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ON A GENERAL CLASS OF SECOND–ORDER, LINEAR, ORDINARY DIFFERENTIAL EQUATIONS SOLVABLE AS A SYSTEM OF FIRST–ORDER EQUATIONS

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Abstract. An approach for solving general second-order, linear, variable-coefficient ordinary differential equations in standard form under initial-value conditions is presented for the case of a specific constant-form relation between the two otherwise arbitrary coefficients. The resulting system of linear equations produces fundamental (or state transition) matrix elements used to create integral- and closed-form solutions for both homogeneous and nonhomogeneous differential equation variants. Two example equations are chosen to illustrate application. A short discussion is presented on the comparison of the theoretical results for these examples with the corresponding symbolic integration outputs provided by several commercial programs which were seen, at times, to be long and unwieldy or even non-existent.

1. Introduction

Given the second-order, linear, nonhomogeneous, ordinary differential equation in standard form [1, 3, 4, 7, 9, 10, 11, 12, 14, 15]

$$\ddot{y}(x)+p(x)\dot{y}(x)+q(x)y(x)=f(x) \tag{1}$$

with excitation $f(x)$ and initial-value conditions

$$y(x_0)=y_0 \text{ and } \dot{y}(x_0)=\dot{y}_0, \tag{2}$$

a relationship between the coefficients of

$$p(x)=aq(x)+\frac{1}{a}, \tag{3}$$

where a is a nonzero real constant and $q(x)$ is an arbitrary real function, leads to a readily obtainable general solution to equations (1) and (2). This result appears to be of practical significance since it provides integral-form general solutions to otherwise difficult-to-solve variable coefficient differential equations such as, for example,

$$\ddot{y}(x)+\left[2x+\frac{1}{2}\right]\dot{y}(x)+xy(x)=f(x) \text{ for parameter } a=2, \tag{4}$$

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or

$$\ddot{y}(x)+\cosh^2(x)\dot{y}(x)+\sinh^2(x)y(x)=f(x) \text{ for } a=1 \tag{5}$$

for arbitrary $f(x)$. All similar second-order differential equations obeying (3) are, of course, also included in the following analysis.

2. Conversion to an Equivalent System of Linear Equations

A general solution to equations (1) and (2) subject to (3) follows from conversion to a system of linear first-order equations by employing state variables $u_1(x)$ and $u_2(x)$ in the form

$$\begin{aligned} y(x)=u_1(x), \quad \dot{y}(x)=\dot{u}_1(x)=u_2(x), \\ \dot{u}_2(x)=-q(x)u_1(x)-p(x)u_2(x)+f(x), \end{aligned} \tag{6}$$

or

$$\begin{bmatrix}\dot{u}_1(x)\\ \dot{u}_2(x)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -q(x) & -p(x)\end{bmatrix}\begin{bmatrix}u_1(x)\\ u_2(x)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}f(x), \tag{7}$$

where the system excitation $f(x)=0$ for the homogeneous case and

$$y(x)=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}u_1(x)\\ u_2(x)\end{bmatrix}. \tag{8}$$

The standard, linear, variable-coefficient system canonical form for this single-input, single-output case in matrix form is

$$\dot{\mathbf{u}}(x)=\mathbf{A}(x)\mathbf{u}(x)+\mathbf{b}f(x) \tag{9}$$

from which matrix $\mathbf{A}(x)$ and the column vector $\mathbf{b}$ are defined by comparison with equation (7). Matrix $\mathbf{A}(x)$ is the companion matrix of the corresponding characteristic polynomial of equation (1) [14]. The standard general solution [14] , [12] to eq. (9) is obtained from the fundamental or state transition matrix $\mathbf{\Phi}(x,x_0)$ as

$$\mathbf{u}(x)=\mathbf{\Phi}(x,x_0)\mathbf{u}(x_0)+\int_{x_0}^{x}\mathbf{\Phi}(x,x')\mathbf{b}f(x')\,dx'. \tag{10}$$

This result is also the variable coefficient version of Duhamel's Formula [14] which exhibits the zero-input and zero-state responses for the state vector $\mathbf{u}(x)$ and single input $f(x)$ in the two terms, respectively, on the right-hand side. For the initial-value problem of (1), (2), since $y(x)=u_1(x)$ and $\dot{y}=u_2(x)$, eq. (11) provides the general solution to (1)

$$y(x)=\Phi_{11}(x,x_0)y(x_0)+\Phi_{12}(x,x_0)\dot{y}(x_0)+\int_{x_0}^{x}\Phi_{12}(x,x')f(x')\,dx' \tag{11}$$

where $\Phi_{ij}(x,x_0)$ are the matrix elements of 2×2 $\mathbf{\Phi}(x,x_0)$. Hence the solution to the nonhomogeneous version of the initial-value problem of equations (1) and (2) is completely determined by the initial conditions, excitation $f(x)$, and the two top-row elements of the fundamental matrix which is itself determinable from its properties as a fundamental matrix. Note also that $\Phi_{ij}(x,x_0)$ is nonzero only for $x\geqslant x_0$ for initial value problems.

3. Linear System Solution $\mathbf{\Phi}(x,x_0)$

As a fundamental matrix, $\mathbf{\Phi}(x,x_0)$ satisfies

$$\frac{d}{dx}\mathbf{\Phi}(x,x_0) = \mathbf{A}(x)\mathbf{\Phi}(x,x_0), \tag{12}$$

where $\mathbf{\Phi}(x_0,x_0) = \mathbf{I}$. Hence, for $\mathbf{A}(x)$ of equations (7) and (9), this leads to the four equations for the Φ_{ij} elements:

$$\frac{d}{dx}\Phi_{11}(x,x_0) = \Phi_{21}(x,x_0) \tag{13}$$

$$\frac{d}{dx}\Phi_{12}(x,x_0) = \Phi_{22}(x,x_0) \tag{14}$$

$$\frac{d}{dx}\Phi_{21}(x,x_0) = -q(x)\Phi_{11}(x,x_0) - p(x)\Phi_{21}(x,x_0) \tag{15}$$

$$\frac{d}{dx}\Phi_{22}(x,x_0) = -q(x)\Phi_{12}(x,x_0) - p(x)\Phi_{22}(x,x_0). \tag{16}$$

Note that by the nature of the resulting $\mathbf{A}(x)$ for general eq. (1), overall coupling among the Φ_{ij} elements is restricted to identical differential equation relations between Φ_{11},Φ_{21} in equations (13) and (15) and Φ_{12},Φ_{22} in equations (14) and (16).

THEOREM 1. *A general* 2×2 *fundamental matrix* $\mathbf{\Phi}(x,x_0)$ *for the second-order, linear, nonhomogeneous differential equation* (1) *with initial-value conditions of* (2) *under the coefficient relationship of* (3) *is given by*

$$\Phi_{11}(x,x_0) = e^{-\frac{x-x_0}{a}} + \frac{1}{a}I_a(x,x_0) \tag{17}$$

$$\Phi_{12}(x,x_0) = I_a(x,x_0) \tag{18}$$

$$\Phi_{21}(x,x_0) = \frac{1}{a}\left[e^{-aQ(x,x_0)} - e^{-\frac{x-x_0}{a}}\right] - \frac{1}{a^2}I_a(x,x_0) \tag{19}$$

$$\Phi_{22}(x,x_0) = e^{-aQ(x,x_0)} - \frac{1}{a}I_a(x,x_0), \tag{20}$$

where

$$Q(x,x_0) = \int_{x_0}^{x} q(x')\,dx' \tag{21}$$

and

$$I_a(x,x_0) = \int_{x_0}^{x} e^{-\frac{(x-x')}{a}} e^{-aQ(x',x_0)}\,dx'. \tag{22}$$

Proof. Under the condition that $p(x) = aq(x) + \frac{1}{a}$ for any nonzero, real constant a, either pair, (13), (15) or (14), (16) can be solved as follows. For (13), (15),

$$\frac{1}{a}\frac{d}{dx}\Phi_{11}(x,x_0) = \frac{1}{a}\Phi_{21}(x,x_0) \quad \text{and}$$

$$\frac{d}{dx}\Phi_{21}(x,x_0) = -q(x)\Phi_{11}(x,x_0) - \left(aq(x) + \frac{1}{a}\right)\Phi_{21}(x,x_0)$$

can be added together to give

$$\frac{1}{a}\frac{d}{dx}\Phi_{11}(x,x_0) + \frac{d}{dx}\Phi_{21}(x,x_0) = -q(x)\Phi_{11}(x,x_0) - aq(x)\Phi_{21}(x,x_0)$$

or

$$\frac{d}{dx}\Phi_{11}(x,x_0) + a\frac{d}{dx}\Phi_{21}(x,x_0) = -aq(x)\left[\Phi_{11}(x,x_0) + a\Phi_{21}(x,x_0)\right] \tag{23}$$

in which the quantity $v_1(x,x_0) = \Phi_{11}(x,x_0) + a\Phi_{21}(x,x_0)$ for constant a acts like a normal mode in that it evolves in a way uncoupled to any other element grouping. Similarly for equations (14), (16), $v_2 = \Phi_{12}(x,x_0) + a\Phi_{22}(x,x_0)$ plays an identical role with the only difference occurring in the initial values of $v_1(x_0,x_0) = 1$ and $v_2(x_0,x_0) = a$.

Eq. (23) is subsequently solved for $v_1(x,x_0)$ and the resulting Φ_{11},Φ_{21} relation substituted into eq. (15) to obtain an uncoupled differential equation for Φ_{21}. When an identical method is used for equations (14), (16), the resulting fundamental or state transition matrix $\mathbf{\Phi}(x,x_0)$ is then found to be comprised of the elements shown in (17) through (20), together with the definitions (21) and (22).

The requisite fundamental matrix condition $\Phi_{ij}(x_0) = \delta_{ij}$ is of course met, and the general solution of equations (1), (2) under the constraint of eq. (3) follows from these results as per eq. (11).

Note that $\Phi_{12}(x,x_0)$ of (18) is the Green's Function or Impulse Response Function of Systems Theory for the initial value problem [12], [1]. Therefore, a general form for particular solutions for these equations from excitation $f(x)$ is denoted by

$$y_p(x) = \int_{x_0}^{x} I_a(x,x')f(x')\,dx' \tag{24}$$

Due to the complexity of eqs. (21), (22), this result would usually require numerical integration except for sufficiently simple $q(x)$, $f(x)$.

REMARK 1. The main result of Theorem 1 can perhaps be further illuminated through an alternative inquiry into the relationship required between coefficients $p(x)$, $q(x)$ in satisfying the pairs of differential equations for Φ_{11}, Φ_{21} and Φ_{12}, Φ_{22}. This can be posed through two problem statements which are subsequently resolved by an appropriate developmental sequence.

PROBLEM 1. Find the resulting $p(x)$, $q(x)$ relation such that, for an explicit $\Phi_{11}(x,x_0)$ related to $\Phi_{21}(x,x_0)$ by eq. (13), $\frac{d}{dx}\Phi_{11} = \Phi_{21}(x,x_0)$ with $\Phi_{ij}(x_0,x_0) = \delta_{ij}$, eq. (15) is also satisfied.

PROBLEM 2. Find the resulting $p(x)$, $q(x)$ relation such that, for an explicit $\Phi_{12}(x,x_0)$ related to $\Phi_{22}(x,x_0)$ by eq. (14), $\frac{d}{dx}\Phi_{12} = \Phi_{22}(x,x_0)$ with $\Phi_{ij}(x_0,x_0) = \delta_{ij}$, eq. (16) is also satisfied.

Since the form of these differential equation pairs is identical, the coefficient relations obtained from each problem must be the same.

These two problems are resolved in the following three steps.

Step 1: For some parameter $a \neq 0$, consider the elementary solution

$$\Phi_{11}(x,x_0) = \phi_{11}(x,x_0) = e^{-\frac{x-x_0}{a}}$$

for which $\phi_{11}(x_0,x_0) = 1$ and $\frac{d}{dx}\phi_{11} = \Phi_{21} = -\frac{1}{a}\phi_{11}$. Then $\left(\frac{d}{dx}\Phi_{21} = \frac{1}{a^2}\phi_{11}\right) = -q\phi_{11} - p\left(-\frac{1}{a}\phi_{11}\right) = \left(-q+\frac{p}{a}\right)\phi_{11}$. This equality requires the p-q relation of eq. (3), but $\Phi_{21}(x_0,x_0) \neq 0$. Hence this choice of $\Phi_{11} = \phi_{11}$ and Φ_{21} does not present a solution to Problem 1.

Step 2: For $\phi_{11}(x,x_0)$ defined above, consider

$$\Phi_{12}(x,x_0) = \int_{x_0}^{x} \phi_{11}(x,x')\, e^{-aQ(x',x_0)}\, dx'$$

from eqs. (18), (21), and (22). Then $\frac{d}{dx}\Phi_{12} = \Phi_{22} = e^{-aQ} - \frac{1}{a}\Phi_{12}$ as in eq. (20), and from this, $\frac{d}{dx}\Phi_{22} = -aq\, e^{-aQ} - \frac{1}{a}\Phi_{22}$. Since the latter must also agree with $\frac{d}{dx}\Phi_{22} = -q\Phi_{12} - p\Phi_{22}$ of eq. (16), equating these last two results and substituting for Φ_{22} from above leads to the p-q relation of eq. (3). Since $\Phi_{12}(x_0,x_0)$ and $\Phi_{22}(x_0,x_0)$ agree with δ_{ij}, these two so-defined fundamental matrix elements satisfy Problem 2 with eq. (3) resulting.

Step 3: For $\phi_{11}(x,x_0)$ defined above, consider

$$\Phi_{11}(x,x_0) = \phi_{11}(x,x_0) + \frac{1}{a}\Phi_{12}(x,x_0)$$

from eqs. (17), (21), and (22). Then $\frac{d}{dx}\Phi_{11} = \Phi_{21} = -\frac{1}{a}\phi_{11} + \frac{1}{a}\frac{d}{dx}\Phi_{12}$. But, from Step 2, $\frac{d}{dx}\Phi_{12} = e^{-aQ} - \frac{1}{a}\Phi_{12}$, which leads to $\frac{d}{dx}\Phi_{21} = \frac{1}{a^2}\phi_{11} - qe^{-aQ} - \frac{1}{a^2}e^{-aQ} + \frac{1}{a^3}\Phi_{12}$. Since the latter must also satisfy eq. (15), $\frac{d}{dx}\Phi_{21} = -q\Phi_{11} - p\Phi_{21}$, substituting for these last two matrix elements and matching terms also demonstrates that $p(x)$, $q(x)$ must be related by constraint (3). Since $\Phi_{11}(x_0,x_0)$ and $\Phi_{21}(x_0,x_0)$ both equal δ_{ij}, these matrix elements constitute a solution pair for Problem 1 under the constraint of eq. (3).

Hence, both problem solutions do lead to the same p-q relation (3) as expected and required.

4. Further Interpretation from the Diagonalization of Matrix A

Additional insight on the implications of the constraint eq. (3), $p(x) = aq(x) + \frac{1}{a}$ on the initial-value problem can be obtained through a diagonalization of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -q(x) & -p(x) \end{bmatrix}$$

of equation (12). The two eigenvalues , $\lambda_{\pm}(x)$, satisfy the characteristic equation

$$\lambda^2 + p(x)\lambda + q(x) = 0 \tag{25}$$

and hence

$$\lambda_{\pm}(x) = -\frac{1}{2}p(x) \pm \frac{1}{2}\sqrt{p^2(x) - 4q(x)}. \tag{26}$$

As usual, the sum of the eigenvalues is $-p(x)$ and their product is $q(x)$. For $p(x) = aq(x) + \frac{1}{a}$, this becomes

$$\begin{aligned}\lambda_{\pm}(x) &= -\frac{1}{2}\left(aq(x) + \frac{1}{a}\right) \pm \frac{1}{2}\sqrt{\left(aq(x) - \frac{1}{a}\right)^2} \\ &= -\frac{1}{2}\left(aq(x) + \frac{1}{a}\right) \pm \frac{1}{2}\left|aq(x) - \frac{1}{a}\right|. \end{aligned} \tag{27}$$

Given $q(x)$, for regions of x and the value of the constant a (either positive or negative) such that $aq(x) - \frac{1}{a} > 0$, then

$$\lambda_{+} = -\frac{1}{a}, \tag{28}$$

$$\lambda_{-}(x) = -aq(x). \tag{29}$$

Note that the sum and product of the eigenvalues are as required and that one eigenvalue is constant for this x-region. For the same a-value but for other regions of x such that $aq(x) - \frac{1}{a} < 0$, then

$$\lambda_{+}(x) = -aq(x), \tag{30}$$

$$\lambda_{-} = -\frac{1}{a}. \tag{31}$$

Lastly, for values of x such that $aq(x) - \frac{1}{a} = 0$, the eigenvalues are equal:

$$\lambda_{+} = \lambda_{-} = -\frac{1}{a}. \tag{32}$$

A key observation is that for a given value of the parameter $a \neq 0$, for each of the separate regions of x defined by $aq(x) - \frac{1}{a}$ being positive, negative, or zero, at least one of the eigenvalues $\lambda_{\pm}$ is constant.

If eq. (5) is used as an illustrative eigenvalue example, $q(x) = \sinh^2(x)$ with parameter $a = 1$. From eq. (27),

$$\lambda_{\pm} = -\frac{1}{2}\left[\sinh^2(x) + 1\right] \pm \frac{1}{2}\left|\sinh^2(x) - 1\right|. \tag{33}$$

Since $\sinh^2(x) = x_p = 1$ for $\pm(x_p = 0.881374)$, $\sinh^2(x) - 1 > 0$ for $|x| > x_p$, and

$$\lambda_{+} = -\frac{1}{a} = -1, \tag{34}$$

$$\lambda_-(x) = -aq(x) = -\sinh^2(x) \tag{35}$$

for this x-region. Similarly, for $|x| < x_p$,

$$\lambda_+(x) = -\sinh^2(x) \text{ and} \tag{36}$$

$$\lambda_- = -\frac{1}{a}, \tag{37}$$

and for $|x| = x_p$,

$$\lambda_+ = \lambda_- = -\frac{1}{a} = -1. \tag{38}$$

For the case in which $aq(x) - \frac{1}{a} \neq 0$ and $\lambda_+ \neq \lambda_-$, the eigenvectors $\mathbf{p}_{1-}(x)$, $\mathbf{p}_{1+}(x)$ can be chosen so as to form the modal matrix $\mathbf{P}_1(x)$

$$\mathbf{P}_1(x) = \left[\, \mathbf{p}_{1-}(x)\ \mathbf{p}_{1+}(x) \,\right] = \begin{bmatrix} 1 & 1 \\ \lambda_- & \lambda_+ \end{bmatrix}, \tag{39}$$

for which $\det\big(\mathbf{P}_1(x)\big) = \Delta_p \neq 0$. From this and the previous observations on the eigenvalues, matrix diagonalization can be shown to provide an alternative yet equivalent pathway to the analysis and results of Section 3 as follows.

PROPOSITION 1. *Diagonalization of matrix* $\mathbf{A}$ *within the fundamental matrix relation of* (12) *under the condition of* (3) *also leads to the general results of equations* (17) *through* (22) *for the initial value problem of* (1) *and* (2).

Proof. Case 1: $\lambda_+ \neq \lambda_-$.

$$\mathbf{A}(x) = \mathbf{P}_1(x)\boldsymbol{\Lambda}(x)\mathbf{P}_1(x)^{-1} = \mathbf{P}_1(x)\begin{bmatrix} \lambda_-(x) & 0 \\ 0 & \lambda_+(x) \end{bmatrix}\mathbf{P}_1(x)^{-1} \tag{40}$$

can be used to transform eq. (12) to

$$\mathbf{P}_1(x)^{-1}\frac{d}{dx}\boldsymbol{\Phi}(x,x_0) = \boldsymbol{\Lambda}(x)\mathbf{P}_1(x)^{-1}\boldsymbol{\Phi}(x,x_0), \tag{41}$$

or

$$\begin{bmatrix} \lambda_+\frac{d}{dx}\Phi_{11} - \frac{d}{dx}\Phi_{21} & \lambda_+\frac{d}{dx}\Phi_{12} - \frac{d}{dx}\Phi_{22} \\ -\lambda_-\frac{d}{dx}\Phi_{11} + \frac{d}{dx}\Phi_{21} & -\lambda_-\frac{d}{dx}\Phi_{12} + \frac{d}{dx}\Phi_{22} \end{bmatrix} = \begin{bmatrix} \lambda_-(x) & 0 \\ 0 & \lambda_+(x) \end{bmatrix}\begin{bmatrix} \lambda_+\Phi_{11} - \Phi_{21} & \lambda_+\Phi_{12} - \Phi_{22} \\ -\lambda_-\Phi_{11} + \Phi_{21} & -\lambda_-\Phi_{12} + \Phi_{22} \end{bmatrix}. \tag{42}$$

At this point, we reconsider the effect on eq. (42) of the specific constraint eq. (3), $p(x) = aq(x) + \frac{1}{a}$, for nonzero real constant a. For the subcase of $aq(x) - \frac{1}{a} > 0$ and the corresponding regions in x for which this is true, the eigenvalues are given by eqs. (28)

and (29). With λ_+ being constant, the two top-row differential equations from eq. (42) are, for the top left element,

$$\left(\lambda_+ = -\frac{1}{a}\right)\frac{d}{dx}\Phi_{11} - \frac{d}{dx}\Phi_{21} = (\lambda_-(x) = -aq(x))\left[\left(\lambda_+ = -\frac{1}{a}\right)\Phi_{11} - \Phi_{21}\right], \quad (43)$$

or

$$\frac{d}{dx}[\Phi_{11} + a\Phi_{21}] = -aq(x)\,[\Phi_{11} + a\Phi_{21}], \quad (44)$$

and for the top right element,

$$\frac{d}{dx}[\Phi_{12} + a\Phi_{22}] = -aq(x)[\Phi_{12} + a\Phi_{22}]\,. \quad (45)$$

Eq. (44) is the same as eq. (23), and eqs. (44) and (45) result in the v_1, v_2 normal mode combinations previously seen in Section 3 that lead to the fundamental matrix elements of eqs. (17) to (22) for use in the solution eq. (11).

For the subcase of $aq(x) - \frac{1}{a} < 0$ and its corresponding x-regions, the eigenvalues are now given by eqs. (30) and (31). The eigenvalues exchange roles from the case above, and we instead consider the two bottom row differential equations from eq. (42). From the bottom left, we have

$$-\left(\lambda_- = -\frac{1}{a}\right)\frac{d}{dx}\Phi_{11} + \frac{d}{dx}\Phi_{21} = \\ \left(\lambda_+(x) = -aq(x)\right)\left[-\left(\lambda_- = -\frac{1}{a}\right)\Phi_{11} + \Phi_{21}\right], \quad (46)$$

which repeats eq. (44) while the bottom right element of (42) correspondingly leads to eq. (45). Once again, the results of Theorem 1 follow and apply to this region for x.

Case 2: $\lambda_+ = \lambda_- = \left(\lambda_a = -\frac{1}{a}\right)$. In this case for which $aq(x) - \frac{1}{a} = 0$, the eigenvalues are repeated and constant, and the diagonalization of eq. (40) must be replaced by the Jordan Canonical Form. For eigenvectors $\mathbf{p}_{2-}$ and $\mathbf{p}_{2+}$ chosen to form the modal matrix $\mathbf{P}_2$, we have

$$\mathbf{P}_2 = \left[\,\mathbf{p}_{2-}\ \mathbf{p}_{2+}\,\right] = \begin{bmatrix} 1 & 1 \\ \lambda_a & \lambda_a + 1 \end{bmatrix} \quad (47)$$

where $\lambda_a = -\frac{1}{a}$. Then

$$\mathbf{A} = \mathbf{P}_2\mathbf{J}\mathbf{P}_2{}^{-1} = \mathbf{P}_2\begin{bmatrix} \lambda_a & 1 \\ 0 & \lambda_a \end{bmatrix}\mathbf{P}_2{}^{-1}. \quad (48)$$

From eq. (12),

$$\mathbf{P}_2{}^{-1}\frac{d}{dx}\boldsymbol{\Phi}(x,x_0) = \mathbf{J}\mathbf{P}_2{}^{-1}\boldsymbol{\Phi}(x,x_0), \quad (49)$$

or

$$\begin{bmatrix} (\lambda_a+1)\frac{d}{dx}\Phi_{11}-\frac{d}{dx}\Phi_{21} & (\lambda_a+1)\frac{d}{dx}\Phi_{12}-\frac{d}{dx}\Phi_{22} \\ -\lambda_a\frac{d}{dx}\Phi_{11}+\frac{d}{dx}\Phi_{21} & -\lambda_a\frac{d}{dx}\Phi_{12}+\frac{d}{dx}\Phi_{22} \end{bmatrix} = \begin{bmatrix} \lambda_a & 1 \\ 0 & \lambda_a \end{bmatrix} \begin{bmatrix} (\lambda_a+1)\Phi_{11}-\Phi_{21} & (\lambda_a+1)\Phi_{12}-\Phi_{22} \\ -\lambda_a\Phi_{11}+\Phi_{21} & -\lambda_a\Phi_{12}+\Phi_{22} \end{bmatrix}. \quad (50)$$

The bottom row left element is seen to produce the differential equation

$$\frac{1}{a}\left(\frac{d}{dx}\Phi_{11}\right)+\frac{d}{dx}\Phi_{21} = -\frac{1}{a}\left[\frac{1}{a}\Phi_{11}+\Phi_{21}\right] \quad (51)$$

which is equal to eqs. (44) and (23) since $-aq(x) = -\frac{1}{a}$. The bottom row right element is correspondingly seen to reproduce eq. (45), and the top row elements of eq. (50) repeat this process for eqs. (44) and (45). Hence the results of Theorem 1 follow and also apply to the regions of x for which $aq(x)-\frac{1}{a}=0$.

In summary, the matrix diagonalization of $\mathbf{A}$ has independently demonstrated the applicability of Theorem 1 for the initial value problem described by eqs. (1), (2), and (3) by separately verifying its equivalence for each region of x demarcated by positive, negative, or zero values of the expression $aq(x)-\frac{1}{a}$.

5. Examples: Application to equations (4) and (5)

If the coefficient $q(x)$ of eq. (1) is directly integrable and hence is assumed to possess an antiderivative function $Q(x)$, then eq. (21) is alternately expressible as

$$Q(x,x_0) = Q(x)-Q(x_0) \quad (52)$$

for $Q(x)=\int q(x')\,dx'$. Consequently result (22) becomes

$$I_a(x,x_0) = e^{-\frac{x}{a}}e^{aQ(x_0)}\int_{x_0}^{x} e^{-aQ(x')+\frac{x'}{a}}\,dx', \quad (53)$$

which redefines the two key fundamental matrix elements $\Phi_{11}(x,x_0)$ and $\Phi_{12}(x,x_0)$ of equations (17), (18) for use in eq. (11). Note that if the integrand within $I_a(x,x_0)$ is also integrable, then at least the homogeneous solution associated with eq. (11) would be of closed form, since both Φ_{11} and Φ_{12} would be. Otherwise, the resulting integral-form solution (53) must be evaluated through numerical rather than symbolic integration. The two sample equations in (4) and (5) are chosen to illustrate application of the results of equations (11) and (17) through (22).

5.1. Example eq. (4), Homogeneous Version

Since (4) has $p(x)=2\big(q(x)=x\big)+\frac{1}{2}$, parameter $a=2$, $Q(x)=\frac{x^2}{2}$, and equations (52) and (53) become

$$Q(x,x_0) = \left(Q(x)=\frac{x^2}{2}\right)-\left(Q(x_0)=\frac{x_0^2}{2}\right) \quad (54)$$

and

$$I_{a=2}(x,x_0) = e^{-\frac{x}{2}} e^{x_0{}^2} \int_{x_0}^{x} e^{-\left((x')^2 - \frac{x'}{2}\right)}\, dx'. \tag{55}$$

Completing the square within the integrand of eq. (55) gives

$$I_{a=2}(x,x_0) = \frac{\sqrt{\pi}}{2} e^{-\frac{x}{2}} e^{x_0^2 + \frac{1}{16}} \left[\frac{2}{\sqrt{\pi}} \int_{x_0}^{x} e^{-(x' - \frac{1}{4})^2}\, dx' \right], \tag{56}$$

or, for eq. (18),

$$\Phi_{12}(x,x_0) = I_{a=2}(x,x_0) = \frac{\sqrt{\pi}}{2} e^{-\frac{x}{2}} e^{x_0^2 + \frac{1}{16}} \left[\operatorname{erf}\left(x - \frac{1}{4} \right) - \operatorname{erf}\left(x_0 - \frac{1}{4} \right) \right]. \tag{57}$$

Similarly, from eq. (17),

$$\Phi_{11}(x,x_0) = e^{-\frac{(x-x_0)}{2}} + \frac{1}{2} I_{a=2}(x,x_0), \tag{58}$$

and hence the homogeneous solution of eq. (4) is, from eq. (11),

$$y_h(x,x_0) = e^{-\frac{(x-x_0)}{2}} y_0 + I_{a=2}(x,x_0) \left[\frac{y_0}{2} + \dot{y}_0 \right] \tag{59}$$

for $I_{a=2}(x,x_0)$ defined by eq. (57).

5.2. Example eq. (4), Nonhomogeneous Version

The corresponding general form for the particular solution $y_p(x,x_0)$ of eq. (4) for any excitation $f(x)$ would entail yet another integration of eq. (57),

$$y_p(x,x_0) = \int_{x_0}^{x} I_{a=2}(x,x') f(x')\, dx', \tag{60}$$

which would generally require numerical integration.

However, of particular interest in linear system analysis is the system step response [12] (for zero initial conditions). Formally, for a unit step excitation applied at $x_1 \geqslant x_0$, $f(x') = u(x' - x_1)$, eq. (60) becomes

$$y_{\text{step}}(x,x_1) = \int_{x_1}^{x} I_{a=2}(x,x')\, dx'. \tag{61}$$

5.3. Example eq. (5), Homogeneous Version

For eq. (5), parameter $a = 1$, $q(x) = \sinh^2(x)$, and equations (52) and (53) become

$$Q(x,x_0) = \int_{x_0}^{x} \sinh^2(x')\, dx' = \frac{1}{4}\left[\sinh(2x) - 2x\right] - \frac{1}{4}\left[\sinh(2x_0) - 2x_0\right] \tag{62}$$

and

$$I_{a=1}(x,x_0) = e^{-x} e^{\frac{1}{4}\left[\sinh(2x_0) - 2x_0\right]} \int_{x_0}^{x} e^{-\frac{1}{4}\sinh(2x') + \frac{3}{2}x'}\, dx'. \tag{63}$$

This last equation is not integrable symbolically and requires numerical integration. The corresponding matrix elements are

$$\Phi_{12}(x,x_0) = I_{a=1}(x,x_0) \tag{64}$$

and

$$\Phi_{11}(x,x_0) = e^{-(x-x_0)} + I_{a=1}(x,x_0). \tag{65}$$

As in the previous example, a concise integral form for the homogeneous solution of eq. (5) is from eq. (11),

$$y_h(x,x_0) = e^{-(x-x_0)}y_0 + I_{a=1}(x,x_0)(y_0+\dot{y}_0). \tag{66}$$

5.4. Example eq. (5), Nonhomogeneous Version

As in Section 5.2, under zero initial conditions, the response to a unit step at $x = x_1$ is

$$y_{\text{step}}(x,x_1) = \int_{x_0}^{x} I_{a=1}(x,x')\left[f(x') = u(x'-x_1)\right]dx' = \int_{x_1}^{x} I_{a=1}(x,x')\,dx' \tag{67}$$

for $I_{a=1}(x,x')$ defined by eq. (63) and for $x_1 \geqslant x_0$.

It has been pointed out that a general form for particular solutions of eq. (1) has been determined in [2]. As presented there, a widely-applicable integral-form result, which is contingent upon finding a Riccati differential equation solution $r(x)$ to

$$\frac{d}{dx}r(x) = q(x) - p(x)r(x) + r^2(x), \tag{68}$$

is given by Theorem 6.1 in [2]

$$y_p(x) = e^{-\int r(x)\,dx}\int e^{\int\left(2r(x)-p(x)\right)dx}\left\{\int f(x)e^{\int(p(x)-r(x))\,dx}\,dx\right\}dx. \tag{69}$$

Note that this method requires two consecutive integrations of the forcing term $f(x)$ of eq. (1) in contrast to the single integral within eq. (11). A more exact rendering of eq. (69) which incorporates the details of the interleaved integration process over the region (x_0,x) is

$$y_p(x) = e^{-\int_{x_3=x_0}^{x} r(x_3)\,dx_3}\int_{x_1=x_0}^{x} e^{\int_{x_3=x_0}^{x_1}\left(2r(x_3)-p(x_3)\right)dx_3}\{F(x_1)\}\,dx_1 \tag{70}$$

where

$$F(x_1) = \int_{x_2=x_0}^{x_1} f(x_2)e^{\int_{x_3=x_0}^{x_2}\left(p(x_3)-r(x_3)\right)dx_3}\,dx_2. \tag{71}$$

As indicated in [2], the special case of eq. (1) under constraint (3) admits the simple Riccati solution $r(x) = \frac{1}{a}$. In employing the $Q(x,x_0)$ definition of eq. (21), eq. (70) becomes

$$y_p(x) = e^{-\frac{x}{a}}\int_{x_1=x_0}^{x} e^{\frac{x_1}{a}-aQ(x_1,x_0)}\left\{\int_{x_2=x_0}^{x_1} e^{aQ(x_2,x_0)}f(x_2)\,dx_2\right\}dx_1 \tag{72}$$

which can be condensed to

$$y_p(x) = e^{-\frac{x}{a}} \int_{x_1=x_0}^{x} \int_{x_2=x_0}^{x_1} U(x_1,x_0)V(x_2,x_0)\,dx_2\,dx_1 \tag{73}$$

for

$$U(x,x_0) = e^{\frac{x}{a}-aQ(x,x_0)} \text{ and } V(x,x_0) = e^{aQ(x,x_0)} f(x). \tag{74}$$

For comparison, substitution of eq. (22) into the particular solution part of eq. (11) gives

$$y_p(x) = e^{-\frac{x}{a}} \int_{x_2=x_0}^{x} \left\{ \int_{x_1=x_2}^{x} e^{\frac{x_1}{a}-aQ(x_1,x_0)}\,dx_1 \right\} e^{aQ(x_2,x_0)} f(x_2)\,dx_2 \tag{75}$$

which can be expressed as

$$y_p(x) = e^{-\frac{x}{a}} \int_{x_2=x_0}^{x} \int_{x_1=x_2}^{x} U(x_1,x_0)V(x_2,x_0)\,dx_1\,dx_2. \tag{76}$$

Eq. (76) is seen to be a double integral equivalent to eq. (73) over the same region in the (x_1,x_2) plane with the order of integration interchanged. Hence the method of [2] and the linear systems method presented here are in exact agreement on the form of the particular solutions of eq. (1) subject to constraint (3).

6. Discussion

A general integral-form solution to the second-order variable coefficient system of equations (1), (2) for arbitrary real function $q(x)$ under the constraint of eq. (3) for $p(x)$ has been provided by equations (11) and (17) through (22). This methodology provides solutions to second-order differential equations such as (4) and (5) and all similar equations following the constraint (3).

An examination of extensive tables of solutions and methods for the linear second order, initial-value problems of equations (1) to (3) such as that provided by Zwillinger [15] and Murphy [11] shows that, of the 569 separate results listed for example in the latter, the solution methodology presented here is clearly applicable as an alternate approach for about seven of the equations. Also, although there is arbitrariness in $q(x)$, the availability of only a single adjustable parameter a, together with the form of eq. (3), restrict $p(x)$ such that almost none of the well-known special function equations of physics falls within its domain of applicability. An exception is the confluent hypergeometric equation [1], $x\ddot{y}(x)+(C-x)\dot{y}(x)-Ay(x)=0$, under the parameter selection $A=C$ for relationship parameter $a=-1$.

More generally, however, from the fundamental matrix methodology presented here, the concise nature of the integral-form or truly closed-form solutions when available for equations (1), (2) adhering to eq. (3) can serve as a superior option to symbolic solutions provided by various commercial mathematics programs. That is, their solutions can, at times, be long and unwieldy or even nonexistent. For example, in the case of eq. (4), MATLAB's `dsolve` command outputs the result, eq. (57), exactly as presented here. However, for eq. (5), it produces a lengthy answer for $\Phi_{12}(x,x_0)$ in

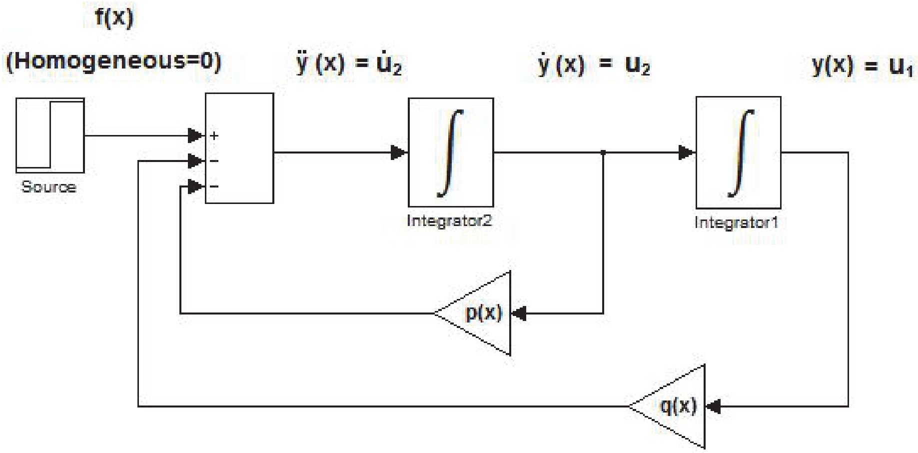

Figure 1: Second-order linear system diagram

terms of the Heun Doubleconfluent function [8], $\mathrm{HeunD}(\alpha,\beta,\gamma,\delta,x)$, stretching over more than one screen. Although the complicated expression using this function represents a closed-form result unlike the integral form eq. (63) appearing in equations (64) and (65), it appears to be difficult to use practically. Alternatively, other versions of MATLAB present solutions for eq. (5) that are also of integral form and involve various combinations of $\tanh(x/2)$ and its integral. Maple also provides long symbolic results with HeunD, and one version of Mathematica failed to provide any solution at all to eq. (5). In contrast, the results generally provided by equations (11) and (17) to (22) are relatively simple and concise and directly encompass both homogeneous and nonhomogeneous forms of the differential equation.

One important area of complementarity between the fundamental matrix methodology presented here and commercial mathematics programs was supplied by the latter's simulation capability. For example, MATLAB's `Simulink` simulation program proved to be more convenient in checking results obtained from equations (11) and (17) to (22) than comparison with various mathematics programs' symbolic outputs. To see this connection, it is useful to recreate an "analog computer" simulation diagram [12] that pictures equations (1) or (6), (7), (8) of the corresponding linear system as shown in Fig. 1. The second-order system is comprised of two integrators, a summing device, and two (negative) feedback amplifiers of variable amplification $p(x)$, $q(x)$. The input $f(x)$ from the source on the left together with any initial energy y_0, $\dot{y}_0$ in the integrators produces an output $y(x)$ on the right. By tracing the signal pathways, for example, leading into the summing device from $p(x)$, $q(x)$, and $f(x)$, it can be seen that $\ddot{y}(x)$ is the output of the summing device that equals $-p(x)\dot{y}(x) - q(x)y(x) + f(x)$ as per eq. (1).

Furthermore, it can also be seen from Fig. 1 that each of the simulation waveforms for Φ_{ij} can be obtained from equation (11) with $f(x) = 0$ through appropriate choice for initial conditions y_0, $\dot{y}_0$. That is, by choosing either $(1,0)$ or $(0,1)$ for $(y_0,\dot{y}_0)$ in integrators $1,2$ of Fig. 1, eq. (11) shows that the former condition produces the output $y(x) = \Phi_{11}(x,x_0)$ and the latter gives $y(x) = \Phi_{12}(x,x_0)$. Similarly, the $\dot{y}(x)$ output leads to Φ_{21} and Φ_{22} as seen from equations (13) and (14). These Φ_{ij} waveforms were then each compared directly to mathematical outputs constructed as systems in `Simulink` as per the corresponding predictions from equations (17) to (22). Agreement was obtained when the differences between these predicted waveforms and the corresponding simulation outputs were found to be zero, as expected. These matrix elements can be graphically constructed from either the theoretical form or simulation data. An illustrative three-dimensional visualization of $\Phi_{11}(x,x_0)$ and $\Phi_{12}(x,x_0)$ for eq. (4) is shown in Figures 2 and 3 respectively. These surfaces display the matrix elements' variation in x for progressing values of x_0, summarizing their behavior in an insightful way. Recall that these elements are nonzero only for $x \geqslant x_0$.

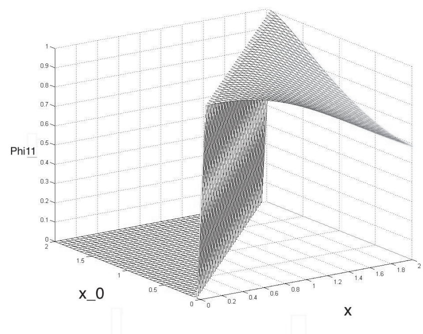

Figure 2: $\Phi_{11}(x,x_0)$ for eq. (4) plotted over the range $(0,2)$ for x and x_0

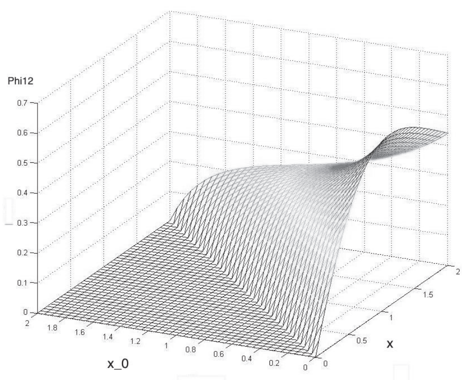

Figure 3: $\Phi_{12}(x,x_0)$ for eq. (4) plotted over the range $(0,2)$ for x and x_0

7. Conclusions

In summary, for those second-order linear equations conforming to the constraint of eq. (3), the fundamental matrix methodology of equations (11) and (17) to (22) provides integral-form results for those equations in both the homogeneous and nonhomogeneous cases. These results are sometimes of closed form and often simpler than symbolic integration results provided by commercial mathematics programs which, for some program versions in some cases, fail to provide any symbolic solutions to equations of type (3) at all.

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