

## VERY WEAK SOLUTIONS OF LINEAR ELLIPTIC PDES WITH SINGULAR DATA AND IRREGULAR COEFFICIENTS

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*Abstract.* In this article it is shown that linear elliptic PDEs admit very weak solutions for rather singular data – like non-integrable right hand sides or singular Neumann boundary conditions – not only in case of continuous coefficients, but even for general bounded measurable coefficients. This is rather astonishing, as under such weak assumptions on the coefficients generally strong solutions do not exist, thus the duality between very weak solutions and strong solutions seems to indicate that very weak solutions do not exist either. We circumvent this problem by using an appropriate functional analytic setting and particularly Hölder continuity of weak solutions established by de Giorgi - Nash - Moser to obtain existence of very weak solutions to singular data for irregular coefficients.

### 1. Introduction

Even for rather singular data, linear elliptic PDEs of second order on smooth domains  $\Omega \subset \mathbb{R}^N$  in divergence form

$$-\operatorname{div}(a_2 \nabla u + ua_1) + a'_1 \cdot \nabla u + a_0 u = f_0 - \operatorname{div}(f_1) \quad \text{in } \Omega \quad (1)$$

may admit solutions, but these may be very weak and not weak solutions. For example, consider BREZIS' problem [2, 3], i.e. Poisson's equation  $-\Delta u = f$  given by (1) with constant coefficients  $a_2 = 1$ ,  $a_1 = 0 = a'_1$ ,  $a_0 = 0$ ,  $f_0 = f$ ,  $f_1 = 0$ , under homogeneous Dirichlet boundary conditions  $u = 0$  on  $\partial\Omega$  for a right hand side (r.h.s.)  $f \in L^1(\Omega, \delta)$ , where  $\delta(x) := \operatorname{dist}(x, \partial\Omega)$  is the distance to the boundary and

$$L^1(\Omega, \delta) := \{f : \Omega \rightarrow \mathbb{R} \text{ measurable} \mid \int_{\Omega} f(x) \delta(x) dx < +\infty\}$$

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is the space of functions which are integrable w.r.t. the weight  $\delta$ . In this case, for every  $f \in L^1(\Omega, \delta)$  existence and uniqueness of a very weak solution  $u \in L^1(\Omega)$  satisfying

$$-\int_{\Omega} u \cdot \Delta v dx = \int_{\Omega} f v dx$$

for every  $v \in C^2(\overline{\Omega})$  with  $v = 0$  on  $\partial\Omega$  is known, but there exist smooth domains  $\Omega$  and right hand sides  $f \in L^1(\Omega, \delta)$ ,  $f \notin L^1(\Omega)$ , such that this very weak solution  $u$  does not have a derivative  $\nabla u \in L^1(\Omega)$ , i.e.  $u \notin W^{1,1}(\Omega)$ , and thus is not a weak solution [1, 5]. Similarly, for Poisson’s equation RAKOTOSON and the author [14] have proved existence of very weak solutions, which satisfy in a rigorous sense *singular Neumann boundary conditions*  $\frac{\partial u}{\partial \bar{n}} = -\infty$  on  $\partial\Omega$  and are generally not weak solutions, see also [17]. The proof in [14] relies on existence of strong solutions to smooth data and thus does not generalize to merely bounded measurable coefficients. Yet, the following one-dimensional example shows that even in case of discontinuous coefficients weak resp. very weak solutions to singular data can exist.

EXAMPLE 1. Let  $a_2(x) := \begin{cases} \frac{1}{2} & \text{if } x < 0 \\ 2 & \text{if } x > 0 \end{cases}$ ,  $a_1 = 0 = a'_1$ ,  $a_0 = 0$ , then

$$u(x) := C - \begin{cases} (1-x)\ln(1-x) + (x+1)\ln(x+1) & \text{if } x < 0 \\ \frac{1}{4}((1-x)\ln(1-x) + (x+1)\ln(x+1)) & \text{if } x > 0 \end{cases}$$

solves for arbitrary  $C \in \mathbb{R}$  the equation  $-(a_2 u')' = \frac{1}{1-x^2}$  on  $\Omega := (-1, 1)$ . Note that the right hand side  $f(x) = \frac{1}{1-x^2}$  is not integrable over  $(-1, 1)$  and that the solution  $u \in C^1(\Omega)$  is not twice differentiable at  $x = 0$  while  $a_2 u' = \frac{1}{2}(\ln(1-x) - \ln(x+1))$  is smooth near  $x = 0$ . The functions  $u$  satisfy

$$\lim_{x \rightarrow \pm 1} u'(x) = \mp \infty, \quad \text{i.e.} \quad \frac{\partial u}{\partial \bar{n}} = -\infty \quad \text{on } \partial\Omega,$$

and the equation is not only satisfied pointwisely, but even in the very weak sense that

$$-\int_{-1}^1 u \cdot (a_2(x)v')' dx = \int_{-1}^1 \frac{1/2}{x-(-1)}(v(x) - v(-1)) dx - \int_{-1}^1 \frac{1/2}{x-1}(v(x) - v(1)) dx$$

holds for every  $v \in C^1([-1, 1])$  with  $v'(-1) = 0 = v'(1)$  such that  $a_2 v' \in C^1([-1, 1])$ . The right hand side of this equation is an example of a singular integral acting on  $v$ . If the equation is changed to  $-(a_2 u')' = \frac{1}{(1-x^2)^2}$ , i.e. the right hand side is squared, then

$$u(x) := C + \begin{cases} \frac{x}{2}(\ln(1-x) - \ln(x+1)) & \text{if } x < 0 \\ \frac{x}{8}(\ln(1-x) - \ln(x+1)) & \text{if } x > 0 \end{cases}$$

is for arbitrary  $C \in \mathbb{R}$  a very weak solution  $u \in L^1(\Omega)$  with  $\lim_{x \rightarrow \pm 1} u'(x) = \mp \infty$ , but the derivative  $u'$  is not integrable over  $(-1, 1)$ , i.e.  $u \notin W^{1,1}(\Omega)$  is not a weak solution.

### 1.1. Outline

For linear elliptic equations with discontinuous coefficients, very weak solutions have been introduced in the classical work of STAMPACCHIA [19]. After fixing notations in section 2, in section 3 we introduce *singular integrals* on  $\Omega \times \partial\Omega$  and make clear, why these singular integrals allow to model singular data like non-integrable right hand sides or infinite outer normal derivatives on the boundary. In section 4 we formulate an abstract functional analytic setting, which allows to deduce existence of very weak solutions via duality from existence of a slightly weakened variant of strong solutions. To prove existence of very weak solutions for irregular coefficients, which neither are uniformly continuous nor of vanishing mean oscillation, such a weakening of the notion of strong solutions is necessary, because existence of twice weakly differentiable solutions to right hand sides  $f_0 \in L^q(\Omega)$ ,  $1 < q < \infty$ ,  $f_1 = 0$ , under homogeneous boundary conditions breaks down for such coefficients. Finally, in section 5 we apply this abstract setting to prove existence of very weak solutions. Particularly, we obtain the following theorem about existence of very weak solutions in the case where singular integrals with kernel  $\underline{h}$  as r.h.s. occur simultaneously with measurable bounded coefficients. In the proof, we use Hölder continuity of weak solutions established by DE GIORGI, NASH and MOSER to guarantee that the singular integral is continuous on a suitable space of test functions. Our main theorem reads as follows.

**THEOREM 1.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded  $C^2$ -domain with Dirichlet resp. Neumann boundary  $\partial\Omega = \Gamma_D \dot{\cup} \Gamma_N$ . Assume that  $a_2 \in L^\infty(\Omega, \mathbb{R}^{N \times N})$  satisfies  $a_2(x)\xi \cdot \xi \geq c|\xi|^2$  with a constant  $c > 0$  for arbitrary  $x \in \Omega$ ,  $\xi \in \mathbb{R}^N$ , and let  $a_1, a'_1 \in L^\infty(\Omega, \mathbb{R}^N)$ ,  $a_0 \in L^\infty(\Omega, \mathbb{R})$ . If the weak realization  $A_w : W_{\Gamma_D}^{1,p}(\Omega) \rightarrow (W_{\Gamma_D}^{1,q}(\Omega))^*$  of (1) under homogeneous Dirichlet boundary conditions  $u = 0$  on  $\Gamma_D$  is a topological isomorphism for  $1 < p < \infty$ ,  $q := p' > \frac{N}{2}$ , then there is an  $\alpha > 0$  such that to every  $f_1 \in L^p(\Omega, \mathbb{R}^N)$  and every measurable function  $\underline{h}$  on  $\Omega \times \Gamma_N$  with  $\underline{h}(x,y)|x-y|^\alpha \in L^1(\Omega \times \Gamma_N)$  as kernel of a singular integral on the r.h.s. of (1) there exists a unique very weak solution  $u \in L^p(\Omega)$  in the sense that*

$$\begin{aligned} & \int_{\Omega} u \left( -\operatorname{div} (a_2^T \nabla v + v a'_1) + a_1 \cdot \nabla v + a_0 v \right) dx \\ &= \int_{\Omega} f_1(x) \cdot \nabla v(x) dx + \int_{\Omega} \int_{\Gamma_N} \underline{h}(x,y) (v(x) - v(y)) dS(y) dx \end{aligned}$$

for every  $v \in W_{\Gamma_D}^{1,q}(\Omega)$  with  $a_2^T \nabla v + v a'_1 \in W^{1,q}(\Omega)$  and  $(a_2^T \nabla v + v a'_1) \cdot \vec{n} = 0$  on  $\Gamma_N$ .

Particularly, if the kernel function  $\underline{h}$  is such that  $h(y) := -\int_{\Omega} \underline{h}(x,y) dx = -\infty$  at points  $y$  in a subset  $\Gamma \subset \Gamma_N$  of positive measure, while  $f_0(x) := \int_{\Gamma_N} \underline{h}(x,y) dS(y)$  is finite for a.e.  $x \in \Omega$ , then the function  $u$  can be considered as a very weak solution of (1) under singular Neumann boundary conditions  $(a_2 \nabla u + u a_1) \cdot \vec{n} = -\infty$  on  $\Gamma$ .

### 2. Preliminaries

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with  $C^2$ -boundary  $\partial\Omega = \Gamma_D \dot{\cup} \Gamma_N$  decomposed into (possibly empty) connected components  $\Gamma_D, \Gamma_N$ . Consider equation (1) with co-

efficients  $a_2 \in L^\infty(\Omega, \mathbb{R}^{N \times N})$ ,  $a_1, a'_1 \in L^\infty(\Omega, \mathbb{R}^N)$ ,  $a_0 \in L^\infty(\Omega, \mathbb{R})$ , where the principal part satisfies the strong ellipticity condition  $a_2(x)\xi \cdot \xi \geq c|\xi|^2$  with a constant  $c > 0$  independent of  $x$ . Note that the term  $-\operatorname{div}(a_2 \nabla u + ua_1)$  in (1) is merely "formal" – as function on  $\Omega$  it generally makes no sense, because products of measurable bounded functions  $a_2$  resp.  $a_1$  with  $\nabla u$  resp.  $u$  have in general merely a distributional derivative. Therefore, we need a weak realization of the linear elliptic PDE (1) as an equation

$$A_w u = b \tag{2}$$

with  $A_w$  defined by

$$\langle A_w u, v \rangle := \int_\Omega (a_2 \nabla u) \cdot \nabla v + ua_1 \cdot \nabla v + va'_1 \cdot \nabla u + a_0 uv \, dx \tag{3}$$

either as continuous linear operator  $A_w : X \rightarrow X^*$  on a Hilbert space  $X$  like e.g.

- (i)  $X := W_0^{1,2}(\Omega) = H_0^1(\Omega)$  for Dirichlet boundary conditions  $u = 0$  on  $\partial\Omega$ ,
- (ii)  $X := W^{1,2}(\Omega) = H^1(\Omega)$  for Neumann boundary conditions on  $\partial\Omega$ ,
- (iii)  $X := W_{\Gamma_D}^{1,2}(\Omega) = \{u \in W^{1,2}(\Omega) \mid u = 0 \text{ on } \Gamma_D\}$  for mixed boundary conditions, where  $\partial\Omega = \Gamma_D \cup \Gamma_N$  is decomposed into a Dirichlet part and a Neumann part,

or as continuous linear operator  $A_w : X_p \rightarrow X_q^*$  between Banach spaces  $X_p$  and  $X_q^*$  like e.g.  $X_p := W_0^{1,p}(\Omega)$  for  $1 < p < \infty$  and  $X_q := W_0^{1,q}(\Omega)$ ,  $q := p' = \frac{p}{p-1}$ , where the dual space of  $X_q = W_0^{1,q}(\Omega)$  is usually denoted by  $X_q^* = W^{-1,p}(\Omega)$ .

In this weak formulation, the right hand side of (1) and the right hand side of inhomogeneous Neumann boundary conditions

$$(a_2 \nabla u + ua_1) \cdot \vec{n} = h + f_1 \cdot \vec{n} \quad \text{on } \Gamma_N \tag{4}$$

constitute the right hand side of (2) defined by

$$\langle b, v \rangle = \int_\Omega f_0(x)v(x) + f_1(x) \cdot \nabla v(x) \, dx + \int_{\Gamma_N} h(y)v(y) \, dS(y), \tag{5}$$

either as a continuous linear functional  $b \in X^*$  on the Hilbert space  $X$  or as continuous linear functional  $b \in X_q^*$  on the Banach space  $X_q$  under appropriate conditions, like e.g.  $f_0 \in L^{(2^*)'}(\Omega, \mathbb{R})$ ,  $f_1 \in L^2(\Omega, \mathbb{R}^N)$ ,  $h \in L^2(\partial\Omega)$  in the case  $X = W^{1,2}(\Omega)$ ,  $\Gamma_N = \partial\Omega$ , due to Sobolev's embedding  $W^{1,2}(\Omega) \subset L^{2^*}(\Omega)$ ,  $\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N}$ , and the trace theorem  $W^{1,2}(\Omega) \ni v \mapsto v|_{\partial\Omega} \in L^2(\partial\Omega)$ .

### 3. Singular integrals

However, note that not  $L^q(\partial\Omega)$  but the Sobolev-Slobodeckij space

$$W^{s,q}(\partial\Omega) := \left\{ v \in L^q(\partial\Omega) \mid \right.$$

$$[v]_{s,q,\partial\Omega} := \left( \int_{\partial\Omega} \int_{\partial\Omega} \frac{|v(x) - v(y)|^q}{|x - y|^{sq+n-1}} dS(y) dS(x) \right)^{\frac{1}{q}} < +\infty \quad (6)$$

with  $s = 1 - \frac{1}{q} = \frac{1}{p}$  is the image of  $W^{1,q}(\Omega)$  under the trace operator  $v \mapsto v|_{\partial\Omega}$ . Therefore, we additionally could add on the right hand side of (5) a singular integral

$$\int_{\partial\Omega} \int_{\partial\Omega} \underline{b}(x,y) (v(x) - v(y)) dS(y) dS(x) \quad (7)$$

with kernel function  $\underline{b}$  satisfying  $\underline{b}(x,y)|x - y|^{s + \frac{n-1}{q}} \in L^p(\partial\Omega \times \partial\Omega)$  but not necessarily  $\underline{b}(x,y) \in L^p(\partial\Omega \times \partial\Omega)$  without destroying  $b \in (W^{1,q}(\Omega))^*$  due to

$$\left| \int_{\partial\Omega} \int_{\partial\Omega} \underline{b}(x,y) |x - y|^{s + \frac{n-1}{q}} \frac{v(x) - v(y)}{|x - y|^{s + \frac{n-1}{q}}} dS(y) dS(x) \right| \leq \| \underline{b}(x,y) |x - y|^{s + \frac{n-1}{q}} \|_{L^p(\partial\Omega \times \partial\Omega)} [v]_{s,q,\partial\Omega}. \quad (8)$$

Note that (7) reduces for kernels  $\underline{b}(x,y) := h_1(x)h_2(y)$  with  $h_1, h_2 \in L^2(\partial\Omega)$  and corresponding  $C_i := \int_{\partial\Omega} h_i(x) dS(x)$ ,  $i = 1, 2$ , to

$$\int_{\partial\Omega} (C_2 h_1(x) - C_1 h_2(x)) v(x) dS(x),$$

i.e. just to the weak realization of an ordinary r.h.s.  $C_2 h_1 - C_1 h_2 \in L^2(\partial\Omega)$  in Neumann boundary conditions (4). Yet, for  $\underline{b}$  such that  $\int_{\partial\Omega} \underline{b}(x,y) dS(x) = \infty$  at points  $y$  in a subset  $\Gamma \subset \partial\Omega$  of positive  $(n - 1)$ -dimensional measure, while  $\int_{\partial\Omega} \underline{b}(x,y) dS(y)$  is finite for a.e.  $x \in \Gamma$ , there formally holds the singular Neumann boundary condition

$$(a_2 \nabla u + ua_1) \cdot \vec{n} = -\infty \quad \text{on } \Gamma \quad (9)$$

for the weak solution  $u \in W^{1,p}(\Omega)$  of (1) obtained by solving (2) with (3) and r.h.s.  $b$  containing a singular integral with such a kernel  $\underline{b}$ . However, note that this kind of singular Neumann boundary condition is due to an additional “fractional divergence” of a (fractionally) non-differentiable function on the right hand side of (4). In fact, assume

$$\underline{b}(x,y) = \frac{h_s(x)}{|x - y|^{s + \frac{n-1}{q}}}$$

for a function  $h_s(x) \in L^p(\partial\Omega)$ , then the singular integral (7) reads as

$$\int_{\partial\Omega} h_s(x) (D_{\partial\Omega}^s v)(x) dS(x)$$

with the operator

$$(D_{\partial\Omega}^s v)(x) := \int_{\partial\Omega} \frac{v(x) - v(y)}{|x - y|^{s + \frac{n-1}{q}}} dS(y)$$

modeling a fractional derivative on  $\partial\Omega$ . Thus, on the right hand side of (4) such a singular integral induces an additional “fractional divergence”  $(D_{\partial\Omega}^s)^* h_s$  of a  $L^p$ -function  $h_s$  on  $\partial\Omega$ , which may not have a finite value at points  $x \in \Gamma \subset \partial\Omega$ . Similarly, singular integrals

$$\int_{\Omega} \int_{\Omega} \underline{b}(x, y) (v(x) - v(y)) \, dy \, dx \tag{10}$$

with kernel function  $\underline{b}$  satisfying  $\underline{b}(x, y)|x - y|^{s+\frac{n}{q}} \in L^p(\Omega \times \Omega)$  can be added on the right hand side of (5) without destroying  $b \in (W^{1,q}(\Omega))^*$ , as  $W^{1,q}(\Omega)$  is continuously embedded into the Sobolev-Slobodeckij space

$$W^{s,q}(\Omega) := \{v \in L^q(\Omega) \mid [v]_{s,q,\Omega} := \left( \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^q}{|x - y|^{sq+n}} \, dy \, dx \right)^{\frac{1}{q}} < +\infty\}, \tag{11}$$

for  $0 < s < 1$ . These singular integrals on  $\Omega \times \Omega$  model “fractional divergences” of (fractionally) non-differentiable functions on the right hand side of (1). For example, if  $\underline{b}(x, y) = \frac{f_s(x)}{|x - y|^{s+\frac{n}{q}}}$  with an  $L^p$ -function  $f_s$  on  $\Omega$ , then (10) is identical with  $\langle (D_{\Omega}^s)^* f_s, v \rangle$  for the operator  $(D_{\Omega}^s v)(x) = \int_{\Omega} \frac{v(x) - v(y)}{|x - y|^{s+\frac{n}{q}}} \, dy$  modeling a fractional derivative on  $\Omega$ .

### 3.1. Singular integrals on $\Omega \times \partial\Omega$

To model not only “fractional divergence” on the the right hand side of the equation (1) resp. Neumann boundary conditions (4), but “true” infinite Neumann boundary conditions (9), instead of singular integrals on  $\Omega \times \Omega$  resp.  $\partial\Omega \times \partial\Omega$  singular integrals on  $\Omega \times \partial\Omega$  can be used. In fact, assume that  $f_0 \in L^p(\Omega)$ ,  $h \in L^p(\partial\Omega)$ , can be simultaneously decomposed by  $f_0(x) = \int_{\partial\Omega} \underline{b}(x, y) \, dS(y)$  and  $h(y) = - \int_{\Omega} \underline{b}(x, y) \, dx$  for some  $\underline{b} \in L^p(\Omega \times \partial\Omega)$ , then in the case  $f_1 = 0$  the standard functional (5) on  $W^{1,q}(\Omega)$  modeling these data is identical with the singular integral

$$\langle b, v \rangle = \int_{\Omega} \int_{\partial\Omega} \underline{b}(x, y) (v(x) - v(y)) \, dS(y) \, dx \tag{12}$$

on  $\Omega \times \partial\Omega$ . To estimate, which kernels  $\underline{b}$  imply  $b \in (W^{1,q}(\Omega))^*$ , note

$$\begin{aligned} & \int_{\Omega} \int_{\partial\Omega \cap \{|x - y| < 1\}} \frac{|v(x) - v(y)|^q}{|x - y|^r} \, dS(y) \, dx \\ &= \int_{\Omega} \int_{\partial\Omega \cap B_1(0)} \left( \frac{|v(x) - v(x + z)|}{|z|^{r/q}} \right)^q \, dS(z) \, dx \\ &\leq \int_{\Omega} \int_{\partial\Omega \cap B_1(0)} \left( \int_0^1 \frac{|\nabla v(x + tz)|}{|z|^{r/q-1}} \, dt \right)^q \, dS(z) \, dx \\ &\leq \int_{\Omega} \int_{\partial\Omega \cap B_1(0)} \int_0^1 \frac{|\nabla v(x + tz)|^q}{|z|^{r-q}} \, dt \, dS(z) \, dx \\ &\leq \int_{\partial\Omega \cap B_1(0)} \int_0^1 \frac{\|\nabla v\|_{L^q(\Omega)}^q}{|z|^{r-q}} \, dt \, dS(z) \end{aligned}$$

$$\leq \int_{\partial\Omega \cap B_1(0)} \frac{\|\nabla v\|_{L^q(\Omega)}^q}{|z|^{r-q}} dS(z) \leq C \|\nabla v\|_{L^q(\Omega)}^q \tag{13}$$

for a constant  $C$  independent of  $v$  provided that  $q + n - 1 > r$  and

$$\begin{aligned} & \int_{\Omega} \int_{\partial\Omega \cap \{|x-y| \geq 1\}} \frac{|v(x) - v(y)|^q}{|x-y|^r} dS(y) dx \\ & \leq 2^{q-1} \int_{\Omega} \int_{\partial\Omega \cap \{|x-y| \geq 1\}} \frac{|v(x)|^q + |v(y)|^q}{|x-y|^r} dS(y) dx \\ & \leq 2^{q-1} \int_{\Omega} \left( \int_{\partial\Omega \cap \{|x-y| \geq 1\}} \frac{1}{|x-y|^r} dS(y) \right) |v(x)|^q dx \\ & \quad + 2^{q-1} \int_{\partial\Omega} \left( \int_{\Omega \cap \{|x-y| \geq 1\}} \frac{1}{|x-y|^r} dx \right) |v(y)|^q dS(y) \\ & \leq C \left( \|v\|_{L^q(\Omega)}^q + \|v\|_{L^q(\partial\Omega)}^q \right). \end{aligned} \tag{14}$$

Thus, even kernels  $\underline{b}$  with  $\underline{b}(x,y)|x-y|^{\alpha+\frac{n-1}{q}} \in L^p(\Omega \times \partial\Omega)$  for  $\alpha < 1$  guarantee that the singular integral (12) is a continuous linear functional acting on  $v \in W^{1,q}(\Omega)$ . Hence, if  $h(y) := -\int_{\Omega} \underline{b}(x,y) dx = -\infty$  at points  $y$  in a subset  $\Gamma \subset \partial\Omega$  of positive  $(n-1)$ -dimensional measure, then formally the singular Neumann boundary condition (9) holds for the weak solution  $u \in W^{1,p}(\Omega)$  of (1) obtained by solving (2) with the weak operator (3) and the functional  $b$  given by the singular integral with kernel  $\underline{b}$ .

Also note that in case of homogeneous Dirichlet boundary conditions  $v = 0$  on  $\partial\Omega$  we can choose  $\underline{b}(x,y) = \frac{1}{|\partial\Omega|} f(x)$ , then (12) reduces to  $\int_{\Omega} f(x)v(x)$  due to  $v(y) = 0$ .

Therefore, non-integrable r.h.s.  $f$  with  $f(x)|x-y|^{\alpha+\frac{n-1}{q}} \in L^p(\Omega \times \partial\Omega)$  for  $\alpha < 1$  and especially  $f$  with  $f(x)\delta(x)^{\alpha+\frac{n-1}{q}} \in L^p(\Omega)$  for  $\delta(x) := \text{dist}(x, \partial\Omega)$  can be considered as singular integrals acting on functions which satisfy homogeneous Dirichlet boundary conditions.

Particularly interesting is the case  $p \searrow 1$ , where  $q \nearrow +\infty$  leads to the space  $C^{0,1}(\overline{\Omega})$  of Lipschitz functions instead of  $W^{1,q}(\Omega)$ . In this case  $\underline{b}$  does not need to lie  $L^1(\Omega \times \partial\Omega)$ , but already  $\underline{b}(x,y)|x-y|^\alpha \in L^1(\Omega \times \partial\Omega)$  is sufficient for (12) being a continuous linear functional on  $C^{0,\alpha}(\overline{\Omega})$  due to

$$\begin{aligned} |\langle b, v \rangle| &= \left| \int_{\Omega} \int_{\partial\Omega} \underline{b}(x,y) |x-y|^\alpha \frac{v(x) - v(y)}{|x-y|^\alpha} dS(y) dx \right| \\ &\leq \left( \int_{\Omega} \int_{\partial\Omega} |\underline{b}(x,y)| |x-y|^\alpha dS(y) dx \right) \left( \sup_{x \in \Omega, y \in \partial\Omega} \frac{|v(x) - v(y)|}{|x-y|^\alpha} \right) \\ &\leq \|\underline{b}(x,y)|x-y|^\alpha\|_{L^1(\Omega \times \partial\Omega)} \|v\|_{C^{0,\alpha}(\overline{\Omega})} \end{aligned}$$

for  $0 < \alpha \leq 1$ . Again, if the kernel  $\underline{b}$  satisfies  $h(y) := -\int_{\Omega} \underline{b}(x,y) dx = -\infty$  at points  $y$  in a subset  $\Gamma \subset \partial\Omega$  of positive  $(n-1)$ -dimensional measure, then formally the singular Neumann boundary condition (9) holds for solutions  $u$  to such r.h.s.  $b$ . Moreover, on a space of functions with vanishing normal derivative on  $\Gamma_N$  there even may be kernels  $\underline{b}$

with worse integrability, which still induce continuous linear functionals, see Appendix B. If  $b$  is allowed to be so singular, then there may not exist a weak solution  $u$  to r.h.s.  $b$  satisfying  $u \in W^{1,1}(\Omega)$ , but just a very weak solution.

This happens in Example 1: The slightly singular  $f(x) = \frac{1}{1-x^2} \notin L^1(-1, 1)$  has a partial fraction decomposition

$$\frac{1}{1-x^2} = \frac{1/2}{x-(-1)} - \frac{1/2}{x-1},$$

which may be viewed as decomposition  $f(x) = \int_{\partial\Omega} \underline{b}(x,y) dS(y)$  with kernel function given by  $\underline{b}(x,-1) := \frac{1/2}{x-(-1)}$ ,  $\underline{b}(x,1) := -\frac{1/2}{x-1}$ , as  $\partial\Omega = \{-1, 1\}$  in the one-dimensional case  $\Omega = (-1, 1)$  and the integration w.r.t. the counting measure is just addition. Note that in this case

$$h(-1) := -\int_{-1}^1 \underline{b}(x,-1) dx = -\infty \quad \text{and} \quad h(1) := -\int_{-1}^1 \underline{b}(x,1) dx = -\infty,$$

because  $\underline{b}(x, \pm 1)$  are not integrable over  $\Omega = (-1, 1)$ . Yet,  $\underline{b}(x, \pm 1)|x - (\pm 1)|^\alpha$  are integrable for every  $\alpha > 0$ , and due to this mild kind of singularity there exist weak solutions  $u$  (which even are continuously differentiable). In contrast, the solutions  $u$  to the rather singular r.h.s.  $f(x) := \frac{1}{(1-x^2)^2}$  in Example 1 do not have an integrable derivative, i.e.  $u \notin W^{1,1}(-1, 1)$ , because the partial fraction decomposition of  $f$  consists of the terms

$$\underline{b}(x,-1) := \frac{1/4}{x-(-1)} + \frac{1/4}{(x-(-1))^2} \quad \text{and} \quad \underline{b}(x,1) := \frac{1/4}{x-1} + \frac{1/4}{(x-1)^2},$$

which merely satisfy  $\underline{b}(x, \pm 1)|x - (\pm 1)|^\alpha \in L^1(-1, 1)$  for  $\alpha > 1$ , but not for  $\alpha \leq 1$ .

Like in Example 1, also in the next example of Poisson’s equations under Neumann boundary conditions solutions are not unique.

**EXAMPLE 2.** Consider Poisson’s equation  $-\Delta u = f$  given by (1) with constant coefficients  $a_2 := 1$ ,  $a_1 := 0$ ,  $a_0 := 0$ ,  $f_0 = f$ ,  $f_1 = 0$ , under inhomogeneous Neumann boundary conditions (4) on the whole boundary  $\Gamma_N := \partial\Omega$ . Right hand sides  $f \in L^1(\Omega)$  resp.  $h \in L^1(\partial\Omega)$  have to satisfy a compatibility condition

$$\int_{\partial\Omega} h(y) dS(y) = - \int_{\Omega} f(x) dx =: C \tag{15}$$

for existence of weak solutions, as the right hand sides encoded in the functional  $b$  defined by (5) has to satisfy  $\langle b, 1 \rangle = 0$ , because the constant functions lie in the kernel of the Neumann Laplacian.  $b$  can be identified with the singular integral (12) by choosing  $\underline{b}(x,y) := \frac{1}{C} f(x) h(y) \in L^1(\Omega \times \partial\Omega)$  in the case  $C \neq 0$  resp.  $\underline{b}(x,y) := \frac{1}{|\partial\Omega|} f(x) - \frac{1}{|\Omega|} h(y)$  in the case  $C = 0$ , as then  $f(x) = \int_{\partial\Omega} \underline{b}(x,y) dS(y)$  and  $h(y) = - \int_{\Omega} \underline{b}(x,y) dx$ . Yet, if  $\underline{b}$  does not lie in  $L^1(\Omega \times \partial\Omega)$  but merely  $\underline{b}(x,y)|x-y| \in L^1(\Omega \times \partial\Omega)$ , then (12)



still defines a continuous linear functional  $b$  on  $C^{0,1}(\overline{\Omega})$ . Existence of very weak solutions for functionals  $b \in C^{0,1}(\overline{\Omega})^*$  is proved for Poisson’s equation in [14], and if the kernel  $\underline{b}$  satisfies  $\int_{\Omega} \underline{b}(x, y) dx = \infty$  at points  $y$  in a subset  $\Gamma \subset \partial\Omega$  of positive measure, then this very weak solution may be considered as a solution to the singular Neumann boundary conditions (9).

While in this example the coefficients are smooth, let us formulate an appropriate framework to prove existence of very weak solutions in the general case, where for simplicity we exclude the case of a non-trivial kernel.

#### 4. Functional analytic setting

Let  $X$  be a Hilbert space, let  $Y$  be a reflexive Banach space and assume that

(A0)  $X$  is continuously embedded into  $Y$  and dense,

i.e. we fix a continuous linear mapping  $\iota : X \rightarrow Y$  which is injective and has a dense range, and via this embedding we consider  $X$  as dense subspace of  $Y$ . As a consequence, the dual mapping  $\iota^*$  embeds  $Y^*$  continuously and dense into  $X^*$ .

Let  $A : X \rightarrow X^*$  be a continuous linear operator and assume that

(A1)  $A : X \rightarrow X^*$  has a continuous inverse  $A^{-1} : X^* \rightarrow X$ .

By Lax-Milgram, (A1) particularly holds, if  $A$  is coercive in the sense that  $\langle Au, u \rangle \geq c \|u\|_X^2$ . As a consequence of (A1), for every  $b \in X^*$  there exists a unique  $u \in X$  such that  $Au = b$ . This  $u \in X$  is called the *weak solution* to the right hand side (r.h.s.)  $b \in X^*$ . Further, (A1) implies the validity of  $\|u\|_X \leq C \|b\|_{X^*}$  for weak solutions  $u$  to r.h.s.  $b$  with a constant  $C < +\infty$  independent of  $u = A^{-1}b$ .

Let  $D := \{u \in X \mid Au \in Y^*\}$  and let us call  $u \in D$  a *strong solution* to the r.h.s.  $b = Au \in Y^*$ . Of course, this notion depends on the choice of  $Y$  and may be weaker than the usual notion even if e.g.  $X = W_0^{1,2}(\Omega)$  and  $Y = L^2(\Omega)$ , because the coefficients of  $A$  may be too irregular to guarantee  $D \subset W^{2,2}(\Omega)$ . Trivially, this notion of a strong solution is

- (i) consistent in the sense that every strong solution  $u \in D$  is a weak solution (to the right hand side  $b = Au \in Y^* \subset X^*$ ),
- (ii) selective in the sense that every weak solution  $u \in X$  satisfying  $u \in D$  is in fact a strong solution (to the r.h.s.  $b = Au \in Y^*$ ),

and for every  $b \in Y^*$  there exists a unique strong solution  $u \in D$  satisfying  $Au = b$ , namely the weak solution  $u \in X^*$  to the r.h.s.  $b \in Y^* \subset X^*$ . Endow  $D$  with the graph norm  $\|u\|_D := \|u\|_X + \|Au\|_{Y^*}$ , then  $D$  is continuously embedded into  $X$  due to  $\|u\|_X \leq \|u\|_D$  and dense in  $X$  by the following lemma.

LEMMA 1. *Under the assumptions (A0) and (A1) the set  $D$  is dense in  $X$ .*

*Proof.* Let  $u \in X$ , then we have to show that there exists a sequence  $u_k \in D$  with  $u_k \rightarrow u$  in  $X$ . For this purpose, let  $b := Au \in X^*$ . As (A0) implies that  $Y^*$  is dense in  $X^*$ , there exists a sequence  $b_k \in Y^*$  such that  $b_k \rightarrow b$  in  $X^*$ . Denote by  $u_k \in X$  the weak solution to  $b_k \in Y^* \subset X^*$ , then  $u_k \in D$  as the r.h.s.  $b_k$  lies in  $Y^*$ . Finally, the continuity of  $A^{-1}$  required in (A1) implies  $u_k = A^{-1}b_k \rightarrow A^{-1}b = u$  in  $X$ , i.e.  $u_k \in D$  is a sequence with  $u_k \rightarrow u$  in  $X$ .  $\square$

Instead of interpreting  $D$  as the set of strong solutions, we alternatively can interpret  $D \subset X$  as the domain of  $A$  when viewed as an operator into  $Y^* \subset X^*$ . To distinguish this operator from  $A$ , we denote it by  $A_s : D \subset X \rightarrow Y^*$ ,  $D \ni u \mapsto Au \in Y^*$ , and call  $A_s$  the strong realization of the linear operator  $A$ . Hereby,  $(D, \|\cdot\|_D)$  is a Banach space iff  $\text{graph}(A_s) \subset X \times Y^*$  is closed, i.e. iff  $A_s : D \subset X \rightarrow Y^*$  is a closed linear operator. Fortunately,  $A_s$  is automatically closed.

LEMMA 2. *Under the assumptions (A0) and (A1) the operator  $A_s : D \subset X \rightarrow Y^*$ ,  $D \ni u \mapsto Au \in Y^*$ , is closed.*

*Proof.* We have to prove that  $u_k \rightarrow u$  in  $X$  and  $Au_k \rightarrow b$  in  $Y^*$  imply  $Au = b$  (then by definition of  $D$  also  $u \in D$  due to  $b \in Y^*$ ). Because  $b \in Y^* \subset X^*$  holds by (A0) and  $A : X \rightarrow X^*$  has a continuous inverse by (A1), from  $Au_k \rightarrow b$  in  $Y^*$  first  $A(u_k - A^{-1}b) \rightarrow 0$  in  $X^*$  and then  $u_k - A^{-1}b \rightarrow 0$  in  $X$  follows. Thus, due to  $u_k \rightarrow u$  in  $X$  we have  $u = A^{-1}b$  and hence  $Au = b$ .  $\square$

If we consider  $A_s$  not as closed linear operator on  $X$  with domain  $D$ , but as a linear operator  $A_s : D \rightarrow Y$  on the Banach space  $(D, \|\cdot\|_D)$ , then  $A_s$  is continuous due to  $\|Au\|_{Y^*} \leq \|u\|_D$ , and continuously invertible by the following Lemma.

LEMMA 3. *Under the assumptions (A0) and (A1) the operator  $A_s$  from  $(D, \|\cdot\|_D)$  to  $Y^*$  has a continuous inverse  $A_s^{-1} : Y^* \rightarrow D$ .*

*Proof.*  $A_s$  is surjective due to existence of strong solutions to r.h.s.  $b \in Y^*$  and injective by uniqueness of strong solutions, i.e. the inverse  $A_s^{-1}$  exists. Moreover,  $A_s$  is an open operator from  $(D, \|\cdot\|_D)$  to  $Y^*$  by the open mapping theorem, thus  $A_s^{-1}$  is continuous from  $Y^*$  to  $D$ .  $\square$

Particularly, the inequality  $\|u\|_D \leq C\|b\|_{Y^*}$  holds for strong solutions  $u$  to r.h.s.  $b$  with a constant  $C < +\infty$  independent of  $u = A_s^{-1}b$ . Using the dual  $A_s^* : Y \rightarrow D^*$  to the continuous linear operator  $A_s : D \rightarrow Y^*$ , we now can easily define very weak solutions (of the dual equation  $A^*u = b$ , which as stated makes sense only for  $b \in X^*$ ) for continuous linear functionals  $b$  on  $(D, \|\cdot\|_D)$  as r.h.s.: Let us call  $u \in Y$  a *very weak solution* to r.h.s.  $b \in D^*$  iff  $A_s^*u = b$ , and correspondingly let us define the very weak realization  $A_v : Y \rightarrow D^*$  of  $A^*$  by  $A_v := A_s^*$ . By this definition,  $u \in Y$  is a very weak solution to r.h.s.  $b \in D^*$  iff  $A_vu = b$ , or equivalently  $\langle Av, u \rangle = \langle b, v \rangle$  for every  $v \in D$ .

THEOREM 2. *Under the assumptions (A0) and (A1), for every  $b \in D^*$  there exists a unique very weak solution  $u \in Y$  (of the dual equation) to the r.h.s.  $b$ .*

*Proof.* For given  $b \in D^*$ , consider the linear functional  $Y^* \ni w \mapsto \langle b, A_s^{-1}w \rangle$ , then due to continuity  $\|A_s^{-1}w\|_D \leq C\|w\|_{Y^*}$  of  $A_s^{-1}$  proved in Lemma 3 this linear functional on  $Y^*$  is continuous. Thus, it can be naturally identified with an element  $u \in Y$  due to reflexivity of  $Y$ . Now this element  $u \in Y$  is a very weak solution to the r.h.s.  $b$ , because by definition  $\langle Av, u \rangle = \langle b, v \rangle$  holds for every  $v \in D$  due to  $A_s^{-1}w = v$  for  $w = Av$ . Moreover, due to  $Y^* = A(D)$  there is at most one very weak solution to the r.h.s.  $b$ .  $\square$

REMARK 1. If  $Y$  is not reflexive, but has a predual  $Y_*$  which is continuously embedded into  $X^*$  and dense, then Theorem 2 holds in the same way with (A0) replaced by this assumption and  $D$  replaced by  $D := \{u \in X \mid Au \in Y_*\}$ .

The notion of a very weak solution is

- (i) consistent in the sense that every weak solution  $u \in X$  is a very weak solution (to the right hand side  $b = A^*u \in X^* \subset D^*$ ),
- (ii) selective in the sense that every very weak solution  $u \in Y$  satisfying  $u \in X$  is in fact a weak solution (to the r.h.s.  $b = A^*u \in X^*$ ).

Finally, as the dual  $A_v$  of the continuous mapping  $A_s$  is itself continuous and moreover bijective by Theorem 2, it has a continuous inverse  $A_v^{-1} : D^* \rightarrow Y$ , and particularly the inequality  $\|u\|_Y \leq C\|b\|_{D^*}$  holds for very weak solutions  $u$  to r.h.s.  $b$  with a constant  $C < +\infty$  independent of  $u = A_v^{-1}b$ .

An alternative to this construction of very weak solutions would be to define the very weak realization  $A_v$  as closure of  $\text{graph}(A^*) \subset X \times X^*$  in the larger space  $Y \times D^*$ . Note that similarly the closure of  $\text{graph}(A_s) \subset D \times Y^*$  in the larger space  $X \times X^*$  is the operator  $A$  itself.

#### 4.1. A framework for Banach spaces

More generally, for Banach space theory instead of Hilbert space theory we could assume that  $X_p, X_q$  are reflexive Banach spaces, and  $A$  is a continuous linear operator such that

(A1)'  $A : X_q \rightarrow X_p^*$  has a continuous inverse  $A^{-1} : X_p^* \rightarrow X_q$ , i.e.  $A$  is a topological isomorphism,

then Theorem 2 holds with (A1) replaced by (A1)' and  $D := \{u \in X_q \mid Au \in Y^*\}$ . Note particularly that (A1)' holds by generalized Lax-Milgram, if  $A$  satisfies the generalized coercivity condition  $\|Au\|_{X_p^*} = \sup_{X_p \ni v \neq 0} \frac{|\langle Au, v \rangle|}{\|v\|_{X_p}} \geq c\|u\|_{X_q}$  for some  $c > 0$ , see SIMADER

[18] or more generally [11] for the case where invertibility is not needed. If  $a_2$  is uniformly continuous and  $a_0$  is sufficiently large, then due to generalized Gårdings inequality the weak realization  $A$  of the linear elliptic PDE (16) in divergence form defined in (17) satisfies (A1)' when considered as operator  $A : W_{\Gamma_D}^{1,q}(\Omega) \rightarrow (W_{\Gamma_D}^{1,p}(\Omega))^*$  for arbitrary  $1 < q < \infty$  and  $p := q'$ . However, if  $a_2$  is merely bounded and measurable,

then GRÖGER [10] proved that there still is a  $Q > 2$  such that this weak realization  $A$  also satisfies (A1)' for  $Q' < q < Q$ , but in general this  $Q$  is rather small, which strongly restricts the applicability of the Banach space framework in case of merely bounded measurable coefficients. Yet, under certain regularity assumptions on the domain the value  $Q > 2$  can be proved to be sufficiently large, e.g. see [7]. Therefore, in Theorem 1 we just require (A1)' in the case  $X_p := W_{\Gamma_D}^{1,p}(\Omega)$ .

**5. Application to singular right hand sides and irregular data**

In this section we apply the abstract theory of section 4 to recover known results about existence of very weak solutions to (1) and to prove new results for irregular coefficients. In section 3 we showed that singular integrals with kernels  $\underline{b}$  on  $\partial\Omega \times \partial\Omega$  satisfying  $\underline{b}(x,y)|x-y|^{\frac{np-n-p+2}{p}} \in L^p(\partial\Omega \times \partial\Omega)$  resp. kernels  $\underline{b}$  on  $\Omega \times \partial\Omega$  satisfying  $\underline{b}(x,y)|x-y|^{\frac{np-n-(1-\alpha)p+1}{p}} \in L^p(\Omega \times \partial\Omega)$  for  $\alpha < 1$  constitute r.h.s.  $b \in (W^{1,q}(\Omega))^*$  which allow to prove existence of weak solutions of (1). However, for more singular kernels existence of weak solutions can not be guaranteed. Therefore, to handle rather singular data, very weak solutions are needed as a generalization of weak solutions.

Note that the formal dual to equation (1) is

$$-\operatorname{div}(a_2^T \nabla v + va_1') + a_1 \cdot \nabla v + a_0 v = f_0 \text{ in } \Omega. \tag{16}$$

As we want to define very weak solutions by duality, we consider the weak realization  $Av = b$  of equation (16) with operator  $A : W_{\Gamma_D}^{1,q}(\Omega) \rightarrow (W_{\Gamma_D}^{1,p}(\Omega))^*$ ,  $q := p'$ , defined by

$$\langle Av, u \rangle := \int_{\Omega} (a_2 \nabla u) \cdot \nabla v + va_1 \cdot \nabla v + va_1' \cdot \nabla u + a_0 uv \, dx. \tag{17}$$

The right hand side is the same as in equation (3), only  $u$  and  $v$  are interchanged on the left hand side, i.e. the weak realization  $A_w$  of the linear PDE (1) defined in equation (3) is the dual operator  $A_w = A^*$  of the  $A$  defined in (17) as weak realization of (16).

**5.1. Regular coefficients**

In the case of uniformly continuous coefficients, if we apply Theorem 2 under the assumption (A1)' in the Banach framework to the operator  $A$  from  $X_q := W_{\Gamma_D}^{1,q}(\Omega)$  to the dual of  $X_p := W_{\Gamma_D}^{1,p}(\Omega)$  defined in (17) for the Lebesgue spaces  $Y^* := L^q(\Omega)$  resp. more generally Lorentz spaces  $Y^* := L^{q,r}(\Omega)$ , then due to the validity of  $D \subset W^{2,q}(\Omega)$  resp.  $D \subset W^{2,L^{q,r}}(\Omega) = \{\phi \in L^{q,r}(\Omega) \mid D^\alpha \phi \in L^{q,r}(\Omega) \text{ for } |\alpha| \leq 2\}$ , i.e. the fact that elements of  $D$  are strong solutions in the usual sense, we obtain very weak solutions to right hand sides  $b \in (W^{2,q}(\Omega))^*$  resp.  $b \in (W^{2,L^{q,r}}(\Omega))^*$ . Particularly, in the case of the Laplacian, this allows to recover a variant of [14, Theorem 1].

**THEOREM 3.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded  $C^2$ -domain with Dirichlet resp. Neumann boundary  $\partial\Omega = \Gamma_D \dot{\cup} \Gamma_N$ . Let  $1 < p < \infty$ ,  $q := p'$ ,  $(a_2 = 1, a_1 = 0 = a_1')$ , and*

assume  $L^\infty(\Omega, \mathbb{R}) \ni a_0 \geq c$  for a constant  $c > 0$ . Then for every  $b \in (W^2L^{q,r}(\Omega))^*$ ,  $1 < r < \infty$ , there exists a unique  $u \in L^{p,r'}(\Omega)$  which satisfies

$$\int_{\Omega} u(-\Delta v + a_0 v) \, dx = \langle b, v \rangle$$

for every  $v \in W^2L^{q,r}(\Omega)$  with  $v = 0$  on  $\Gamma_D$  and  $\frac{\partial v}{\partial \bar{n}} = 0$  on  $\Gamma_N$ .

### 5.2. Irregular boundary

Let  $\Omega \subset \mathbb{R}^N$  be a bounded Lipschitz domain, and denote by  $\phi_1 > 0$  the (normed) eigenfunction to the first Dirichlet eigenvalue  $\lambda_1 > 0$  of  $-\Delta$  on  $\Omega$ . For  $C^2$ -domains,  $\phi_1(x)$  behaves like  $\delta(x)$  near the boundary, so that  $L^1(\Omega, \delta) = L^1(\Omega, \phi_1)$ . However, as noted by MCKENNA and REICHEL [13], for general Lipschitz domains the notion of a very weak solution of BREZIS' problem  $-\Delta u = f$  in  $\Omega$ ,  $u = 0$  on  $\partial\Omega$ , to r.h.s.  $f \in L^1(\Omega, \phi_1)$  has to be slightly changed due to irregularity of the boundary: A very weak solution  $u \in L^1(\Omega, \phi_1)$  should satisfy

$$-\int_{\Omega} u \cdot \Delta v \, dx = \int_{\Omega} f v \, dx \tag{18}$$

for every  $v \in W_0^{1,2}(\Omega)$  with  $-\Delta v = \eta$ , where  $\eta : \Omega \rightarrow \mathbb{R}$  is measurable and satisfies  $\eta/\phi_1 \in L^\infty(\Omega)$ , or equivalently

$$\int_{\Omega} u \cdot \eta \, dx = \int_{\Omega} f(-\Delta)^{-1} \eta \, dx.$$

In fact, in general there is no  $u \in L^1(\Omega)$  such that (18) holds for all  $v \in C^2(\bar{\Omega})$  with  $v = 0$  on  $\partial\Omega$ , and the solution of  $-\Delta v = 1$  in  $\Omega$ ,  $v = 0$  on  $\partial\Omega$  is not  $C^2$ .

In our setting, due to the definition of  $D$  our very weak solutions automatically coincide with the notion of MCKENNA and REICHEL. Moreover, by remark 1 we can consider the space

$$Y := M(\Omega, \phi_1) = \{ \mu \text{ signed measure on } \Omega \mid \int_{\Omega} \phi_1 d|\mu| < \infty \}$$

of signed measures which are not necessarily finite (near the boundary), but have a finite norm  $\|\mu\|_Y := \int_{\Omega} \phi_1 d|\mu|$  w.r.t. weight  $\phi_1$ . A predual of this space is the space

$$Y_* := C_0(\Omega, \phi_1) = \{ \eta \in C(\Omega, \mathbb{R}) \mid \eta/\phi_1 \in C(\Omega, \mathbb{R}) \}$$

of continuous functions  $\eta$  such that also  $\eta/\phi_1$  is continuous (particularly,  $\eta = 0$  on  $\partial\Omega$ , and  $\eta(x)$  is not allowed to decrease more slowly than  $\phi_1(x)$  as  $x$  approaches the boundary), endowed with the norm  $\|\eta\|_{Y_*} := \|\eta/\phi_1\|_\infty$ . By the maximum principle, every solution  $v$  of  $-\Delta v = \eta$  to r.h.s.  $\eta \in Y_*$  satisfies  $v/\phi_1 \in L^\infty(\Omega)$ . Thus, every  $f \in L^1(\Omega, \phi_1)$  satisfies  $f \in D^*$ , and hence we obtain the following result by the non-reflexive variant of Theorem 2.

**THEOREM 4.** *Let  $\Omega \subset \mathbb{R}^N$  be a bounded Lipschitz domain. Then for every r.h.s.  $f \in L^1(\Omega, \phi_1)$  there exists a unique measure  $u \in M(\Omega, \phi_1)$  which satisfies*

$$-\int_{\Omega} \Delta v du = \int_{\Omega} f v dx$$

for every  $v \in W_0^{1,2}(\Omega)$  with  $-\Delta v = \eta$  in  $\Omega$ , where  $\eta \in C(\Omega, \mathbb{R})$  is such that  $\eta/\phi_1 \in C(\Omega, \mathbb{R})$ .

Finally, as in [13] it is possible to verify that the measure  $u$  even lies in  $L^1(\Omega, \phi_1)$ .

### 5.3. Bounded measurable coefficients

In this subsection we prove existence of very weak solutions to (1) under mixed boundary conditions, i.e. Dirichlet boundary conditions  $u = 0$  on  $\Gamma_D$  and Neumann boundary conditions (4) on  $\Gamma_N$ , for singular right hand sides and irregular coefficients. Particularly, we prove Theorem 1.

As in 5.1, we consider  $Y := L^p(\Omega)$  and consider the strong realization  $A_S : D \rightarrow Y^*$  of equation (16) defined on

$$D := \{v \in W_{\Gamma_D}^{1,q}(\Omega) \mid Av \in L^q(\Omega)\}.$$

Observe that r.h.s.  $b \in Y^*$  can not anymore realize nonhomogeneous Neumann boundary conditions, i.e. our strong solutions  $v \in D$  satisfy not only  $v = 0$  on  $\Gamma_D$  but also homogeneous Neumann boundary conditions

$$(a_2^T \nabla v + va_1') \cdot \vec{n} = 0 \quad \text{on } \Gamma_N. \tag{19}$$

Note carefully that for merely bounded coefficients our strong solutions are not as strong as usual although  $f_0 \in Y^* = L^q(\Omega)$  holds for the r.h.s. in (16). Particularly, our notion of strong solutions does not imply  $v \in W^{2,q}(\Omega)$ , but just that equation (16) is satisfied in  $Y^* = L^q(\Omega)$ . Yet, if  $q > \frac{N}{2}$ , then  $v \in D \subset C^{0,\alpha}(\Omega)$  holds for some  $\alpha > 0$ , i.e. strong solutions of (16) to r.h.s.  $f_0 \in L^q(\Omega)$  with  $q > \frac{N}{2}$  under mixed homogeneous boundary conditions  $u = 0$  on  $\Gamma_D$  and (19) on  $\Gamma_N$  are Hölder continuous w.r.t. some exponent  $\alpha > 0$  by DE GIORGI [4], NASH [16] and MOSER [15], who proved (interior) Hölder regularity, by GILBARG and TRUDINGER [8] for the case of Dirichlet boundary conditions, and by TROIANELLO [20] for allowing Neumann boundary conditions on a component  $\Gamma_N$  of  $\partial\Omega$ , see also [9]. From these regularity results, we obtain as a consequence of Theorem 2 the following result on existence and uniqueness of very weak solutions  $u \in Y = L^p(\Omega)$  to singular integrals  $b \in (C^{0,\alpha}(\Omega))^*$ .

**THEOREM 5.** *Let  $1 < p < \infty$ ,  $q := \frac{p}{p-1} > \frac{N}{2}$ , and assume that  $a_2 \in L^\infty(\Omega, \mathbb{R}^{N \times N})$  satisfying  $a_2(x)\xi \cdot \xi \geq c|\xi|^2$  with a constant  $c > 0$  for arbitrary  $x \in \Omega$ ,  $\xi \in \mathbb{R}^N$ , and  $a_1, a_1' \in L^\infty(\Omega, \mathbb{R}^N)$ ,  $a_0 \in L^\infty(\Omega, \mathbb{R})$ , are such that the weak realization of (16) given by the operator  $A : W_{\Gamma_D}^{1,q}(\Omega) \rightarrow (W_{\Gamma_D}^{1,p}(\Omega))^*$  defined in (17) has a continuous inverse. Then there is an  $\alpha > 0$  such that for every  $b \in (C^{0,\alpha}(\Omega))^*$  there exists a unique solution  $u \in L^p(\Omega)$  of  $A_S u = b$  with the very weak realization  $A_S$  of (1) induced by  $A$ .*

The main Theorem 1 is just a corollary of Theorem 5. It precisely points out, that for a sum of  $\operatorname{div}(f_1)$ ,  $f_1 \in L^p(\Omega)$ , and a singular integral with kernel  $\underline{b}$  on  $\Omega \times \Gamma_N$  satisfying  $\underline{b}(x,y)|x-y|^\alpha \in L^1(\Omega \times \Gamma_N)$  as r.h.s.  $b$ , the very weak solution  $u \in L^p(\Omega)$  satisfies

$$\begin{aligned} & \int_{\Omega} u (-\operatorname{div}(a_2^T \nabla v + va_1') + a_1 \cdot \nabla v + a_0 v) dx \\ &= \int_{\Omega} f_1(x) \cdot \nabla v(x) dx + \int_{\Omega} \int_{\Gamma_N} \underline{b}(x,y) (v(x) - v(y)) dS(y) dx \end{aligned} \tag{20}$$

for every  $v \in W_{\Gamma_D}^{1,q}(\Omega)$  with  $a_2^T \nabla v + va_1' \in W^{1,q}(\Omega)$  and  $(a_2^T \nabla v + va_1') \cdot \vec{n} = 0$  on  $\Gamma_N$ , where  $v \in C^{0,\alpha}(\Omega)$  is automatically Hölder continuous due to the validity of the equation  $-\operatorname{div}(a_2^T \nabla v + va_1') + a_1 \cdot \nabla v + a_0 v = f_0$  with a function  $f_0 \in L^q(\Omega)$ .

Note that  $A$  has a continuous inverse, if  $A$  satisfies a generalized coercivity condition, and such a condition holds in turn for  $a_0 \geq c$  with sufficiently large  $c > 0$ . Further, if  $\int_{\Omega} \underline{b}(x,y) dx = \infty$  holds at points  $y$  in a subset  $\Gamma \subset \Gamma_N$  of positive  $(n-1)$ -dimensional measure,, then  $u$  formally satisfies the singular Neumann boundary condition  $(a_2 \nabla u + ua_1) \cdot \vec{n} = -\infty$  on  $\Gamma$ . However, rigorously we can merely say that equation (20) holds, where the singular Neumann boundary condition is modeled via the kernel function  $\underline{b}$  of a singular integral.

Yet, in two aspects this result is unsatisfying: On the one hand, the optimal Hölder exponent  $\alpha > 0$  of solutions to (16) with r.h.s.  $f_0 \in L^q(\Omega)$  for  $q > \frac{N}{2}$  under mixed homogeneous boundary conditions  $u = 0$  on  $\Gamma_D$  and (19) on  $\Gamma_N$  is not known explicitly, it depends on the dimension  $N$ , the domain  $\Omega$  and the ellipticity ratio. On the other hand, it would be nice to have a similar result for  $b$  not only in the dual space of  $C^{0,\alpha}(\Omega)$ , but also in the dual space of the completion  $W^{s,q}(\Omega; \partial\Omega)$  (see appendix A) of  $C^1(\overline{\Omega})$  w.r.t. the norm  $\|\cdot\|_{q,\Omega} + [\cdot]_{s,q,\Omega;\partial\Omega}$ , where

$$[v]_{s,q,\Omega;\partial\Omega} := \left( \int_{\Omega} \int_{\partial\Omega} \frac{|v(x) - v(y)|^q}{|x - y|^{sq+n-1}} dS(y) dx \right)^{\frac{1}{q}}.$$

However, for such a generalization a regularity result is needed, which guarantees that solutions to (16) with r.h.s.  $f_0 \in L^q(\Omega)$  for  $q > \frac{N}{2}$  under mixed homogeneous boundary conditions  $u = 0$  on  $\Gamma_D$  and (19) on  $\Gamma_N$  lie in this space. Yet, by the generality of the functional analytic setting developed in section 4, whenever you have a more advanced regularity result for solutions of (16) to r.h.s.  $f_0 \in L^q(\Omega)$  like e.g. the suggested regularity result, then there is a corresponding existence and uniqueness theorem for very weak solutions  $u \in L^p(\Omega)$  to r.h.s.  $b$  in the dual of the space where the regular solutions are from.

## 6. Conclusion

In section 3 integrability conditions on the kernels  $\underline{b}$  of singular integrals have been given, which guarantee existence of weak solutions  $u \in W^{1,p}(\Omega)$  to singular data. However, if the data is so singular that these integrability conditions are not satisfied

anymore, then merely very weak solutions  $u \in L^p(\Omega)$  exist, and using the general functional analytic framework formulated in section 4, we were able to prove in section 5 the existence of very weak solutions to singular integrals as r.h.s. which are continuous functionals on  $C^{0,\alpha}(\Omega)$ . Hereby, the index  $\alpha > 0$  comes from the Hölder regularity of solutions to r.h.s. in  $L^q(\Omega)$ ,  $q > \frac{N}{2}$ , established by DE GIORGI, NASH and MOSER, and may be very small. An extension of our result to other settings seems to be possible, e.g. to coefficients  $a_2 \in L^\infty(\Omega)$  which additionally lie in  $BV(\Omega)$  or  $W^{1,r}(\Omega)$  for  $1 < r \leq N$ , where continuity of principal coefficients does not hold in general, but where solutions to r.h.s.  $f \in L^q(\Omega)$  seem to satisfy  $u \in C^{0,\alpha}(\Omega)$  for arbitrary  $\alpha < 1$ , so that very weak solutions seem to exist for r.h.s.  $b$  in the dual space of  $C^{0,\alpha}(\Omega)$  with arbitrary  $\alpha < 1$ . Further, our methods seem to allow unbounded lower order coefficients even in case of bounded measurable principal coefficients (for the case of Lipschitz continuous coefficients see [6]). Yet, these claims seem to be open and still have to be verified.

### A. Generalization of Sobolev-Slobodecki spaces

While the Sobolev-Slobodeckij spaces  $W^{s,q}(\Omega)$  resp.  $W^{s,q}(\partial\Omega)$ ,  $0 < s < 1$ ,  $1 < q < \infty$  with seminorms

$$[v]_{s,q,\Omega} := \left( \int_{\Omega} \int_{\Omega} \frac{|v(x) - v(y)|^q}{|x - y|^{sq+n}} dy dx \right)^{\frac{1}{q}} \quad \text{resp.}$$

$$[v]_{s,q,\partial\Omega} := \left( \int_{\partial\Omega} \int_{\partial\Omega} \frac{|v(x) - v(y)|^q}{|x - y|^{sq+n-1}} dS(y) dS(x) \right)^{\frac{1}{q}}$$

are well-known, are identical with the Besov spaces  $B^{s,q,q}(\Omega)$  resp.  $B^{s,q,q}(\partial\Omega)$  and particularly are real interpolation spaces, the completion  $W^{s,q}(\Omega; \partial\Omega)$  of  $C^1(\overline{\Omega})$  w.r.t. the norm  $\|\cdot\|_{q,\Omega} + [\cdot]_{s,q,\Omega;\partial\Omega}$ , where

$$[v]_{s,q,\Omega;\partial\Omega} := \left( \int_{\Omega} \int_{\partial\Omega} \frac{|v(x) - v(y)|^q}{|x - y|^{sq+n-1}} dS(y) dx \right)^{\frac{1}{q}}.$$

occurs in our study of singular integrals on  $\Omega \times \partial\Omega$  in section 3.1, but does not seem to be well-known. We do not know much more about the spaces  $W^{s,q}(\Omega; \partial\Omega)$  than the following Lemma and Corollary. These follow from the estimates (13), (14), which particularly show that  $[\cdot]_{s,q,\Omega;\partial\Omega}$  is finite on  $C^1(\overline{\Omega})$ .

LEMMA 4.  $W^{1,q}(\Omega)$  is embedded into  $W^{s,q}(\Omega; \partial\Omega)$  for every  $0 < s < 1$ .

COROLLARY 1. For every  $0 < s < 1$  the space  $W^{s,q}(\Omega; \partial\Omega)$  is an intermediate space between  $L^q(\Omega)$  and  $W^{1,q}(\Omega)$ .



### B. Singular Integrals on spaces of functions with vanishing normal derivative

Consider singular integrals on  $\Omega \times \partial\Omega$  of the form

$$\langle b, v \rangle = \int_{\Omega} \int_{\partial\Omega} \underline{b}_t(x, y) (v(x) - v(y + \langle x - y, \vec{n}(y) \rangle \vec{n}(y))) + \underline{b}_n(x, y) (v(y + \langle x - y, \vec{n}(y) \rangle \vec{n}(y)) - v(y)) dS(y) dx \tag{21}$$

with the tangential part satisfying  $\underline{b}_t(x, y)|x - y|^\alpha \in L^1(\Omega \times \partial\Omega)$ , but the normal part merely satisfying  $\underline{b}_n(x, y)|\langle x - y, \vec{n}(y) \rangle|^{1+\alpha} \in L^1(\Omega \times \partial\Omega)$  for some  $\alpha < 1$ . Then  $b$  is continuous on the space of functions  $v \in C^{1,\alpha}(\overline{\Omega})$  which satisfy  $\frac{\partial v}{\partial \vec{n}} = 0$  on  $\partial\Omega$  due to

$$\sup_{x \in \Omega, y \in \partial\Omega} \left| \frac{v(y + \langle x - y, \vec{n}(y) \rangle \vec{n}(y)) - v(y) - \frac{\partial v}{\partial \vec{n}}(y) \langle x - y, \vec{n}(y) \rangle}{|\langle x - y, \vec{n}(y) \rangle|^{1+\alpha}} \right| < +\infty.$$

Note that in the case  $\underline{b}_t = \underline{b}_n$  the singular integral (21) is identical with (12).

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