geqslant t_1. \tag{17}$$

It is evident that $g_k(t) = h_k(t) - h_k(t_1) = h_k^0(t) - h_k^0(t_1)$, $t \geqslant t_1$, $k=\overline{2,n}$, or due to (9)

$$g_k(t) = h_k^0(\xi_k + m_k T_k) - h_k^0(t_1) + h_k^0(t), \ t \geqslant t_1, \ k=\overline{2,n}. \tag{18}$$

By virtue of (14) we will assume $\delta > 0$ so small that

$$\sum_{k=2}^{n} |h_k^0(\xi_k + m_k T_k) - h_k^0(t_1)| \leqslant \frac{M}{4}. \tag{19}$$

Denote $h(t) \equiv \int\limits_{t_1}^{t} H(\tau)d\tau$, $t \geqslant t_1$ (then $h(t) = \sum\limits_{k=0}^{n} g_k(t)$, $t \geqslant t_1$). From (16) it follows, that $g_0(t) \geqslant -\frac{M}{4}$, $t \geqslant t_1$, and from (19) we have: $\sum\limits_{k=2}^{n}[h_k(\xi_k + m_k T_k) - h_k(t_1)] \geqslant -\frac{M}{4}$. Then taking into account (12), (16), (17) and (16) that we will get: $\int\limits_{t_1}^{t} h(\tau)d\tau = \int\limits_{t_1}^{t} g_0(\tau)d\tau + (t-t_1)[-h_1^0(\xi_0) + \sum\limits_{k=2}^{n}(h_k^0(\xi_k + +m_k T_k) - h_k^0(t_1))] + \sum\limits_{k=1}^{n}\int\limits_{t_1}^{t} h_k^0(\tau)d\tau \geqslant \frac{M}{2}(t - t_1) + \sum\limits_{k=1}^{n}\int\limits_{t_1}^{t} h_k^0(\tau)d\tau, t \geqslant t_1$. From here and from (8) it follows that if $f(t) \equiv 1$ and $t_0 = t_1$ then for Eq. (4) the conditions 1) and 2) of Theorem 1 are fulfilled. Since $g_1(t),...,g_n(t)$ are periodic functions we have

$$g_k(t) = \overline{g_k} + g_k^0(t), \quad t \geqslant t_1. \tag{20}$$

where $\overline{g_k} \equiv \frac{1}{T_k} \int\limits_{t_1}^{t_1+T_k} g_k(\tau)d\tau,\ k = \overline{1,n}$,

$$\int\limits_{t_1}^{t_1+T_k} g_k^0(\tau)d\tau = 0. \tag{21}$$

Let us take $\lambda \equiv -\int\limits_{t_1}^{+\infty} H_0(\tau)d\tau - \sum\limits_{k=1}^{n} \overline{g_k}$. Then from (20) it follows that

$$\lambda + \int\limits_{t_1}^{t} H(\tau)d\tau = -\int\limits_{t}^{+\infty} H_0(\tau)d\tau + \sum_{k=1}^{n} g_k^0(t), \quad t \geqslant t_1. \tag{22}$$

Since $g_1^0(t) \not\equiv 0$, we can find the maximum and minimum points. Let η_+ and η_- be maximum and minimum point of $g_1^0(t)$ on $[t_1;t_1+T_1]$ respectively. Obviously $g_1^0(t) = h_1^0(t)$, $t \geqslant t_1$. Therefore

$$g_1^0(\eta_+) \geqslant M, \quad g_1^0(\eta_-) \leqslant -M. \tag{23}$$

Let $g_k^0(\eta_k) = 0,\ \eta_k \in [t_1;t_1+T_k],\ k = \overline{2,n}$ (the existence of $\eta_k (k = \overline{2,n})$ follows from (21)). Chose $\Delta > 0$ so small that $\Delta < \delta$ and that

$$g_1^0(t) > \frac{M}{2}, \quad |t - \eta_+| \leqslant 2\Delta; \tag{24}$$

$$g_1^0(t) < -\frac{M}{2} \quad |t - \eta_-| \leqslant 2\Delta; \tag{25}$$

$$|g_k^0(t)| \leqslant \frac{M}{8n}, \quad |t - \eta_k| \leqslant 2\Delta, \quad k = \overline{2,n} \tag{26}$$

By virtue of Lemma 1 we chose infinitely large sequences of positive integers $\{n_j^{\pm}\}_{j=1}^{+\infty}$, $\{m_{kj}^{\pm}\}_{j=1}^{+\infty}$, $k = \overline{2,n}$ such, that $|n_j^{\pm}T_1 + \eta_{\pm} - (m_{kj}^{\pm}T_k + \eta_k)| < \Delta,\ k = \overline{2,n},\ j = 1,2,\ldots$. Then from (13), (22), (24) and (26) it follows, that $\lambda + \int\limits_{t_1}^{t} H(\tau)d\tau \geqslant \frac{M}{4}$, $t \in [n_j^{+}T_1 + \eta_+ - \Delta; n_j^{+}T_1 + \eta_+ + \Delta]$, $j = 1,2,\ldots$, and from (15), (20), (22), (25) and (26) it follows that $\lambda + \int\limits_{t_1}^{t} H(\tau)d\tau \leqslant -\frac{M}{4}$, $t \in [n_j^{-}T_1 + \eta_- - \Delta; n_j^{-}T_1 + \eta_- + \Delta]$, $j = 1,2,\ldots$. From here it follows that if $t_0 = t_1$ then for Eq. (4) the conditions 3) and 4) of Theorem 1 are fulfilled. The theorem is proved. □

REMARK 2. Due to remark 1 we conclude that Theorem 2 is a generalization of Corollary 1 from [12].

REMARK 3. Let $\Delta H(t)$ be a real valued continuous function on $[t_0;+\infty)$ such that $\sup\limits_{t\geqslant t_1}\left|\int\limits_{t_1}^{t} \Delta H(\tau)d\tau\right| < \frac{M}{8}$, for each $t_1 \geqslant T$, and for some $T \geqslant t_0$ (the number M is defined in the proof of Theorem 2). Slightly changing the proof of Theorem 1 it can be shown that the equation

$$\phi''(t) + [H(t) + \Delta H(t)]\phi(t) = 0, \quad t \geqslant t_0.$$

is oscillatory if $H(t)$ satisfies of the conditions of Theorem 2.

EXAMPLE 1. Consider the generalized Mathieu equation

$$\phi''(t) + \Big[a + \sum_{k=1}^{+\infty} a_k \cos(\lambda_k t + \omega_k) + \sum_{k=1}^{m} b_k t^{\alpha_k} \cos(\mu_k t^{\beta_k}) + \sum_{k=1}^{p} c_k t^{\gamma_k} \sin(\nu_k t^{\delta_k})\Big]\phi(t) = 0, \tag{27}$$

$t \geqslant t_0$, where a, a_k, λ_k, ω_k $(k = 1,2,\dots)$, b_k, α_k, μ_k, β_k $(k = \overline{1,m})$, c_k, γ_k, ν_k, δ_k $(k = \overline{1,p})$ are some real constants. We suppose that $a_k\lambda_k \neq 0$, $k = 1,2,\dots$, $b_k\mu_k \neq 0$, $k = \overline{1,m}$, $c_k\nu_k \neq 0$, $k = \overline{1,p}$. Let $a \geqslant 0$, $\frac{1}{\lambda_1} + \dots + \frac{1}{\lambda_n}$ be rational independent, $\sum\limits_{k=1}^{+\infty} |a_k| < +\infty$, $2\sum\limits_{k=n+1}^{+\infty} \big|\frac{a_k}{\lambda_k}\big| < \frac{1}{8} \max\limits_{1 \leqslant k \leqslant n} \big\{\big|\frac{a_k}{\lambda_k}\big|\big\}$, $\alpha_k - \beta_k + 1 < 0$, $k = \overline{1,m}$, $\lambda_k - \delta_k + 1 < 0$, $k = \overline{1,p}$. Here $H_0(t) = \sum\limits_{k=1}^{m} b_k t^{\alpha_k} \cos(\mu_k t^{\beta_k}) + \sum\limits_{k=1}^{p} c_k t^{\gamma_k} \sin(\nu_k t^{\delta_k})$, $H_k(t) = \frac{a}{n} + a_k \cos(\lambda_k t + \omega_k)$, $k = \overline{1,n}$, $H(t) = \sum\limits_{k=0}^{n} H_k(t)$, $\Delta H(t) = \sum\limits_{k=n+1}^{+\infty} a_k \cos(\lambda_k t + \omega_k)$, $t \geqslant t_0$, $M = \max\limits_{1 \leqslant k \leqslant n} \big\{\big|\frac{a_k}{\lambda_k}\big|\big\}$. We can check the convergence of $\int\limits_{t_0}^{+\infty} H_0(\tau)d\tau$ by integrating by parts of the elementary integrals $\int\limits_{t_0}^{t} \tau^{\alpha_k} \cos(\mu_k \tau^{\beta_k})d\tau$, $k = \overline{1,m}$, $\int\limits_{t_0}^{t} \tau^{\gamma_k} \sin(\nu_k \tau^{\delta_k})d\tau$, $k = \overline{1,p}$, and then tending t to $+\infty$. Obviously the minimal periods of $H_1(t),\dots,H_n(t)$ are $T_1 = \frac{2\pi}{\lambda_1},\dots,T_n = \frac{2\pi}{\lambda_n}$ respectively. Therefore from the rational independence of $\frac{1}{\lambda_1},\dots,\frac{1}{\lambda_n}$ it follows the rational independence of $T_1,\dots,T_n$, It is not difficult to check that $\int\limits_{t_0}^{t_0+T_k} H_k(\tau)d\tau = \frac{a}{n} \geqslant 0$, $k = \overline{1,n}$. From the convergence of $\sum\limits_{k=n+1}^{+\infty} |a_k|$ it follows that $\Delta H(t)$ is a continuous function on $[t_0;+\infty)$ and $\big|\int\limits_{t_1}^{t} \Delta H(\tau)d\tau\big| \leqslant \sum\limits_{k=n+1}^{+\infty} \big|\frac{a_k}{\lambda_k}[\sin(\lambda_k t + \omega_k) - \sin(\lambda_k t_1 + \omega_k)]\big| \leqslant 2\sum\limits_{k=n+1}^{+\infty} \big|\frac{a_k}{\lambda_k}\big|$, $t \geqslant t_1 \geqslant t_0$. Without loss of generality we can take that $\big|\frac{a_1}{\lambda_1}\big| = \max\limits_{1 \leqslant k \leqslant n} \big\{\big|\frac{a_k}{\lambda_k}\big|\big\}$ and $t_0 = -\frac{\omega_1}{\lambda_1}$. Then obviously (recall the definition of M; see above) $M = \big|\min\limits_{t \in [t_0;t_0+T_1]} \int\limits_{t_0}^{t} a_1 \cos(\lambda_1 \tau + \omega_1)d\tau\big| = \max\limits_{t \in [t_0;t_0+T_1]} \int\limits_{t_0}^{t} a_1 \cos(\lambda_1 \tau + \omega_1)d\tau = \big|\frac{a_1}{\lambda_1}\big|$. Hence we have $\sup\limits_{t \geqslant t_1} \big|\int\limits_{t_1}^{t} \Delta H(\tau)d\tau\big| < \frac{M}{8}$, $t \geqslant t_1 \geqslant t_0$.. Thus we see that all the conditions of Theorem 2 and Remark 3 are fulfilled for Eq. (27). Therefore Eq. (27) is oscillatory.

It is easy to verify that the oscillation criterion of Ph. Hartman (see [2], p. 138, Theorem 52) is not applicable to Eq. (27) (in general to Eq. (4)). and the oscillation criterion of I. V. Kamenev [3] is not applicable to Eq (27) for $a = 0$ (in general to Eq. (4) for $\sum\limits_{k=1}^{n} \frac{1}{T_k} \int\limits_{t_0}^{t_0+T_k} H_k(\tau)d\tau = 0$). It is hard to verify (if it is possible) the oscillatory behavior of Eq. (27) by using oscillation criteria of J. S. W. Wong (see [2], p. 100, Theorem 1), Y. Jan (see [4], Theorem 1), M. K. Kwong (see [6], p. 16, Theorem 11), Q. Kong (see [7], p. 265, Theorem 2.3), A. Elbert (see [10], p. 2, Theorem 2).

REMARK 4. **(a hypothesis)**. If T_1 and T_2 are rational dependent then $F_1(t) \equiv H_1(t) + H_2(t),\ t \geqslant t_0,$ is a periodic function with some period $\widetilde{T}_1 > 0$. We conjecture that this statement is true for the general case, i. e. if $T_1, ..., T_n$ are rational dependent then there exist periodic functions $F_1(t), ..., F_m(t),\ (m < n)$ with rational independent periods $\widetilde{T}_1, ..., \widetilde{T}_m$ respectively such that $\sum\limits_{k=1}^{n} H_k(t) = \sum\limits_{k=1}^{m} F_k(t),\ t \geqslant t_0$. If this statement is true then it is easy to show that $\sum\limits_{k=1}^{n} \frac{1}{T_k} \int\limits_{t_0}^{t_0+T_k} H_k(\tau)d\tau = \sum\limits_{k=1}^{m} \frac{1}{\widetilde{T}_k} \int\limits_{t_0}^{t_0+\widetilde{T}_k} F_k(\tau)d\tau$, and therefore the condition of rational independence of $T_1, ..., T_n$ can be omitted from the formulation of Theorem 2.

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